The Co-PI index of some compound graphs*

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Abstract. Let G be a graph and u be a vertex of G. $T_G(u)$ is the sum of distances from u to all the other vertices in graph G, $T(u) = T_G(u) = \sum_{v \in V} d_G(u,v)$. The Co-PI index [1] is defined as $Co\text{-}PI_v(G) = \sum_{uv \in E(G)} |T(u) - T(v)|$. In this paper, we give some upper bounds for the Co-PI indices of the join, composition, disjunction, symmetric difference and corona graph $G_1 \circ G_2$.

Keywords: CO-PI index; compound graphs; corona graph

1 Introduction

This paper we only consider simple connected graphs without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, the distance between the vertices u and v of $V(G)$ is denoted by $d_G(u,v)$ and it is defined as the number of edges in a shortest path connecting the vertices u and v, $d_G(u)$ denote the degree of vertex u in graph G.

A topological index is a real number related to a graph which does not depend on the label or the pictorial representation of the graph. The Wiener index [2] is one of the most studied topological indices, both from theoretical point of view and applications, see for details [3–5]. The Zagreb indices [18] have been introduced more than thirty years ago by Gutman and Trinajestic. They are defined as $M_1(G) = \sum_{v \in V(G)} deg_G^2(v)$, $M_2(G) = \sum_{uv \in E(G)} deg_G(u) deg_G(v)$. We encourage the reader to consult [7–12] for historical background, computational techniques and mathematical properties of Zagreb indices. Recently, Hasani et al. [13] introduced a new topological index similar to the vertex version of PI index. This index is called the

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Co-PI index of G and defined as: $Co\text{-}PI_v(G) = \sum_{e \in E(G)} |n_u(e) - n_v(e)|$, here the summation goes over all edges of G. The transmission of a vertex u in graph G, denoted by $T_G(u)$, is the sum of distances from it to all the other vertices in graph G, i.e., $T(u) = T_G(u) =: \sum_{v \in V} d_G(u, v)$. According to the definition of transmission, Guifu Su et al. [1] give the equivalent definition of the Co-PI index: $Co\text{-}PI_v(G) = \sum_{uv \in E(G)} |T(u) - T(v)|$, then determine the eigenvalues of Co-PI matrices and their Laplacians of Cartesian product graphs, including bounds on the second and third Co-PI spectral moment of a graph. The explicit formulae for the Co-PI index of Cartesian product graphs are also presented.

Now we recall that some definitions of several compound graphs. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1, V_2 and edge sets E_1, E_2 is the graph union $G_1 \bigcup G_2$ together with all the edges joining V_1 and V_2. If $G = \underbrace{H + ... + H}_{n}$, then we denote G by nH.

The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex setsV_1, V_2 and edge sets E_1, E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2).

The symmetric difference $G_1 \bigoplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and $E(G_1 \bigoplus G_2) = \{(u_1, u_2), (v_1, v_2) \mid u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \text{ but not both}\}$.

The disjunction $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_1, v_1) is adjacent with (u_2, v_2), whenever $u_1u_2 \in E(G_1)$ or $v_1v_2 \in E(G_2)$.

Given two graphs G_1 (with vertices $1, 2, ..., n$ and edges $e_1, e_2, ..., e_n$) and G_2. The vertex corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking n copies of G_2 and for each vertice of G_1 must connected every vertex of G_2.

In [14], Chen ming have researched Total Coloring of Join Graph $C_n \bigvee K_{n-3}$. Khalifeh et al. [15] computed vertex and edge PI indices of the join and composition of graphs. Mehdi Eliasi et al. [16] study the Wiener index of some graph operations. Mehdi Eliasi and Bijan Taeri [17] present the Wiener indices of four new sums of graphs. The hyper-Wiener index [18] of graph operations is presented by M. H. Khalifeh. Yaoping HOU and Waichee SHIU [19] give the spectrum of the edge corona of two graphs. In [20] Sagan et al. computed some exact formulae for the Wiener Polynomal of various graph operations containing Cartesian product, composition, join, disjunction and symmetric difference of graphs.

In this paper, we give some upper bounds for the Co-PI indices of the join, composition, disjunction, symmetric difference and corona graph

$G_1 \circ G_2$.

2 Main results

In this section, some exact formulae for the Co-PI indices of the join, composition, disjunction, symmetric difference and corona graphs are presented. We first give the Co-PI index of the join graph. The following two Lemmas related to distance properties of join graphs are required.

Lemma 1 ([18]) *Let G_1 and G_2 be graphs, then we have*

$$d_{G_1+G_2}(u,v)= \begin{cases} 0, & u=v; \\ 1, & uv \in E(G_1) \text{ or } uv \in E(G_2) \text{ or } u \in V(G_1),\ v \in V(G_2); \\ 2, & \text{otherwise.} \end{cases}$$

Lemma 2 *Let G_1 and G_2 be connected graphs, $G = G_1 + G_2$, then*
i)$u \in G_1, T_G(u) = |V_2| - d_{G_1}(u) + 2|V_1| - 2$;
ii)$v \in G_2, T_G(v) = |V_1| - d_{G_2}(v) + 2|V_2| - 2$.

Proof: It follows the definition of join graph that each of its edges contain either one vertex in G_1 one vertex in G_2, or two vertices in G_1 or G_2.
i)if $u \in G_1$, then v lies either in G_1 or in G_2.

$$\begin{aligned} T_G(u) &= \sum_{u\in V(G),v\in V(G)} d_G(u,v) \\ &= \sum_{u\in V(G_1),v\in V(G_1)} d_G(u,v) + \sum_{u\in V(G_1),v\in V(G_2)} d_G(u,v) \\ &= |V_2| - d_{G_1}(u) + 2|V_1| - 2. \end{aligned}$$

ii)if $v \in G_2$, then s lies either in G_1 or in G_2.

$$\begin{aligned} T_G(v) &= \sum_{v\in V(G),s\in V(G)} d_G(v,s) \\ &= \sum_{v\in V(G_2),s\in V(G_1)} d_G(v,s) + \sum_{v\in V(G_2),s\in V(G_2)} d_G(v,s) \\ &= |V_1| - d_{G_2}(v) + 2|V_2| - 2. \end{aligned}$$

Theorem 3 *Let G_1 and G_2 be connected graphs, then*
$Co\text{-}PI_v(G_1+G_2) \leq M_1(G_1)+M_1(G_2)+|V_1|^2|V_2|-|V_2|^2|V_1|+2|V_1||E(G_2)|-2|V_2||E(G_1)|$.

Proof: Set $V(G_1) = \{u_1, u_2, ..., u_n\}$ and $V(G_2) = \{v_1, v_2, ..., v_n\}$. Applying Lemma 1 and Lemma 2, we have:

$Co\text{–}PI_v(G_1+G_2)$

$$\begin{aligned}
&= \sum_{uv\in E(G_1)} |d_{G_1}(u)-d_{G_1}(v)| + \sum_{st\in E(G_2)} |d_{G_2}(s)-d_{G_2}(t)| \\
&\quad + \sum_{\substack{w\in V(G_1)\\ l\in V(G_2)}} ||V_1|-|V_2|-d_{G_1}(w)+d_{G_2}(l)| \\
&\leq \sum_{uv\in E(G_1)} |d_{G_1}(u)|+|d_{G_1}(v)| + \sum_{st\in E(G_2)} |d_{G_2}(s)|+|d_{G_2}(t)| \\
&\quad + \sum_{\substack{w\in V(G_1)\\ l\in V(G_2)}} ||V_1|-|V_2|-d_{G_1}(w)+d_{G_2}(l)| \\
&= M_1(G_1)+M_1(G_2)+|V_1|^2|V_2|-|V_2|^2|V_1|+2|V_1||E(G_2)| \\
&\quad -2|V_2||E(G_1)|.
\end{aligned}$$

Lemma 4 *[18] Let G_1 and G_2 be graphs, if G_1 is connected with $|V(G_1)| > 1$ and $G=G_1[G_2]$, then for any two vertices $(u_1,v_1),(u_2,v_2)\in V(G)$, we have*

$$d_G((u_1,v_1),(u_2,v_2)) = \begin{cases} d_{G_1}(u_1,u_2), & u_1\neq u_2;\\ 0, & u_1=u_2 \text{ and } v_1=v_2;\\ 1, & u_1=u_2 \text{ and } v_1v_2\in E_2;\\ 2, & u_1=u_2 \text{ and } v_1v_2\notin E_2.\end{cases}$$

Theorem 5 *Let G_1 and G_2 be two graphs, if G_1 is connected with $|V(G_1)| > 1$ and $G=G_1[G_2]$, then we have: $Co\text{-}PI_v(G)\leq |V_2|^3 Co\text{-}PI_v(G_1)+(|V_2|^2-|V_2|+|V_1|)M_1(G_2)$.*

Proof: It follows from the definition of G that each of its edges is either of the form $u_i\sim u_k$ or $u_i=u_k$ and $v_j\sim v_l$, then

$$\begin{aligned}
&T((u_i,v_j))-T((u_k,v_l)) \\
=&[\sum_{(x,y)\in V(G)} d_G((u_i,v_j),(x,y)) - \sum_{(a,s)\in V(G)} d_G((u_k,v_l),(a,s))] \\
&=[\sum_{\substack{(x,y)\in V(G)\\ (u_i,x)\in E(G_1)}} d_{G_1}(u_i,x) + \sum_{\substack{(x,y)\in V(G)\\ u_i=x,(v_j,y)\in E(G_2)}} 1 + \sum_{\substack{(x,y)\in V(G)\\ u_i=x,(v_j,y)\notin E(G_2)}} 2]
\end{aligned}$$

$$-[\sum_{\substack{(a,s)\in V(G)\\(u_k,a)\in E(G_1)}} d_{G_1}(u_k,a)+\sum_{\substack{(a,s)\in V(G)\\u_k=a,(v_l,s)\in E(G_2)}} 1+\sum_{\substack{(a,s)\in V(G)\\u_k=a,(v_l,s)\notin E(G_2)}} 2]$$
$$= [|V_2|T_{G_1}(u_i)+d_{G_2}(v_j)+(|V_2|-1-d_{G_2}(v_j))\times 2]-[|V_2|T_{G_1}(u_k)+d_{G_2}(v_l)+(|V_2|-1-d_{G_2}(v_l))\times 2]$$
$$= |V_2|(T_{G_1}(u_i)-T_{G_1}(u_k))+(d_{G_2}(v_l)-d_{G_2}(v_j))$$

This implies that: $|T((u_i,v_j))-T((u_k,v_l))|$

$$= ||V_2|(T_{G_1}(u_i)-T_{G_1}(u_k))+(d_{G_2}(v_l)-d_{G_2}(v_j))|$$
$$\leq ||V_2|(T_{G_1}(u_i)-T_{G_1}(u_k))|+|d_{G_2}(v_l)-d_{G_2}(v_j)|$$

Therefore, the Co-PI of composition graph G is given by

$$Co\text{–}PI_v(G)=\sum_{((u_i,v_j),(u_k,v_l))\in E(G)} |T((u_i,v_j)-T((u_k,v_l))|$$
$$\leq |V_1|M_1(G_2)+|V_2|^2Co\text{–}PI_v(G_1)+|V_2|(|V_2|-1)(|V_2|\,Co\text{–}PI_v(G_1)+M_1(G_2))$$
$$= |V_2|^3Co\text{–}PI_v(G_1)+(|V_2|^2-|V_2|+|V_1|)M_1(G_2).$$

Lemma 6 ([18]) *Let G_1 and G_2 be connected graphs, Then*

$$d_{G_1\vee G_2}((a,b)(c,d))=\begin{cases}0, & a{=}c \text{ and } b{=}d;\\ 1, & ac\in E(G_1) \text{ or } bd\in E(G_2);\\ 2, & \text{otherwise.}\end{cases}$$

Theorem 7 *Let G_1 and G_2 be connected graphs, $G=G_1\bigvee G_2$, then Co-$PI_v(G)\leq |V_1|^2M_1(G_1)+|V_2|^2M_1(G_2)+2|V_2||E(G_2)|M_1(G_1)+2|V_1||E(G_1)|\,M_1(G_2)-M_1(G_1)M_1(G_2)$.*

Proof: Let us consider an edge $e=((u_i,v_j),(u_k,v_l))$. We have $T((u_i,v_j))-T((u_k,v_l))$

$$=[\sum_{(x,y)\in V(G)} d_G((u_i,v_j),(x,y))-\sum_{(a,s)\in V(G)} d_G((u_k,v_l),(a,s))]$$
$$=[\sum_{\substack{(x,y)\in V(G)\\(u_i,x)\in E(G_1)or\\(v_j,y)\in E(G_2)}} d_G((u_i,v_j),(x,y))+\sum_{\substack{(x,y)\in V(G)\\(u_i,x)\notin E(G_1)and\\(v_j,y)\notin E(G_2)}} d_G((u_i,v_j),(x,y))]$$
$$-[\sum_{\substack{(a,s)\in V(G)\\(u_k,a)\in E(G_1)or\\(v_l,s)\in E(G_2)}} d_G((u_k,v_l),(a,s))+\sum_{\substack{(a,s)\in V(G)\\(u_k,a)\notin E(G_1)and\\(v_l,s)\notin E(G_2)}} d_G((u_k,v_l),(a,s))]$$
$$=|V_1|(d_{G_2}(v_l)-d_{G_2}(v_j))+|V_2|(d_{G_1}(u_k)-d_{G_1}(u_i))+d_{G_1}(u_i)d_{G_2}(v_j)-d_{G_1}(u_k)d_{G_2}(v_l)$$

This implies that $|T((u_i,v_j)) - T((u_k,v_l))|$

$$= ||V_1|(d_{G_2}(v_l) - d_{G_2}(v_j)) + |V_2|(d_{G_1}(u_k) - d_{G_1}(u_i)) + d_{G_1}(u_i)d_{G_2}(v_j) - d_{G_1}(u_k)d_{G_2}(v_l)|$$
$$\leq ||V_1|d_{G_2}(v_l) - d_{G_2}(v_j)| + |V_2||d_{G_1}(u_k) - d_{G_1}(u_i)| + |d_{G_1}(u_i)d_{G_2}(v_j) - d_{G_1}(u_k)d_{G_2}(v_l)|$$

Therefore, the Co-PI of G is given by $Co\text{–}PI_v(G)$

$$= \sum_{((u_i,v_j),(u_k,v_l))\in E(G)} |T((u_i,v_j)) - T((u_k,v_l))|$$
$$\leq |V_1|^2M_1(G_2) + |V_2|^2M_1(G_1) + \sum_{i=1}^{|V_1|}\sum_{j=1}^{|V_2|} d_G(u_i,v_j)d_{G_1}(u_i)d_{G_2}(v_j)$$
$$= |V_1|^2M_1(G_2) + |V_2|^2M_1(G_1) + 2|V_2||E(G_2)|M_1(G_1) + 2|V_1||E(G_1)| M_1(G_2) - M_1(G_1)M_1(G_2).$$

Lemma 8 ([18]) *Let G_1 and G_2 be connected graphs, then*
$$d_{G_1\bigoplus G_2}((a,b),(c,d)) = \begin{cases} 0, & a=c \text{ and } b=d; \\ 1, & ac\in E(G_1) \text{ or } bd\in E(G_2), \text{ but not both}; \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 9 *Let G_1 and G_2 be connected graphs, $G = G_1 \bigoplus G_2$, then Co–$PI_v(G) \leq |V_1|^2M_1(G_1) + |V_2|^2M_1(G_2) + 4|V_2||E(G_2)|M_1(G_1) + 4|V_1||E(G_1)| M_1(G_2) - 2M_1(G_1)M_1(G_2)$.*

Proof: According to the definition of G, each of its edges is either $(u_i,v_j) \in E(G_1)$ or $(u_k,v_l) \in E(G_2)$, but not both. Without loss of generality, let us consider an edge $e = (u_i,v_j)(u_k,v_l)$, then $T((u_i,v_j)) - T((u_k,v_l))$

$$= [\sum_{(x,y)\in V(G)} d_G((u_i,v_j),(x,y)) - \sum_{(a,s)\in V(G)} d_G((u_k,v_l),(a,s))]$$
$$= [\sum_{\substack{(x,y)\in V(G)\\(u_i,x)\in E(G_1)\\(v_j,y)\notin E(G_2)}} d_G((u_i,v_j),(x,y)) + \sum_{\substack{(x,y)\in V(G)\\(u_i,x)\notin E(G_1)\\(v_j,y)\in E(G_2)}} d_G((u_i,v_j),(x,y))$$
$$+ \sum_{\substack{(x,y)\in V(G)\\(u_i,x)\in E(G_1)\\(v_j,y)\in E(G_2)}} d_G((u_i,v_j),(x,y)) + \sum_{\substack{(x,y)\in V(G)\\(u_i,x)\notin E(G_1)\\(v_j,y)\notin E(G_2)}} d_G((u_i,v_j),(x,y))]$$
$$-[\sum_{\substack{(a,s)\in V(G)\\(u_k,a)\in E(G_1)\\(v_l,s)\notin E(G_2)}} d_G((u_k,v_l),(a,s)) + \sum_{\substack{(a,s)\in V(G)\\(u_k,a)\notin E(G_1)\\(v_l,s)\in E(G_2)}} d_G((u_k,v_l),(a,s))$$

$$+ \sum_{\substack{(a,s)\in V(G)\\(u_k,a)\in E(G_1)\\(v_l,s)\in E(G_2)}} d_G((u_k,v_l),(a,s)) + \sum_{\substack{(a,s)\in V(G)\\(u_k,a)\notin E(G_1)\\(v_l,s)\notin E(G_2)}} d_G((u_k,v_l),(a,s))]$$

$$= |V_1|(d_{G_2}(v_l) - d_{G_2}(v_j)) + |V_2|(d_{G_1}(u_k) - d_{G_1}(u_i)) + 2(d_{G_1}(u_i)d_{G_2}(v_j) - d_{G_1}(u_k)d_{G_2}(u_l))$$

This implies that $|T((u_i,v_j)) - T((u_k,v_l))|$

$$= ||V_1|(d_{G_2}(v_l) - d_{G_2}(v_j)) + |V_2|(d_{G_1}(u_k) - d_{G_1}(u_i)) + 2(d_{G_1}(u_i)d_{G_2}(v_j) - d_{G_1}(u_k)d_{G_2}(u_l))|$$

$$\leq |V_1||d_{G_2}(v_l) - d_{G_2}(v_j)| + |V_2||d_{G_1}(u_k) - d_{G_1}(u_i)| + 2|d_{G_1}(u_i)d_{G_2}(v_j) - d_{G_1}(u_k)d_{G_2}(u_l)|$$

Therefore, the Co-PI of G is given by $Co\text{–}PI_v(G)$

$$= \sum_{((u_i,v_j),(u_k,v_l))\in E(G)} |T((u_i,v_j)) - T((u_k,v_l))|$$

$$\leq |V_1|^2 M_1(G_2) + |V_2|^2 M_1(G_1) + \sum_{i=1}^{|V_1|}\sum_{j=1}^{|V_2|} 2d_G(u_i,v_j)d_{G_1}(u_i)d_{G_2}(v_j)$$

$$= |V_1|^2 M_1(G_2) + |V_2|^2 M_1(G_1) + 4|V_2||E(G_2)|M_1(G_1) + 4|V_1||E(G_1)| M_1(G_2) - 2M_1(G_1)M_1(G_2).$$

Lemma 10 *Let G_1 and G_2 be connected graphs, $G = G_1 \circ G_2$, then*
i)$u \in G_1, T_G(u_i) = (|V_2|+1)T_{G_1}(u_i) + |V_1||V_2|$;
ii)$v \in G_2, T_G(v_j) = (|V_2|+1)T_{G_1}(u_i) - d_{G_2}(v_j) + 2|V_1||V_2| + |V_1|$.

Proof: In graph G, there are three kinds of edges: two end vertices in G_1, two end vertices in G_2, and one end vertex in G_1, another end vertex in G_2.

$$i) T_G(u_i) = \sum_{v_j\in G} d_G(u_i,v_j)$$

$$= \sum_{v_j\in G_1} d_G(u_i,v_j) + \sum_{\substack{v_i\in V(G_2)\\u_i\sim v_j}} d_G(u_i,v_j) + \sum_{\substack{v_i\in V(G_2)\\u_i\not\sim v_j}} d_G(u_i,v_j)$$

$$= (|V_2|+1)T_{G_1}(u_i) + |V_1||V_2|.$$

$$ii) T_G(v_j) = \sum_{u_i\in G} d_G(u_i,v_j)$$

$$= \sum_{\substack{v_i\in G_2\\v_i\sim v_j}} d_G(u_i,v_j) + \sum_{\substack{v_i\in G_2\\v_i\not\sim v_j}} d_G(u_i,v_j) + \sum_{u_i\in G_1} d_G(u_i,v_j)$$

$$= (|V_2|+1)T_{G_1}(u_i) - d_{G_2}(v_j) + 2|V_1||V_2| + |V_1|.$$

Theorem 11 *Let G_1 and G_2 be connected graphs, and $G = G_1 \circ G_2$, then $Co\text{-}PI_v(G) \leq (|V_2|+1)Co\text{-}PI(G_1) + 2|V_1||E(G_2)| + |V_1|^2|V_2|^2 + |V_1|^2|V_2| + M_1(G_2)$.*

Proof: By lemma 10, we have $Co\text{–}PI_v(G)$

$$= \sum_{(u_i,v_j)\in E(G)} |T(u_i) - T(v_j)| + \sum_{(v_i,v_j)\in E(G_2)} |d_{G_2}(v_i)| + |d_{G_2}(v_j)|$$

$$\leq (|V_2|+1)Co\text{–}PI_v(G_1) + 2|V_1||E(G_2)| + |V_1|^2|V_2|^2 + |V_1|^2|V_2| + 2M_1(G_2).$$

3 Concluding Remarks:

Let $G = G_1 \square G_2$ be the cartesian product of two graphs G_1 and G_2, many topological indices of G all have concise expression, such as: the CO-PI index, Wiener index, the PI index and the Szeged index. In this paper, we first attempt to obtain some explicit expressions for the CO-PI indices of the join, composition, disjunction, symmetric difference and corona graph. But it is regret that we only obtain some upper bounds for the CO-PI indices of the five graph operations with the order or the size of the original graphs. In fact, it is very difficult to give concise expressions for the CO-PI indices of the five graph operations, even if giving some tight upper bounds for them are still very difficult.

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