On two Curious Sums of Triple Multiplication of Binomial Coefficients*

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Abstract. By computer-assisted approaches and inductive arguments, two curious sums of triple multiplication of binomial coefficients are established in the present paper. The two curious sums arise in proving Melham's conjecture on odd power sums of Fibonacci numbers, which was confirmed by Xie, Yang and the present author. However, being different from their's technical way, the method used in the paper is more elementary.

Key words: Fibonacci numbers; Lucas numbers; Melham's conjecture; Binomial coefficients

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1 Introduction

We always denote respectively by $\mathbb{N}$ and $\mathbb{N}^*$ the sets of natural numbers $\{0,1,2,\cdots\}$ and positive integral numbers $\{1,2,3,\cdots\}$ throughout the paper.

The two curious sums in the paper are defined as follows,

$$F(s,m,n)=\sum_{j=s}^{m}(2j+1)^{2n+1}(-1)^j\binom{2m+1}{m-j}\binom{j+s}{2s}, \tag{1.1}$$

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and

$$G(s,m,n;x) = \sum_{j=s}^{m}(1+2j)^2(-1)^j\binom{2m+1}{m-j}\binom{j+s}{2s}\binom{n+x(2j+1)-1}{2n-1} \tag{1.2}$$

where $s, m, n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Before giving our main results, it is maybe interesting to recall some background concerning the above two curious sums (1.1) and (1.2). As a matter of fact, they came from the proof in our paper [8], in which gave a solution to a conjecture due to Melham [2, Conjecture 2.1]. Precisely, it is the following conjecture:

Conjecture 1.1 *Let $m \geq 1$ be a positive integer. Then the sum*

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{j=1}^{n} F_{2j}^{2m+1} \tag{1.3}$$

can be expressed as $(F_{2n+1}-1)^2 P_{2m-1}(F_{2n+1})$, where $P_{2m-1}(x)$ is a polynomial of degree $2m-1$ with integer coefficients and here F_n denote the nth Fibonacci number which satisfies the following recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2,$$

with the initial values $F_0 = 0$ and $F_1 = 1$, and L_n is the nth Lucas number satisfies the same recurrence relation as F_n but with the initial values $L_0 = 2$ and $L_1 = 1$.

Actually, Melham posed his conjectures in 1998, and afterwards many authors made inroads into some of the challenges it poses. The interested reader can see the following references [4, 5, 6, 7] for more details. Recently, Xie and Yang and the author of the present paper [8] completely solved the conjecture 1.1. During attacking the Conjecture 1.1, we encountered the key summations (1.1) and (1.2), which were proved technically by using the following well-known combinatorial identity

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n = 0, \quad \text{for } n < k.$$

However, in this paper we give an elementary proof of the two sums (1.1) and (1.2) by using computer algebra methods [3, 9] and inductive argument, which is distinguished from the technical way in [8].

The main results of the present paper is the following theorem and corollary.

Theorem 1.2 *Given nonnegative integers m and s with $m > s$, and for all $n \in [0, m-s-1]$, we have $F(s,m,n) = 0$.*

As an application of the theorem, we have the following corollary,

Corollary 1.3 *Given nonnegative integers m and s for which $m > s$, for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \cdots, m-s-1$, we have $G(s,m,n;x) = 0$, i.e., it is independent of real number x.*

Remark 1.4 *As we will see that Theorem 1.2 is the key point in the paper, and Corollary 1.3 can be derived from it.*

The rest of the paper is organized as follows. We first give some preliminaries in the second section, and then the corresponding proofs of Theorem 1.2 and Corollary 1.3 are put forward in the third section.

2 Preliminaries

For the sake of convenience, in the sequel $F(s,m,n)$ and $G(s,m,n;x)$ always represents the summation formulaes (1.1) and (1.2), respectively.

Note that it is hard to compute directly the summation formula (1.1) by mathematical softwares since the summand in the summation formula is not a hypergeometric term with respect to j in general. Therefore, it is naturally to think that we can get the value of $F(s,m,n)$ if we can find a proper recurrence of $F(s,m,n)$ and its basic values. Fortunately, we find them! So, in this section, we first compute its basic value $F(s,m,0)$ by using Zeilberger's creative telescoping method [9] and then give a recurrence of $F(s,m,n)$.

Lemma 2.1 *Given nonnegative integer m, and for all $s = 0, 1, \cdots, m-1$, we have $F(s,m,0) = 0$, i.e.,*

$$\sum_{j=s}^{m} (-1)^j (1+2j) \binom{2m+1}{m-j} \binom{j+s}{2s} = 0. \tag{2.1}$$

Proof. Let

$$\begin{aligned}
&\sum_{j=s}^{m}(-1)^j(1+2j)\binom{2m+1}{m-j}\binom{j+s}{2s}\\
&=(-1)^s\sum_{j=0}^{m-s}(-1)^j(1+2j+2s)\binom{2m+1}{m-s-j}\binom{j+2s}{2s}\\
&=(-1)^s\sum_{j=0}^{n}(-1)^j(1+2j+2s)\binom{2n+2s+1}{n-j}\binom{j+2s}{2s}.
\end{aligned}$$

Since the summand

$$(-1)^j(1+2j+2s)\binom{2n+2s+1}{n-j}\binom{j+2s}{2s}$$

is hypergeometric, we can compute the recurrence relation of the following form by creative telescoping method,

$$\Delta_j h(j)=(-1)^j(1+2j+2s)\binom{2n+2s+1}{n-j}\binom{j+2s}{2s},$$

where

$$h(j)=\frac{(n+2s+1+j)j(-1)^{j+1}}{n}\binom{2n+2s+1}{n-j}\binom{j+2s}{2s}.$$

and Δ_j is an operator for difference with respect to the indeterminant j of the summand.

Therefore,

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^j(1+2j+2s)\binom{2n+2s+1}{n-j}\binom{j+2s}{2s}\\
&=\sum_{j=0}^{n}[h(j+1)-h(j)]\\
&=h(n+1)-h(0)\\
&=0.
\end{aligned}$$

As desired. ∎

The following lemma is important for us to proceed the inductive argument. Additionally, it is worthy to mention that we first found, by using the method in [3], the recurrence (2.3) but not the following recursion (2.2). Actually, when replacing $(-1)^j$ for $(-x)^j$ in the summand of the summation formula $F(s,m,n;x)$, which is defined in Lemma 2.2, we obtained the following recursion (2.2) unexpectedly.

Lemma 2.2 *Let*

$$F(s,m,n;x) = \sum_{j=s}^{m}(2j+1)^{2n+1}(-x)^j\binom{2m+1}{m-j}\binom{j+s}{2s}.$$

Then for any given nonnegative integers m and s with $m > s$, and $n \in \mathbb{N}$, $F(s,m,n;x)$ satisfies the following recurrence:

$$F(s,m,n+1;x) = (8+24s+16s^2)F(s+1,m,n;x)+(1+2s)^2F(s,m,n;x). \tag{2.2}$$

Proof. Consider the coefficient of x^j and note that

$$[x^j]F(s,m,n+1;x) - [x^j](8+24s+16s^2)F(s+1,m,n;x) - [x^j]F(s,m,n;x)$$
$$= (-1)^j(2j+1)^{2n+1}\binom{2m+1}{m-j}\left[(2j+1)^2\binom{j+s}{2s} - (1+2s)^2\binom{j+s}{2s} - 8(1+s)(1+2s)\binom{j+s+1}{2s+2}\right]$$
$$= \frac{(-1)^j(2j+1)^{2n+1}(j+s)!}{(j-s-1)!(2s)!}\binom{2m+1}{m-j}\left[\frac{(2j+1)^2-(1+2s)^2}{j-s} - 4(j+s+1)\right]$$
$$= 0.$$

Hence, the recurrence (2.2) holds and thus we complete the proof. ∎

As a corollary of the Lemma 2.2, we have

Corollary 2.3 *For any given nonnegative integers m and s with $m > s$, and $n \in \mathbb{N}$, $F(s,m,n)$ satisfies the recurrence:*

$$F(s,m,n+1) = (8+24s+16s^2)F(s+1,m,n)+(1+2s)^2F(s,m,n). \tag{2.3}$$

With the above preliminaries in hand, we are now in a position to prove Theorem 1.2 and Corollary 1.3.

3 Proofs of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. We prove the theorem in two steps.
The first step is to prove $F(s,m,n) = 0$ for $n \le m-s-1$. By Lemma 2.1, we have $F(s,m,0) = 0$ for given $m > s$, as desired.

By Corollary 2.3 and Lemma 2.1, it follows that $F(s,m,1)=0$ if $m>s+1$. This is equivalent to $F(s,m,1)=0$ if $m-s-1=m-(s+1)\geq 1=n$. So it is valid for $n=1$.

Suppose that $F(s,m,k)=0$ for $k\leq m-s-1$. We next prove that $F(s,m,k+1)=0$ for $k+1\leq m-s-1$. By Corollary 2.3, $F(s,m,k+1)=(8+24s+16s^2)F(s+1,m,k)$. By inductive hypothesis, $F(s,m,k+1)=0$ since $F(s+1,m,k)=0$ if $m-(s+1)-1=m-s-2\geq k$. In other words, $F(s,m,k+1)=0$ if $k+1\leq m-s-1$. By inductive argument, we have done the first step.

The second step is to show that $F(s,m,n)\neq 0$ if $n=m-s$, for given m and s satisfying $m>s$. We confirm the fact by induction on $m-s$.

For $m-s=1$, first note that $F(s,s+1,1)=-8(-1)^s(1+s)(1+2s)(3+2s)\neq 0$ since $s\geq 0$. Suppose that $F(s,s+k,k)\neq 0$ for $m-s=k\geq 2$, we need to prove it holds for $m-s=k+1$.

Since we have known that $F(s,m,m-s-1)=0$ by the first step, and hence $F(s,s+k+1,k)=0$, thus we have

$$\begin{aligned} F(s,s+k+1,k+1) &= (8+24s+16s^2)F(s+1,s+k+1,k) \\ &\quad +(1+4s+4s^2)F(s,s+k+1,k) \\ &= (8+24s+16s^2)F(s+1,s+k+1,k). \end{aligned}$$

Since here $m-s=(s+k+1)-(s+1)=k$, by inductive hypothesis $F(s+1,s+k+1,k)\neq 0$. Hence $F(s,m,m-s)\neq 0$ for $k+1=m-s$. By inductive argument, we complete the second step.

Based on the above two steps, we are done. ∎

Proof of Corollary 1.3. It is trivial for $n=0$. Consider that the formula $G(s,m,n;x)$ can be expressed in the following way,

$$G(s,m,n;x)=\frac{xH(x^2)}{(2n-1)!},$$

where $H(x)=\sum_{j=s}^{m}(-1)^j(2j+1)^{2n+1}\binom{2m+1}{m-j}\binom{j+s}{2s}\prod_{k=1}^{n-1}\left(x-\frac{k^2}{(2j+1)^2}\right)$.

To show Corollary 1.3, it is equivalent to show $H(x)=0$ for $n=1,2,\cdots,m-s-1$ and $x\in\mathbb{R}$. Note that

$$H(x)=(-1)^{n-1}\sum_{l=0}^{n-1}(-x)^l\sum_{k_1,k_2,\cdots,k_{n-1-l}}(k_1k_2\cdots k_{n-1-l})^2\cdot F(s,m,l+1),$$

where $l=0,1,\cdots n-1$ and the second sum in above formulae ranges over all subsets $\{k_1,k_2,\cdots,k_{n-1-l}\}$ of $\{1,2,\cdots,n-1\}$ for all $0\leq l\leq n-1$.

By Theorem1.2, $F(s, m, l+1) = 0$ for $l = 0, 1, \cdots, m-s-2$. Therefore, $H(x) = 0$ if $0 \leq n-1 \leq m-s-2$, i.e., $1 \leq n \leq m-s-1$, as desired. ∎

Consider that the summation formulae (1.1) should have a combinatorial interpretation, but we are unable to find one. So we leave it as an open problem.

Problem 3.1 *Is there a combinatorial interpretation of the sum* (1.1)*?*

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