Automorphisms of the alternating group network

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Abstract

Let A_n be the alternating group of degree n with $n \geq 4$. Set $T = \{(1\ 2\ 3), (1\ 3\ 2), (1\ 2)(3\ i) \mid 4 \leq i \leq n\}$. The *alternating group network*, denoted by AN_n, is defined as the Cayley graph on A_n with respect to T. Some properties of AN_n have been investigated in [Appl. Math.—JCU. Ser. A 14 (1998) 235-139; IEEE Tran. Comput. 55 (2006) 1645-1648; Inform. Process Lett. 110 (2010) 403-409; J. Supercomput. 54 (2010) 206-228]. In this paper, it is shown that the full automorphism group of AN_n is the semi-direct product $R(A_n) \rtimes \mathrm{Aut}(A_n, T)$, where $R(A_n)$ is the right regular representation of A_n and $\mathrm{Aut}(A_n, T) = \{\alpha \in \mathrm{Aut}(A_n) \mid T^\alpha = T\} \cong S_{n-3} \times S_2$.

Key Words: Automorphism; Cayley graph; Vertex-tansitive

1 Introduction

We follow [11, 12] for graph theoretical terminology and notation not defined here. Throughout the paper graphs are undirected finite connected without loops or multiple edges. For a graph G, denote by $V(G)$, $E(G)$, $A(G)$ and $\mathrm{Aut}(G)$ the vertex-set, edge-set, arc-set and full automorphism group of G, respectively. A graph G is said to be *vertex-transitive*, *edge-transitive* or *arc-transitive* if $\mathrm{Aut}(G)$ acts transitively on $V(G)$, $E(G)$ or $A(G)$, respectively.

For a set V and a group G with identity element 1, an *action* of G on V is a mapping $V \times G \to V$, $(v, g) \mapsto v^g$, such that $v^1 = v$ and $(v^g)^h = v^{gh}$ for $v \in V$ and $g, h \in G$. The subgroup $K = \{g \in G \mid v^g = v, \forall v \in V\}$ of G is called the *kernel* of G acting on V. For two groups K, H, if H acts on K (as a set) such that $(xy)^h = x^h y^h$ for any $x, y \in K$ and $h \in H$, then H is said to *act* on K as a group. In this case, we use $K \rtimes H$ to denote the *semi-direct product* of K by H with respect to the action.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$ (where 1 is the identity element of G), the *Cayley graph* $\mathrm{Cay}(G,S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. The automorphism group $\mathrm{Aut}(\mathrm{Cay}(G,S))$ of $\mathrm{Cay}(G,S)$ contains the right regular representation $R(G)$ of G, the action of G on itself by right multiplication, as a subgroup. Thus, $\mathrm{Cay}(G,S)$ is vertex-transitive. Furthermore, $\mathrm{Aut}(G,S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G,S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$. Xu [13, Proposition 1.5] proved that $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Aut}(\mathrm{Cay}(G,S))_1 = \mathrm{Aut}(G,S)$.

Ji [6] proposed a new type of alternating group graph, AN_n, which improve the initial alternating group graphs AG_n advocated by Jwo et al. [7]. Let A_n be the alternating group of degree n with $n \geq 3$. Set $T = \{(1\ 2\ 3), (1\ 3\ 2), (1\ 2)(3\ i) \mid 4 \leq i \leq n\}$. The *alternating group network*, denoted by AN_n, is defined as the Cayley graph

$$AN_n = \mathrm{Cay}(A_n, T). \tag{1}$$

It has been shown that the alternating group network has many nice structures and properties. For example, Ji [6] proved that AN_n has a node degree that is smaller by a factor of about 2 while maintaining a diameter comparable to that of AG_n, is maximally fault tolerant, and shares some of the positive structural attributes of the well-known star graph. Chen et al. [1] characterized the distance between any two nodes in AN_n and presented an optimal (shortest-path) routing algorithm for AN_n. Zhou [16] determined the h-extra connectivity of AN_n. Zhou et al. [18] showed that in a given AN_n, there exist $n-1$ parallel paths between any pair of nodes. They also showed that the wide diameter of AN_n is at most one unit greater than the known lower bound $D+1$, where D is the network diameter. Zhou et al. [17] studied the conditional diagnosability of AN_n.

However, the full automorphism group of AN_n remained unknown. Computing the automorphism group of a graph is a very difficult topic in algebraic graph theory, and there are a lot of work along this line, see, for example, [3, 4, 8, 9, 13]. For the automorphism groups of Cayley graphs having connection with interconnection networks, there are also some interesting work. For example, Feng [2] proved that for any minimal generating set S of transpositions of the symmetric group S_n of degree n, the Cayley graph $\mathrm{Cay}(S_n, S)$ is normal, that is, $\mathrm{Aut}(\mathrm{Cay}(S_n,S)) = R(S_n) \rtimes \mathrm{Aut}(S_n, S)$. Note that this family of graphs contained the well-known bubble-sort graph, modified bubble-sort graph and star graph. The author [14] characterized the full automorphism group of the alternating group graph. For more results on the applications of Cayley graphs to interconnection networks, we refer the reader to [5, 10, 15].

In this article, we completely determined the full automorphism group of AN_n. The following is the main result.

Theorem 1.1 *The full automorphism group* $\mathrm{Aut}(AN_n)$ *of* AN_n *is isomorphic to* $R(A_n) \rtimes \mathrm{Aut}(A_n, T)$. *Furthermore,*

$$\mathrm{Aut}(A_n,T) = \langle\sigma((1\ 2))\rangle \times \langle\sigma((4\ 5)),\sigma((4\ 6)),\cdots,\sigma((4\ n))\rangle \cong S_2 \times S_{n-3},$$

where $\sigma(g)$ *is the automorphism of* A_n *induced by the conjugacy action of* g *for* $g \in S_n$.

Remark As a consequence of Theorem 1.1, AN_n is a normal Cayley graph which is neither arc-transitive nor edge-transitive.

2 Proof of Theorem 1.1

Let $A = \mathrm{Aut}(AN_n)$ and let A_e be the stabilizer of the identity element e of A_n in A. For any $4 \le i < j \le n$, by an easy computation, we have the following:

$$\begin{array}{l}(1\ 2\ 3)(1\ 2)(3\ i) = (2\ i\ 3), (1\ 3\ 2)(1\ 2)(3\ i) = (1\ i\ 3),\\ (1\ 2)(3\ i)(1\ 2\ 3) = (1\ 3\ i), (1\ 2)(3\ i)(1\ 3\ 2) = (2\ 3\ i),\\ (1\ 2)(3\ i)(1\ 2)(3\ j) = (3\ i\ j), (1\ 2)(3\ j)(1\ 2)(3\ i) = (3\ j\ i).\end{array} \tag{2}$$

We now prove three claims.

Claim 1 There are no 4-cycles in AN_n.

By the vertex-transitivity of AN_n, it suffices to show that there is no 4-cycle in AN_n passing through e. For any vertex $u \in V(AN_n)$, denote by $N(u)$ the neighborhood of u in AN_n. By Eq. (2),

$$\begin{array}{lcl} N((1\ 2\ 3)) & = & \{e, (1\ 3\ 2)\ (1\ 3\ j) \mid 4 \le j \le n\}\\ N((1\ 3\ 2)) & = & \{e, (1\ 2\ 3), (2\ 3\ j) \mid 4 \le j \le n\}\\ N((1\ 2)(3\ i)) & = & \{e, (2\ i\ 3), (1\ i\ 3), (3\ j\ i) \mid j \ne i, 4 \le j \le n\}(4 \le i \le n).\end{array}$$

From this it is seen that for any two elements in T, they have exactly one common neighbor, that is, e. Consequently, there is no 4-cycle passing through e.

Claim 2 For any $g, h \in T - \{(1\ 2\ 3), (1\ 3\ 2)\}$, there is a unique 6-cycle in AN_n passing through e, g, h, that is, $(e, g, hg, ghg = hgh, gh, h, e)$.

Assume that $g = (1\ 2)(3\ k)$, $h = (1\ 2)(3\ \ell)$ with $k \ne \ell$ and $4 \le k, \ell \le n$. By Eq. (2), $gh \ne hg$. An easy computation gives $ghg = hgh = (1\ 2)(k\ \ell)$. This implies that $(e, g, hg, ghg = hgh, gh, h, e)$ is a 6-cycle.

Let $C := (e, g, g_1g, g_2g_1g = h_2h_1h, h_1h, h, e)$ be an arbitrary 6-cycle in X passing through e, g, h, where g_1, g_2, h_1, $h_2 \in T$. Then $g_2 \neq h_2$, $g_2g_1 \neq e$, $h_2h_1 \neq e$, $g_1 \neq g, h_1 \neq h$. Since $g_2g_1g = h_2h_1h$, one has $g_2g_1 = h_2h_1(3\ \ell\ k)$.

Suppose $h_2 \neq h$. Then $h_2 = (1\ 2\ 3)$, $(1\ 3\ 2)$ or $(1\ 2)(3\ s)$ for some $4 \leq s \leq n, s \neq \ell$. Assume $h_2 = (1\ 2\ 3)$. Then $h_1 \neq (1\ 3\ 2)$. If $h_1 = h_2$ then $h_2h_1 = (1\ 3\ 2)$, and then $g_2g_1 = h_2h_1(3\ \ell\ k) = (1\ \ell\ k\ 3\ 2)$. This is impossible by Eq. (2). Let $h_1 = (1\ 2)(3\ t)$ for some $4 \leq t \leq n, t \neq \ell$. By Eq. (2), $h_2h_1 = (2\ t\ 3)$. Then $g_2g_1 = h_2h_1(3\ \ell\ k)$ is equal to $(2\ t\ \ell\ k\ 3)$ for $t \neq k$ or to $(2\ 3)(k\ \ell)$ for $t = k$. Again, by Eq. (2), this can not happen. Similarly, $h_2 \neq (1\ 3\ 2)$. Assume $h_2 = (1\ 2)(3\ s)$ for some $4 \leq s \leq n, s \neq \ell$. Then $h_2h_1 = (1\ 3\ s), (2\ 3\ s)$ or $(3\ s\ t)$ for some $4 \leq t \leq n, t \neq s, \ell$. If $s = k$, then g_2g_1 would be equal to one of the following: $(1\ 3\ s)(3\ \ell\ k) = (1\ \ell\ k), (2\ 3\ s)(3\ \ell\ k) = (2\ \ell\ k)$ and $(3\ s\ t)(3\ \ell\ k) = (k\ t\ \ell)$. This is impossible by Eq. (2). If $s \neq k$, then g_2g_1 would be equal to one of the following: $(1\ 3\ s)(3\ \ell\ k) = (1\ \ell\ k\ 3\ s), (2\ 3\ s)(3\ \ell\ k) = (2\ \ell\ k\ 3\ s)$ and $(3\ s\ t)(3\ \ell\ k) = (3\ s)(k\ \ell)$ (for $t = k$) or $(3\ s\ t\ \ell\ k)$ (for $t \neq k$). This is also impossible by Eq. (2). Thus, $h_2 = h$.

By the same argument in the above paragraph, one may conclude that $g_2 = g$. Then $gg_1g = hh_1h$, and hence $g_1 = (hg)^{-1}h_1(hg)$. If $h_1 = (1\ 2\ 3)$ or $(1\ 3\ 2)$ then $g_1 = (1\ 2\ \ell)$ or $(1\ \ell\ 2)$, which is contrary to the fact that $g_1 \in T$. If $h_1 = (1\ 2)(3\ s)$ for some $s \neq k$ and $4 \leq s \leq n$, then $g_1 = (1\ 2)(\ell\ s) \notin T$, a contradiction. Thus, $h_1 = g$, and hence $g_1 = h$. It follows that $C = (e, g, hg, ghg = hgh, gh, h, e)$ is the unique 6-cycle in AN_n passing through e, g, h, as claimed.

Claim 3 For any $g \in \{(1\ 2\ 3), (1\ 3\ 2)\}$ and $h \in T - \{(1\ 2\ 3), (1\ 3\ 2)\}$, there is a unique 6-cycle in X passing through e, g, h, that is, $(e, g, hg, ghg = hg^{-1}h, g^{-1}h, h, e)$.

Without loss of generality, assume $g = (1\ 2\ 3)$ and $h = (1\ 2)(3\ i)$ with $4 \leq i \leq n$. It is easy to check that $(e, g, hg, ghg = hg^{-1}h, g^{-1}h, h, e)$ is a 6-cycle. By an easy computation, one may obtain that the set of vertices at distance 2 from g is

$$\begin{aligned}\Theta = \ & \{(1\ 2)(3\ j) \mid 4 \leq j \leq n\} \cup \\ & \{(1\ 2\ s), (1\ s)(2\ 3), (2\ 3\ t), (3\ t\ s) \mid t \neq s, 4 \leq t, s \leq n\}.\end{aligned}$$

Also, the set of vertices at distance 2 from h is

$$\begin{aligned}\Lambda_i = \ & \{(1\ 2\ 3), (1\ 3\ 2), (1\ 2)(3\ j) \mid j \neq i, 4 \leq j \leq n\} \cup \\ & \{(1\ i\ 3\ \ell\ 2), (1\ i\ 2), (1\ 2\ i\ 3\ \ell), (1\ 2\ i), (1\ 2\ \ell\ i\ 3), \\ & (1\ \ell\ i\ 3\ 2), (1\ 2)(i\ \ell), (1\ 2)(3\ t\ \ell\ i) \mid \ell \neq i, t \neq i, t \neq \ell, 4 \leq \ell, t \leq n\}.\end{aligned}$$

Let C be an arbitrary 6-cycle passing through e, g, h. Then in C there is a vertex, say u, at distance 3 from e. Clearly, $u \in \Theta \cap \Delta$. The only possible case is $u = (1\ 2\ i)$. By Claim 1, there are no 4-cycles in AN_n. It follows that C must be $(e, g, hg, ghg = hg^{-1}h, g^{-1}h, h, e)$, as claimed.

Now we are ready to complete the proof. Let A_e^* be the kernel of A_e acting on T. Then $A_e/A_e^* \lesssim S_{n-1}$. Take $g \in T$. If $g = (1\ 2\ 3)$, then by Claim 3, for any $h = (1\ 2)(3\ i)$ with $4 \le i \le n$, $(e, g, hg, ghg = hg^{-1}h, g^{-1}h, h, e)$ is the unique 6-cycle in X passing through e, g, h. Since A_e^* fixes e, g and h, it also fixes each vertex in this 6-cycle. In particular, A_e^* fixes $hg = (1\ 3\ i)$ and $g^{-1}h = (1\ i\ 3)$. Clearly, A_e^* fixes $(1\ 3\ 2)$ because $(1\ 3\ 2) \in T$. So, A_e^* fixes each neighbor of $g = (1\ 2\ 3)$. If $g = (1\ 3\ 2)$, by the same argument as above, A_e^* fixes each neighbor of g and also fixes $g^{-1}h = (2\ i\ 3)$. If $g = (1\ 2)(3\ i)$ for some $4 \le i \le n$, then by Claim 2, for any $h = (1\ 2)(3\ j)$ with $j \ne i$, $(e, g, hg, ghg = hgh, gh, h, e)$ is the unique 6-cycle in AN_n passing through e, g, h. Since A_e^* fixes e, g, h, it must fixes $hg = (3\ j\ i)$. Remember that A_e^* also fixes $(1\ i\ 3)$ and $(2\ i\ 3)$. Now we know that A_e^* also fixes each neighbor of $g = (1\ 2)(3\ i)$. By the connectivity and the vertex-transitivity of AN_n, A_e^* fixes all vertices of AN_n, and so $A_e^* = 1$. Since $(e, (1\ 2\ 3), (1\ 3\ 2), e)$ is the unique triangle in subgraph induced by $\{e\} \cup T$, one has $A_e \lesssim S_2 \times S_{n-3}$.

For any $g \in S_n$, let $\sigma(g)$ denote the automorphism of A_n induced by the conjugacy action of g. It is easy to check that $\sigma((4\ i))(5 \le i \le n)$ and $\sigma((1\ 2))$ are in $\mathrm{Aut}(A_n, T)$. Furthermore, by the elementary group theory,

$$\langle \sigma((1\ 2)) \rangle \times \langle \sigma((4\ 5)), \sigma((4\ 6)), \cdots, \sigma((4\ n)) \rangle \cong S_2 \times S_{n-3}.$$

This forces that

$$\mathrm{Aut}(A_n, T) = \langle \sigma((1\ 2)) \rangle \times \langle \sigma((4\ 5)), \sigma((4\ 5)), \cdots, \sigma((4\ n)) \rangle,$$

and hence $A = R(A_n) \rtimes \mathrm{Aut}(A_n, T)$. □

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