Upper bound on the diameter of a total domination vertex critical graph

Michitaka Furuya[1,*]
Naoya Kato[1]

[1]*Department of Mathematical Information Science, Tokyo University of Science*
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

Abstract

A vertex of a graph is said to be total domination critical if its deletion decreases the total domination number. A graph is said to be total domination vertex critical if all of its vertices except the supporting vertices are total domination vertex critical. We show that if G is a connected total domination vertex critical graph with total domination number $k \geq 4$, then the diameter of G is at most $\lfloor \frac{5k-7}{3} \rfloor$.

Key words and phrases. total domination vertex critical graph, total domination, diameter.
AMS 2010 Mathematics Subject Classification. 05C69.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

---

*e-mail:michitaka.furuya@gmail.com

Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $v \in V(G)$, we let $d_G(v)$, $N_G(v)$, and $N_G[v]$ denote the degree of v, the open neighborhood of v, and the closed neighborhood of v, respectively; thus $d_G(v) = |N_G(v)|$ and $N_G[v] = N_G(v) \cup \{v\}$. The minimum degree of G is denoted by $\delta(G)$. We let $\overline{G}$ denote the complement of G. For $v, u \in V(G)$, we let $d_G(v, u)$ denote the distance between v and u. For $v \in V(G)$ and a non-negative integer i, we let $N_G^{(i)}(v) = \{u \in V(G) | d_G(v, u) = i\}$. For $v \in V(G)$, we define the *eccentricity* $\mathrm{ecc}_G(v)$ of v in G by $\mathrm{ecc}_G(v) = \max\{d_G(v, u) | u \in V(G)\}$; thus $\mathrm{ecc}_G(v)$ is the maximum integer i for which $N_G^{(i)}(v) \neq \emptyset$. The *diameter* of G is defined to be the maximum of $\mathrm{ecc}_G(v)$ as v ranges over $V(G)$, and is denoted by $\mathrm{diam}(G)$. A vertex is called an *endvertex* if it has degree one, and a vertex is called a *supporting vertex* if it is adjacent to an endvertex. We let $S(G)$ denote the set of supporting vertices of G. For terms and symbols not defined here, we refer the reader to [1].

Let G be a graph with no isolated vertex. For two subsets X, Y of $V(G)$, we say that X totally dominates Y (or X γ_t-dominates Y for short) if $Y \subseteq \bigcup_{v \in X} N_G(v)$. A subset of $V(G)$ which totally dominates $V(G)$ is called a *total dominating set* of G. The minimum cardinality of a total dominating set of G is called the *total domination number* of G, and is denoted by $\gamma_t(G)$. We have $\gamma_t(G) \geq 2$ unless G is empty (note that we discuss the total domination number of a graph only when the graph under consideration has no isolated vertex). A total dominating set of G having cardinality $\gamma_t(G)$ is called a *γ_t-set* of G. A vertex $v \in V(G) - S(G)$ is said to be *total domination critical* (*γ_t-ciritical*) if $\gamma_t(G - v) < \gamma_t(G)$. We say that G is *total domination vertex critical* (*γ_t-critical*) if every vertex of $G - S(G)$ is γ_t-critical in G. If G is γ_t-critical and $\gamma_t(G) = k$, G is said to be k-γ_t-critical. Various properties of γ_t-critical graphs were explored in [3, 4, 5].

Goddard et al. [2] proved the following result.

Theorem A *Let $k \geq 3$ be an integer, and let G be a connected k-γ_t-critical graph.*

(i) *If $k = 3$, then the diameter of G is at most 3.*

(ii) *If $4 \le k \le 8$, then the diameter of G is at most $\lfloor \frac{5k-7}{3} \rfloor$.*

(iii) *If $k \ge 9$, then the diameter of G is at most $2k - 3$.*

They showed that the bound in Theorem A is best possible for each $3 \le k \le 8$ by constructing a k-γ_t-critical graph attaining the bound. They also constructed a k-γ_t-critical graph with diameter $\frac{5k-7}{3}$ for each $k \equiv 2 \pmod 3$, and conjectured that for each $k \ge 4$, every connected k-γ_t-critical graph has diameter at most $\lfloor \frac{5k-7}{3} \rfloor$. In this paper, we prove the following theorem, which shows that the conjecture of Goddard et al. is true.

Theorem 1.1 *Let $k \ge 3$ be an integer, and let G be a connected k-γ_t-critical graph.*

(i) *If $k = 3$, then the diameter of G is at most 3.*

(ii) *If $k \ge 4$, then the diameter of G is at most $\lfloor \frac{5k-7}{3} \rfloor$.*

In Section 3, we show that the bound is best possible for all $k \ge 4$ by constructing infinitely many k-γ_t-critical graphs with diameter $\lfloor \frac{5k-7}{3} \rfloor$ for each fixed k. In our proof of Theorem 1.1, we make use of the following lemma, which is proved in [2].

Lemma 1.2 *Let $k \ge 3$ be an integer, and let G be a connected k-γ_t-critical graph with $\delta(G) = 1$. Then the diameter of G is at most k.*

The following observation is useful for our arguments.

Observation 1.3 *Let G be a γ_t-critical graph with $\delta(G) \ge 2$, and let $v \in V(G)$. Then the following hold.*

(i) $\gamma_t(G - v) = \gamma_t(G) - 1$.

(ii) *There exists a γ_t-set which contains v.*

2 Proof of Theorem 1.1

Let k, G be as in Theorem 1.1, and let d denote the diameter of G. We may assume that $k \geq 4$. By way of contradiction, suppose that $d > \frac{5k-7}{3} (> 4)$. By Lemma 1.2, $\delta(G) \geq 2$. Let $x \in V(G)$ be a vertex with $\mathrm{ecc}_G(x) = d$. Let $X_i = N_G^{(i)}(x)$ for each $i \geq 0$, and let $U_j = X_0 \cup \cdots \cup X_j$ for each $j \geq 0$. For $j \geq 2$, a γ_t-set S of G is called j-sufficient if $|S \cap U_j| \geq (3j + 10)/5$.

Claim 2.1 *For some $j \geq 2$, there exists a γ_t-set which is j-sufficient.*

Proof. Suppose that for all $j \geq 2$, no γ_t-set is j-sufficient. Take $x_1 \in X_1$, and let S_{x_1} be a γ_t-set of $G - x_1$. Since S_{x_1} γ_t-dominates $U_1 - \{x_1\}$, $|S_{x_1} \cap U_2| \geq 2$. Let $S = S_{x_1} \cup \{x\}$. Note that S is a γ_t-set of G. Since S is not 3-sufficient, $|S \cap U_3| < (3 \cdot 3 + 10)/5 = 19/5 < 4$. Since $|S \cap U_3| \geq |S_{x_1} \cap U_2| + |\{x\}| \geq 3$, this forces $|S \cap U_3| = 3$ and $S_{x_1} \cap X_3 = \emptyset$.

Take $x_4 \in X_4$, and let S_{x_4} be a γ_t-set $G - x_4$. Since S_{x_4} γ_t-dominates U_1, $|S_{x_4} \cap U_2| \geq 2$. Take $x_3 \in X_3 \cap N_G(x_4)$, and let $S' = S_{x_4} \cup \{x_3\}$. Note that S' is a γ_t-set of G. Since S' is not 3-sufficient, $|S' \cap U_3| < 19/5$. Since $|S' \cap U_3| \geq |S_{x_4} \cap U_2| + |\{x_3\}| \geq 3$, this forces $|S' \cap U_3| = 3$, $|S_{x_4} \cap U_3| = 2$ and $S_{x_4} \cap X_3 = \emptyset$. Since S' is not 5-sufficient, $|S' \cap U_5| < (3 \cdot 5 + 10)/5 = 5$, and hence $|S_{x_4} \cap (X_4 \cup X_5)| = |S' \cap U_5| - |S' \cap U_3| \leq 4 - 3 = 1$. Now if $S_{x_4} \cap X_4 \neq \emptyset$, then since $S_{x_4} \cap X_3 = \emptyset$, the unique vertex in $S_{x_4} \cap X_4$ cannot be γ_t-dominated by S_{x_4}, a contradiction. Thus $S_{x_4} \cap X_4 = \emptyset$.

Subclaim 2.1.1 *The set $(S_{x_4} \cap U_3) \cup (S - U_3)$ is a total dominating set of G.*

Proof. Since S_{x_4} γ_t-dominates $V(G) - \{x_4\}$ and $S_{x_4} \cap X_4 = \emptyset$, each vertex in U_3 is adjacent to a vertex in $S_{x_4} \cap U_3$. Since S γ_t-dominates $V(G)$ and $S \cap X_3 = \emptyset$, each vertex in $V(G) - U_3$ is adjacent to a vertex in $S - U_3$. Hence the desired conclusion holds. □

We have $|(S_{x_4} \cap U_3) \cup (S - U_3)| = |S_{x_4} \cap U_3| + |S - U_3| = 2 + (k-3) = k - 1$. This together with Subclaim 2.1.1 contradicts the assumption that

$\gamma_t(G) = k$, which completes the proof of Claim 2.1. □

Having Claim 2.1 in mind, let $m \geq 2$ denote the maximum integer such that there exists an m-sufficient γ_t-set. Let S_1 be an m-sufficient γ_t-set. Then $|S_1 \cap U_m| \geq (3m + 10)/5$, and we also have $|S_1 \cap U_{m+1}| < (3(m + 1) + 10)/5$ by the maximality of m. Since $d > (5k - 7)/3$ and $k \geq |S_1 \cap U_m| \geq (3m + 10)/5$, it follows that $d \geq m + 2$.

Claim 2.2 $S_1 \cap X_{m+1} = \emptyset$.

Proof. Since $|S_1| \geq (3m+10)/5$ and $|S_1 \cap U_{m+1}| < (3m+13)/5$, it follows that $|S_1 \cap X_{m+1}| = |S_1 \cap U_{m+1}| - |S_1 \cap U_m| < (3m+13)/5 - (3m+10)/5 = 3/5$, which implies $S_1 \cap X_{m+1} = \emptyset$. □

Recall that $d \geq m+2$. If $|S_1 \cap (X_{m+2} \cup X_{m+3})| \geq 2$, then $|S_1 \cap U_{m+3}| \geq |S_1 \cap U_m| + 2 \geq (3m+10)/5 + 2 > (3(m+3)+10)/5$, which contradicts the maximality of m. Thus $|S_1 \cap (X_{m+2} \cup X_{m+3})| \leq 1$. Since S_1 γ_t-dominates X_{m+2} and $S_1 \cap X_{m+1} = \emptyset$, $S_1 \cap (X_{m+2} \cup X_{m+3}) \neq \emptyset$. If $S_1 \cap X_{m+2} \neq \emptyset$, then S_1 does not γ_t-dominate the vertex in $S_1 \cap X_{m+2}$, a contradiction. Thus $S_1 \cap X_{m+2} = \emptyset$, and hence $|S_1 \cap X_{m+3}| = 1$. Write $S_1 \cap X_{m+3} = \{w_{m+3}\}$. Since $S_1 \cap X_{m+1} = S_1 \cap X_{m+2} = \emptyset$, w_{m+3} is adjacent to every vertex in X_{m+2}. Since S_1 γ_t-dominates w_{m+3}, $|S_1 \cap X_{m+3}| = 1$ and $S_1 \cap X_{m+2} = \emptyset$, $S_1 \cap X_{m+4} \neq \emptyset$. In particular, $d \geq m + 4$.

Let S_2 be a γ_t-set of $G - w_{m+3}$. Note that S_2 is not a γ_t-set of G. Since w_{m+3} is adjacent to every vertex in X_{m+2}, this implies $S_2 \cap X_{m+2} = \emptyset$.

Claim 2.3 *The set $(S_2 \cap U_{m+2}) \cup (S_1 - U_{m+2})$ is a total dominating set of G.*

Proof. Since S_2 γ_t-dominates $V(G) - \{w_{m+3}\}$, each vertex in U_{m+1} is adjacent to a vertex in $S_2 \cap U_{m+2}$. Since S_1 γ_t-dominates $V(G)$ and $S_1 \cap X_{m+1} = S_1 \cap X_{m+2} = \emptyset$, each vertex in $V(G) - U_{m+1}$ is adjacent to a vertex in $S_1 - U_{m+2}$. Hence the desired conclusion holds. □

If $|S_2 \cap U_{m+2}| \leq |S_1 \cap U_{m+2}| - 1$, then $|(S_2 \cap U_{m+2}) \cup (S_1 - U_{m+2})| \leq |S_1 \cap U_{m+2}| + |S_1 - U_{m+2}| - 1 = k - 1$ which, in view of Claim 2.3, contradicts the assumption that $\gamma_t(G) = k$. Thus

$$|S_2 \cap U_{m+2}| \geq |S_1 \cap U_{m+2}|. \tag{2.1}$$

Suppose that $|X_{m+3}| \geq 2$. Since S_2 γ_t-dominates $(X_{m+3} - \{w_{m+3}\}) \cup X_{m+4}$ and $S_2 \cap X_{m+2} = \emptyset$, $|S_2 \cap (X_{m+3} \cup X_{m+4} \cup X_{m+5})| \geq 2$. Let $w_{m+2} \in X_{m+2}$. Then by (2.1), $|(S_2 \cup \{w_{m+2}\}) \cap U_{m+5}| \geq |S_1 \cap U_{m+2}| + 3 \geq (3m+10)/5 + 3 = (3(m+5)+10)/5$. Since $S_2 \cup \{w_{m+2}\}$ is a γ_t-set of G, $S_2 \cup \{w_{m+2}\}$ is $(m+5)$-sufficient, which contradicts the maximality of m. Thus $|X_{m+3}| = 1$. In particular, w_{m+3} is adjacent to every vertex in X_{m+4}.

Take $w_{m+4} \in X_{m+4}$, and let S_3 be a γ_t-set of $G - w_{m+4}$. Note that $S_3 \cap X_{m+3} = \emptyset$ and $S_3 \cup \{w_{m+3}\}$ is a γ_t-set of G. Suppose that $S_3 \cap X_{m+2} \neq \emptyset$.

Claim 2.4 *The set $(S_3 \cap U_{m+2}) \cup (S_2 - U_{m+2})$ is a total dominating set of G.*

Proof. Since S_3 γ_t-dominates $V(G) - \{w_{m+4}\}$ and $S_3 \cap X_{m+3} = \emptyset$, each vertex in U_{m+2} is adjacent to a vertex in $S_3 \cap U_{m+2}$. Since $S_3 \cap X_{m+2} \neq \emptyset$, $X_{m+3} = \{w_{m+3}\}$ and w_{m+3} is adjacent to every vertex in X_{m+2}, this implies that $S_3 \cap U_{m+2}$ γ_t-dominates U_{m+3}. Since S_2 γ_t-dominates $V(G) - \{w_{m+3}\}$, each vertex in $V(G) - U_{m+3}$ is adjacent to a vertex in $S_2 - U_{m+2}$. Hence the desired conclusion holds. □

If $|S_3 \cap U_{m+2}| \geq |S_1 \cap U_{m+2}| + 1$, then $|(S_3 \cup \{w_{m+3}\}) \cap U_{m+3}| \geq |S_1 \cap U_{m+2}| + 2 \geq (3m+10)/5 + 2 > (3(m+3)+10)/5$, which contradicts the maximality of m. Thus $|S_3 \cap U_{m+2}| \leq |S_1 \cap U_{m+2}|$. By (2.1), $|S_2 - U_{m+2}| = (k-1) - |S_2 \cap U_{m+2}| \leq (k-1) - |S_1 \cap U_{m+2}|$. Consequently $|(S_3 \cap U_{m+2}) \cup (S_2 - U_{m+2})| \leq |S_1 \cap U_{m+2}| + ((k-1) - |S_1 \cap U_{m+2}|) = k-1$ which, in view of Claim 2.4, contradicts the assumption that $\gamma_t(G) = k$. Thus $S_3 \cap X_{m+2} = \emptyset$.

Since S_3 γ_t-dominates X_{m+3} and $S_3 \cap X_{m+2} = S_3 \cap X_{m+3} = \emptyset$, $S_3 \cap X_{m+4} \neq \emptyset$ and $|S_3 \cap (X_{m+4} \cup X_{m+5})| \geq 2$.

Claim 2.5 *The set $(S_3 \cap U_{m+2}) \cup (S_1 - U_{m+2})$ is a total dominating set of G.*

Proof. Since S_3 γ_t-dominates $V(G) - \{w_{m+4}\}$, each vertex in U_{m+1} is adjacent to a vertex in $S_3 \cap U_{m+2}$. Since S_1 γ_t-dominates $V(G)$ and $S_1 \cap X_{m+1} = S_1 \cap X_{m+2} = \emptyset$, each vertex in $V(G) - U_{m+1}$ is adjacent to a vertex in $S_1 - U_{m+2}$. Hence the desired conclusion holds. □

If $|S_3 \cap U_{m+2}| \geq |S_1 \cap U_{m+2}|$, then $|(S_3 \cup \{w_{m+3}\}) \cap U_{m+5}| \geq |S_1 \cap U_{m+2}| + 3 \geq (3m + 10)/5 + 3 = (3(m+5) + 10)/5$, which contradicts the maximality of m. Thus $|S_3 \cap U_{m+2}| \leq |S_1 \cap U_{m+2}| - 1$. Therefore $|(S_3 \cap U_{m+2}) \cup (S_1 - U_{m+2})| \leq |S_1 \cap U_{m+2}| + |S_1 - U_{m+2}| - 1 = k - 1$ which, in view of Claim 2.5, contradicts the assumption that $\gamma_t(G) = k$.

This completes the proof of Theorem 1.1. □

3 Examples

In this section, we prove a theorem concerning the construction of γ_t-critical graphs (Theorem 3.8), and then use the theorem to construct examples which show the sharpness of the bound in Theorem 1.1.

In our construction, we make use of the coalescence of graphs. Let A_1 and A_2 be graphs. For $i = 1, 2$, let x_i be a vertex of A_i. Under this notation, we let $(A_1 \bullet A_2)(x_1, x_2 : x)$ denote the graph obtained from A_1 and A_2 by identifying x_1 and x_2 into a vertex labelled x. We call $(A_1 \bullet A_2)(x_1, x_2 : x)$ the *coalescence of* A_1 *and* A_2 *via* x_1 *and* x_2. We first give some properties of the total domination number of the coalescence of graphs.

Lemma 3.1 *For each $i = 1, 2$, let A_i be a graph with $\delta(A_i) \geq 2$, and x_i be a vertex of A_i. Let $G = (A_1 \bullet A_2)(x_1, x_2 : x)$.*

(i) *If x_i is γ_t-critical in A_i for some $i \in \{1, 2\}$, then $\gamma_t(G) \leq \gamma_t(A_1) + \gamma_t(A_2) - 1$.*

(ii) *We have $\gamma_t(G) \geq \gamma_t(A_1) + \gamma_t(A_2) - 2$. Further, if $\gamma_t(G) = \gamma_t(A_1) + \gamma_t(A_2) - 2$, then $\gamma_t(A_i - N_{A_i}[x_i]) = \gamma_t(A_i) - 2$ for some $i \in \{1, 2\}$.*

(iii) *If $\gamma_t(G) = \gamma_t(A_1) + \gamma_t(A_2) - 1$ and both A_1 and A_2 are γ_t-critical, then G is γ_t-critical.*

Proof.

(i) Let S_i be a γ_t-set of $A_i - x_i$, and let S_{3-i} be a γ_t-set of A_{3-i}. If $x_{3-i} \in S_{3-i}$, let $S = ((S_1 \cup S_2) - \{x_{3-i}\}) \cup \{x\}$; if $x_{3-i} \notin S_{3-i}$, let $S = S_1 \cup S_2$. Then S is a total dominating set of G. We also have $|S| = |S_i| + |S_{3-i}| = \gamma_t(A_i - x_i) + \gamma_t(A_{3-i}) \leq \gamma_t(A_1) + \gamma_t(A_2) - 1$. Hence $\gamma_t(G) \leq \gamma_t(A_1) + \gamma_t(A_2) - 1$.

(ii) It suffices to show that $\gamma_t(G) = \gamma_t(A_1) + \gamma_t(A_2) - 2$ and $\gamma_t(A_i - N_{A_i}[x_i]) = \gamma_t(A_i) - 2$ for some $i \in \{1, 2\}$ under the assumption that $\gamma_t(G) \leq \gamma_t(A_1) + \gamma_t(A_2) - 2$. Thus assume $\gamma_t(G) \leq \gamma_t(A_1) + \gamma_t(A_2) - 2$. Let S be a γ_t-set of G. Since S γ_t-dominates x, $S \cap N_G(x) \neq \emptyset$. We may assume that $S \cap N_{A_1}(x_1) \neq \emptyset$. Suppose that $x \notin S$. Then $S \cap V(A_1)$ is a total dominating set of A_1 and $S \cap V(A_2)$ is a total dominating set of $A_2 - x_2$. Hence $|S \cap V(A_1)| \geq \gamma_t(A_1)$ and $|S \cap V(A_2)| \geq \gamma_t(A_2 - x_2)$. Since removing a vertex can decrease the total domination number at most by one, we get $\gamma_t(G) = |S| = |S \cap V(A_1)| + |S \cap V(A_2)| \geq \gamma_t(A_1) + \gamma_t(A_2 - x_2) \geq \gamma_t(A_1) + \gamma_t(A_2) - 1$. This contradicts the assumption that $\gamma_t(G) \leq \gamma_t(A_1) + \gamma_t(A_2) - 2$. Thus $x \in S$. Note that $((S - \{x\}) \cap V(A_1)) \cup \{x_1\}$ is a total dominating set A_1, and hence $|((S - \{x\}) \cap V(A_1)) \cup \{x_1\}| \geq \gamma_t(A_1)$. If $S \cap N_{A_2}(x_2) \neq \emptyset$, then $((S - \{x\}) \cap V(A_2)) \cup \{x_2\}$ is a total dominating set A_2, and hence $|((S - \{x\}) \cap V(A_2)) \cup \{x_2\}| \geq \gamma_t(A_2)$, which implies that $\gamma_t(G) = |S| = |((S - \{x\}) \cap V(A_1)) \cup \{x_1\}| + |((S - \{x\}) \cap V(A_2)) \cup \{x_2\}| - 1 \geq \gamma_t(A_1) + \gamma_t(A_2) - 1$, a contradiction. Thus $S \cap N_{A_2}(x_2) = \emptyset$. Consequently $(S - \{x\}) \cap V(A_2)$ is a subset of $V(A_2) - N_{A_2}[x_2]$ and γ_t-dominates $V(A_2) - N_{A_2}[x_2]$. Since $\gamma_t(G) \leq \gamma_t(A_1) + \gamma_t(A_2) - 2$, we now obtain $\gamma_t(A_2 - N_{A_2}[x_2]) \leq |(S - \{x\}) \cap V(A_2)| = |S| - |((S - \{x\}) \cap V(A_1)) \cup \{x_1\}| \leq \gamma_t(G) - \gamma_t(A_1) \leq (\gamma_t(A_1) + \gamma_t(A_2) - 2) - \gamma_t(A_1) = \gamma_t(A_2) - 2$. Since we clearly have $\gamma_t(A_2) \leq \gamma_t(A_2 - N_{A_2}[x_2]) + 2$, this forces $\gamma_t(A_2 - N_{A_2}[x_2]) = \gamma_t(A_2) - 2$ and $\gamma_t(G) = \gamma_t(A_1) + \gamma_t(A_2) - 2$, as desired.

(iii) Let $v \in V(G)$. We prove that $\gamma_t(G - v) \leq \gamma_t(G) - 1$.

Case 1: $v = x$.

For each i, let S_i be a γ_t-set of $A_i - x_i$. Then $S_1 \cup S_2$ is a total dominating set of $G - x$. We also have $|S_1 \cup S_2| = |S_1| + |S_2| = \gamma_t(A_1 - x_1) + \gamma_t(A_2 - x_2) \le \gamma_t(A_1) + \gamma_t(A_2) - 2 = \gamma_t(G) - 1$. Hence $\gamma_t(G - v) \le \gamma_t(G) - 1$.

Case 2: $v \ne x$.

Without loss of generality, we may assume that $v \in V(A_1) - \{x_1\}$. Let S_1 be a γ_t-set of $A_1 - v$, and let S_2 be a γ_t-set of $A_2 - x_2$. If $x_1 \in S_1$, let $S = (S_1 - \{x_1\}) \cup S_2 \cup \{x\}$; if $x_1 \notin S_1$, let $S = S_1 \cup S_2$. Then S is a total dominating set of $G - v$. We also have $|S| \le \gamma_t(A_1) + \gamma_t(A_2) - 2 = \gamma(G) - 1$. Hence $\gamma_t(G - v) \le \gamma_t(G) - 1$, as desired. □

As a preparation for our construction, we describe the definition of the graph Q constructed by Goddard et al. in [2].

For each $i = 1, 2$ and $j = 1, 2$, let $P^{(i,j)} = x_1^{(i,j)} x_2^{(i,j)} x_3^{(i,j)} x_4^{(i,j)}$ be a path of order 4. Let $U = \{u_1, u_2\}$. Let $E_1 = \bigcup_{i=1,2} \{u_i x_l^{(i,1)} \mid 1 \le l \le 4\}$, $E_2 = \bigcup_{i=1,2} \{x_l^{(i,1)} x_{l'}^{(i,2)} \mid (l, l') \notin \{(1,2), (2,4), (3,1), (4,3)\}\}$ and $E_3 = \{xx' \mid x \in V(P^{(1,2)}), x' \in V(P^{(2,2)})\}$. Let Q be the graph defined by

$$V(Q) = \Big(\bigcup_{i=1,2} \Big(\bigcup_{j=1,2} V(P^{(i,j)}) \Big) \Big) \cup U$$

and

$$E(Q) = \Big(\bigcup_{i=1,2} \Big(\bigcup_{j=1,2} E(P^{(i,j)}) \Big) \Big) \cup \Big(\bigcup_{1 \le i \le 3} E_i \Big)$$

(see Figure 1). Note that $Q - u_1 \simeq Q - u_2$, $Q - N_Q[u_1] \simeq Q - N_Q[u_2]$ and $Q - (\{u_1\} \cup V(P_{1,1}) \cup V(P_{1,2})) \simeq Q - (\{u_2\} \cup V(P_{2,1}) \cup V(P_{2,2}))$. By inspection, we see that $\gamma_t(Q[V(P^{(2,1)}) \cup V(P^{(2,2)})]) = 2$, and $\{x_2^{(2,2)}, x_3^{(2,2)}\}$ and $\{x_2^{(2,1)}, x_3^{(2,1)}\}$ are the only γ_t-sets of $Q[V(P^{(2,1)}) \cup V(P^{(2,2)})]$.

Lemma 3.2 ([2]) *We have* $\gamma_t(Q) = 4$ *and* $diam(Q) = d_Q(u_1, u_2) = 5$.

Lemma 3.3 (i) $\gamma_t(Q - N_Q[u_1]) = 3$.

(ii) *There exists a* γ_t*-set of* $Q - N_Q[u_1]$ *which contains* u_2.

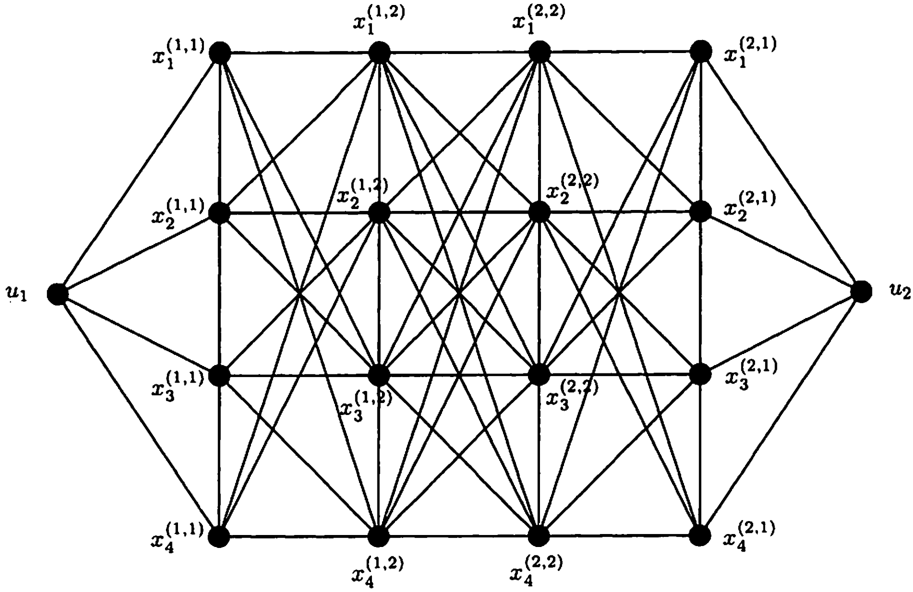

Figure 1: Graph Q

(iii) *Every vertex of* $V(Q)-(N_Q[u_1]\cup V(P^{(1,2)}))$ *is* γ_t*-critical in* $Q-N_Q[u_1]$.

Proof. We first prove (i) and (ii). Suppose that $\gamma_t(Q-N_Q[u_1])=2$. Then there exists a total dominating set S of $Q-N_Q[u_1]$ with $|S|=2$. Since $\mathrm{ecc}_{Q-N_Q[u_1]}(u_2)=3$, we see that $|S\cap V(P^{(2,1)})|=|S\cap V(P^{(2,2)})|=1$. This implies that S is a total dominating set of $Q[V(P^{(2,1)})\cup V(P^{(2,2)})]$, which contradicts the fact that $\{x_2^{(2,2)},x_3^{(2,2)}\}$ and $\{x_2^{(2,1)},x_3^{(2,1)}\}$ are the only γ_t-sets of $Q[V(P^{(2,1)})\cup V(P^{(2,2)})]$. Thus $\gamma_t(Q-N_Q[u_1])\geq 3$. This together with the fact that $\{u_2,x_1^{(2,1)},x_1^{(2,2)}\}$ is a total dominating set of $Q-N_Q[u_1]$ yields (i) and (ii).

We next prove (iii). Let $x\in V(Q)-(N_Q[u_1]\cup V(P^{(1,2)}))$. We show that x is γ_t-critical in $Q-N_Q[u_1]$. Without loss of generality, we may assume that $x\in\{u_2\}\cup\{x_j^{(2,i)}\mid 1\leq i\leq 2, 1\leq j\leq 2\}$. If $x=u_2$, then $\{x_2^{(2,2)},x_3^{(2,2)}\}$ is a total dominating set of $(Q-N_Q[u_1])-x$. Similarly $\{x_3^{(2,1)},x_2^{(2,2)}\}$, $\{x_4^{(2,1)},x_4^{(2,2)}\}$, $\{x_3^{(2,1)},x_3^{(2,2)}\}$ or $\{x_1^{(2,1)},x_4^{(2,2)}\}$ is a total dominating set of $(Q-N_Q[u_1])-x$ according as $x=x_1^{(2,1)}$, $x_2^{(2,1)}$, $x_1^{(2,2)}$ or $x_2^{(2,2)}$. Thus (iii) is proved. $\square$

For each $i=1,2$, let B_i be a γ_t-critical graph with $\delta(B_i)\geq 2$, and let $b_i\in V(B_i)$ be a vertex with $\gamma_t(B_i-N_{B_i}[b_i])\geq\gamma_t(B_i)-1$ (the construction of such graphs will be given later; see Propositions 3.9 through 3.16). Let $a\geq 0$ be an integer. Let $G^{(0)}$ be a graph isomorphic to B_1, and let $z_2^{(0)}$ be the vertex of $G^{(0)}$ corresponding to b_1 in B_1. Let $G^{(a+1)}$ be a graph isomorphic to B_2, and let $z_1^{(a+1)}$ be the vertex of $G^{(a+1)}$ corresponding to b_2 in B_2. For each $1\leq i\leq a$, let $G^{(i)}$ be a graph isomorphic to Q, and let $z_1^{(i)}$ and $z_2^{(i)}$ be the vertices of $G^{(i)}$ corresponding to u_1 and u_2, respectively. Let $Z_1^{(a)}(B_1;b_1)$ be the graph obtained by concatenating $G^{(0)},G^{(1)},\cdots,G^{(a)}$ by letting $G^{(i)}$ and $G^{(i+1)}$ coalesce via $z_2^{(i)}$ and $z_1^{(i+1)}$ (we let $z^{(i)}$ denote the vertex of $Z_1^{(a)}(B_1;b_1)$ arising from $z_2^{(i)}$ and $z_1^{(i+1)}$ through their identification) for each $0\leq i\leq a-1$. Let $Z_2^{(a)}(B_1,B_2;b_1,b_2)=(Z_1^{(a)}(B_1;b_1)\bullet G^{(a+1)})(z_2^{(a)},z_1^{(a+1)}:z^{(a)})$. Note that $Z_1^{(0)}(B_1;b_1)=G^{(0)}$, $Z_2^{(0)}(B_1,B_2;b_1,b_2)\simeq(B_1\bullet B_2)(b_1,b_2:b)$ and $Z^{(a)}(B_1,B_2;b_1,b_2)\simeq Z^{(a)}(B_2,B_1;b_2,b_1)$.

Lemma 3.4 *There exists a total dominating set of $Z_1^{(a)}(B_1; b_1)$ of cardinality $\gamma_t(B_1) + 3a$ which contains $z_2^{(a)}$.*

Proof. By Observation 1.3(ii), there exists a γ_t-set $S^{(0)}$ of $G^{(0)}$ with $z_2^{(0)} \in S^{(0)}$. For each $1 \le i \le a$, there exists a γ_t-set $S^{(i)}$ of $G^{(i)} - N_{G^{(i)}}[z_1^{(i)}]$ with $z_2^{(i)} \in S^{(i)}$ by Lemma 3.3(ii). Then $S = ((\bigcup_{0 \le i \le a} S^{(i)}) - \{z_2^{(i)} \mid 0 \le i \le a-1\}) \cup \{z^{(i)} \mid 0 \le i \le a-1\}$ is a total dominating set of $Z_1^{(a)}(B_1; b_1)$. We also have $|S| = \gamma_t(G_0) + \sum_{1 \le i \le a} \gamma_t(G^{(i)} - N_{G^{(i)}}[z_1^{(i)}]) = \gamma_t(B_1) + 3a$ by Lemma 3.3(i). Hence S is a desired set. $\square$

Lemma 3.5 (i) *We have $\gamma_t(Z_1^{(a)}(B_1; b_1)) = \gamma_t(B_1) + 3a$.*

(ii) *There exists a γ_t-set of $Z_1^{(a)}(B_1; b_1)$ which contains $z_2^{(a)}$.*

(iii) *We have $\gamma_t(Z_1^{(a)}(B_1; b_1) - N_{Z_1^{(a)}(B_1;b_1)}[z_2^{(a)}]) \ge \gamma_t(B_1) + 3a - 1$.*

Proof. For each i, let $Z^{(i)} = Z_1^{(i)}(B_1; b_1)$. We proceed by induction on a. If $a = 0$, then we get the desired results by the choice of B_1 and Observation 1.3. Thus we may assume that $a \ge 1$. Note that $Z^{(a)} \simeq (Z^{(a-1)} \bullet Q)(z_2^{(a-1)}, u_1 : x)$. By the induction assumption,

$$\gamma_t(Z^{(a-1)} - N_{Z^{(a-1)}}[z_2^{(a-1)}]) \ge \gamma_t(Z^{(a-1)}) - 1. \tag{3.1}$$

By Lemma 3.3, $\gamma_t(Q - N_Q[u_1]) \ge 3 = \gamma_t(Q) - 1$. Hence by (3.1) and Lemma 3.1(ii), $\gamma_t(Z^{(a)}) \ge \gamma_t(Z^{(a-1)}) + \gamma_t(Q) - 1$. Consequently $\gamma_t(Z^{(a)}) \ge (\gamma_t(B_1) + 3(a-1)) + 4 - 1 = \gamma_t(B_1) + 3a$ by the induction assumption. This together with Lemma 3.4 implies that (i) and (ii) hold. Note that $Z^{(a)} - N_{Z^{(a)}}[z_2^{(a)}] \simeq (Z^{(a-1)} \bullet (Q - N_Q[u_2]))(z_2^{(a-1)}, u_1 : x)$. We clearly have $\gamma_t((Q - N_Q[u_2]) - N_Q[u_1]) \ge 2 = \gamma_t(Q - N_Q[u_2]) - 1$ by Lemma 3.3(i). Hence by (3.1), Lemma 3.1(ii) and the induction assumption, $\gamma_t(Z^{(a)} - N_{Z^{(a)}}[z_2^{(a)}]) \ge \gamma_t(Z^{(a-1)}) + \gamma_t(Q - N_Q[u_2]) - 1 = \gamma_t(B_1) + 3(a-1) + 3 - 1 = \gamma_t(B_1) + 3a - 1$. This proves (iii). $\square$

Lemma 3.6 *We have $\gamma_t(Z_2^{(a)}(B_1, B_2; b_1, b_2)) \ge \gamma_t(B_1) + \gamma_t(B_2) + 3a - 1$.*

Proof. Let $Z^{(a)} = Z_1^{(a)}(B_1; b_1)$. Recall that $Z_2^{(a)}(B_1, B_2; b_1, b_2) = (Z^{(a)} \bullet G^{(a+1)})(z_2^{(a)}, z_1^{(a+1)} : z^{(a)}) \simeq (Z^{(a)} \bullet B_2)(z_2^{(a)}, b_2 : x)$. By Lemma 3.5, $\gamma_t(Z^{(a)} - N_{Z^{(a)}}[z_2^{(a)}]) \geq \gamma_t(Z^{(a)}) - 1$. By the choice of B_2, $\gamma(B_2 - N_{B_2}[b_2]) \geq \gamma_t(B_2) - 1$. Hence by Lemmas 3.1(ii) and 3.5, $\gamma_t(Z_2^{(a)}(B_1, B_2; b_1, b_2)) \geq \gamma_t(Z^{(a)}) + \gamma_t(B_2) - 1 = \gamma_t(B_1) + 3a + \gamma_t(B_2) - 1$, as desired. □

Lemma 3.7 *Let $Z^{(a)} = Z_2^{(a)}(B_1, B_2; b_1, b_2)$. Then $\gamma_t(Z^{(a)} - v) \leq \gamma_t(B_1) + \gamma_t(B_2) + 3a - 2$ for every $v \in V(Z^{(a)})$.*

Proof. Recall that $Z_2^{(0)}(B_1, B_2; b_1, b_2) \simeq (B_1 \bullet B_2)(b_1, b_2 : b)$. By Lemmas 3.1(i) and 3.6, $\gamma_t(Z_2^{(0)}(B_1, B_2; b_1, b_2)) = \gamma_t(B_1) + \gamma_t(B_2) - 1$. Hence $Z_2^{(0)}(B_1, B_2; b_1, b_2)$ is γ_t-critical by Lemma 3.1(iii). Consequently the lemma holds for $a = 0$. Thus we may assume that $a \geq 1$.

Case 1: $v \in (V(G^{(0)}) - \{z_2^{(0)}\}) \cup \{z^{(0)}\}$ or $v \in (V(G^{(a+1)}) - \{z_1^{(a+1)}\}) \cup \{z^{(a)}\}$.

Without loss of generality, we may assume that $v \in (V(G^{(0)}) - \{z_2^{(0)}\}) \cup \{z^{(0)}\}$. If $v \neq z^{(0)}$, let S_0' be a γ_t-set of $G^{(0)} - v$; if $v = z^{(0)}$, let S_0' be a γ_t-set of $G^{(0)} - z_2^{(0)}$. If $z_2^{(0)} \in S_0'$, let $S_0 = (S_0' - \{z_2^{(0)}\}) \cup \{z^{(0)}\}$; if $z_2^{(0)} \notin S_0'$, let $S_0 = S_0'$. Let S_1 be a γ_t-set of $G^{(1)} - (N_{G^{(1)}}[z_2^{(1)}] \cup \{z_1^{(1)}\})$. By Lemma 3.3(i),(iii), $|S_1| = 2$. For each $2 \leq i \leq a$, there exists a γ_t-set S_i of $G^{(i)} - N_{G^{(i)}}[z_2^{(i)}]$ which contains $z_1^{(i)}$ by Lemma 3.3(ii). By Observation 1.3(ii), there exists a γ_t-set S_{a+1} of $G^{(a+1)}$ which contains $z_1^{(a+1)}$. Then $S = ((\bigcup_{0 \leq i \leq a+1} S_i) - \{z_1^{(i)} \mid 2 \leq i \leq a+1\}) \cup \{z^{(i)} \mid 1 \leq i \leq a\}$ is a total dominating set of $Z^{(a)} - v$. We also have $|S| = (\gamma_t(B_1) - 1) + 2 + \sum_{2 \leq i \leq a} \gamma_t(G^{(i)} - N_{G^{(i)}}[z_2^{(i)}]) + \gamma_t(B_2) = \gamma_t(B_1) + \gamma_t(B_2) + 3a - 2$. Hence $\gamma_t(Z^{(a)} - v) \leq \gamma_t(B_1) + \gamma_t(B_2) + 3a - 2$.

Case 2: $v \notin (V(G^{(0)}) - \{z_2^{(0)}\}) \cup (V(G^{(a+1)}) - \{z_1^{(a+1)}\}) \cup \{z^{(0)}, z^{(a)}\}$.

Let $0 \leq i_0 \leq a-1$ be the integer such that $d_{Z^{(a)}}(z^{(0)}, z^{(i_0)}) \leq d_{Z^{(a)}}(z^{(0)}, v) < d_{Z^{(a)}}(z^{(0)}, z^{(i_0+1)})$. Replacing the roles of $G^{(0)}$ and $G^{(a+1)}$ by each other if necessary, we may assume $d_{Z^{(a)}}(v, z^{(i_0)}) < d_{Z^{(a)}}(v, z^{(i_0+1)})$. If $v \neq z^{(i_0)}$, let S_{i_0+1} be a γ_t-set of $(G^{(i_0+1)} - N_{G^{(i_0+1)}}[z_2^{(i_0+1)}]) - v$; if $v = z^{(i_0)}$, let S_{i_0+1} be a γ_t-set of $(G^{(i_0+1)} - N_{G^{(i_0+1)}}[z_2^{(i_0+1)}]) - z_1^{(i_0+1)}$. By Lemma 3.3(i),(iii),

$|S_{i_0+1}| = 2$. If $v \neq z^{(i_0)}$, then the eccentricity of $z_1^{(i_0+1)}$ in $(G^{(i_0+1)} - N_{G^{(i_0+1)}}[z_2^{(i_0+1)}]) - v$ is 3, and hence $z_1^{(i_0+1)} \notin S_{i_0+1}$; if $v = z^{(i_0)}$, we clearly have $z_1^{(i_0+1)} \notin S_{i_0+1}$. Thus $z^{(i_0+1)} \notin S_{i_0+1}$ in either case. For each $i_0 + 2 \leq i \leq a$, there exists a γ_t-set S_i of $G^{(i)} - N_{G^{(i)}}[z_2^{(i)}]$ which contains $z_1^{(i)}$ by Lemma 3.3(ii). By Observation 1.3(ii), there exists a γ_t-set S_{a+1} of $G^{(a+1)}$ which contains $z_1^{(a+1)}$.

Subcase 2.1: $i_0 = 0$.

Let S_0 be a γ_t-set of $G^{(0)} - z_2^{(0)}$. Then $S = ((\bigcup_{0\leq i\leq a+1} S_i) - \{z_1^{(i)} \mid 2 \leq i \leq a+1\}) \cup \{z^{(i)} \mid 1 \leq i \leq a\}$ is a total dominating set of $Z^{(a)} - v$. We also have $|S| = (\gamma_t(B_1) - 1) + 2 + \sum_{2\leq i\leq a} \gamma_t(G^{(i)} - N_{G^{(i)}}[z_2^{(i)}]) + \gamma_t(B_2) = \gamma_t(B_1) + \gamma_t(B_2) + 3a - 2$. Hence $\gamma_t(Z^{(a)} - v) \leq \gamma_t(B_1) + \gamma_t(B_2) + 3a - 2$.

Subcase 2.2: $1 \leq i_0 \leq a - 1$.

By Observation 1.3(ii), there exists a γ_t-set S_0 of $G^{(0)}$ which contains $z_2^{(0)}$. For each $1 \leq i \leq i_0 - 1$, there exists a γ_t-set S_i of $G^{(i)} - N_{G^{(i)}}[z_1^{(i)}]$ which contains $z_2^{(i)}$ by Lemma 3.3(ii). Let S_{i_0} be a γ_t-set of $(G^{(i_0)} - N_{G^{(i_0)}}[z_1^{(i_0)}]) - z_2^{(i_0)}$. By Lemma 3.3(iii), $|S_{i_0}| = 2$. Then $S = (\bigcup_{0\leq i\leq a+1} S_i - (\{z_2^{(i)} \mid 0 \leq i \leq i_0 - 1\} \cup \{z_1^{(i)} \mid i_0 + 2 \leq i \leq a+1\})) \cup \{z^{(i)} \mid 0 \leq i \leq i_0 - 1 \text{ or } i_0 + 1 \leq i \leq a\}$ is a total dominating set of $Z^{(a)} - v$. We also have $|S| = \gamma_t(B_1) + \sum_{1\leq i\leq i_0-1} \gamma_t(G^{(i)} - N_{G^{(i)}}[z_1^{(i)}]) + 2 + 2 + \sum_{i_0+2\leq i\leq a} \gamma_t((G^{(i)} - N_{G^{(i)}}[z_2^{(i)}]) + \gamma_t(B_2) = \gamma_t(B_1) + \gamma_t(B_2) + 3a - 2$. Hence $\gamma_t(Z^{(a)} - v) \leq \gamma_t(B_1) + \gamma_t(B_2) + 3a - 2$, as desired. □

By Observation 1.3(i) and Lemmas 3.6 and 3.7, we get the following theorem.

Theorem 3.8 *Let $a \geq 0$ be an integer. For each $i = 1, 2$, let B_i be a connected γ_t-critical graph with $\delta(B_i) \geq 2$, and let $b_i \in V(B_i)$ be a vertex with $\gamma_t(B_i - N_{B_i}[b_i]) \geq \gamma_t(B_i) - 1$. Then $Z_2^{(a)}(B_1, B_2; b_1, b_2)$ is a $(\gamma_t(B_1) + \gamma_t(B_2) + 3a - 1)$-$\gamma_t$-critical graph with diameter* $\max\{ecc_{B_1}(b_1) + ecc_{B_2}(b_2) + 5a, diam(B_1), diam(B_2)\}$.

In the remainder of this section, we construct candidates for B_1 and B_2, and apply Theorem 3.8 to them. We first construct 3-γ_t-critical graphs with diameter 3. Let $m \geq 2$ be an integer and, for $i = 1, 2$, let $C^{(i)} = x_1^{(i)} x_2^{(i)} \cdots x_{5m}^{(i)} x_1^{(i)}$ be a cycle of order $5m$. For each $1 \leq j \leq 5$, let $X_j^{(1)} = \{x_l^{(1)} \mid l \equiv j (\text{mod } 5)\}$ and $X_j^{(2)} = \{x_l^{(2)} \mid l \equiv 2j - 1 (\text{mod } 5)\}$. Note that $X_1^{(2)} = \{x_l^{(2)} \mid l \equiv 1 (\text{mod } 5)\}$, $X_2^{(2)} = \{x_l^{(2)} \mid l \equiv 3 (\text{mod } 5)\}$, $X_3^{(2)} = \{x_l^{(2)} \mid l \equiv 5 (\text{mod } 5)\}$, $X_4^{(2)} = \{x_l^{(2)} \mid l \equiv 2 (\text{mod } 5)\}$ and $X_5^{(2)} = \{x_l^{(2)} \mid l \equiv 4 (\text{mod } 5)\}$. Let $Y = \{y_1, y_2\}$. Let $E_1 = \bigcup_{1 \leq i \leq 5} (\bigcup_{j \neq i} \{xx' \mid x \in X_i^{(1)}, x' \in X_j^{(2)}\})$ and $E_2 = \bigcup_{i=1,2} \{y_i x \mid x \in V(C^{(i)})\}$. Let G_m be the graph defined by

$$V(G_m) = V(C^{(1)}) \cup V(C^{(2)}) \cup Y$$

and

$$E(G_m) = E(\overline{C^{(1)}}) \cup E(\overline{C^{(2)}}) \cup E_1 \cup E_2.$$

The graph G_m is depicted in Figure 2. In the figure, two solid lines indicate that for each $i = 1, 2$, all edges between y_i and $\overline{C^{(i)}}$ are present; dotted lines indicate that no edge between the two sets joined by a dotted line is present, and all other edges between $\overline{C^{(1)}}$ and $\overline{C^{(2)}}$ are present; dashed lines indicate that for each $i = 1, 2$, all edges inside $\overline{C^{(i)}}$ are present except for a perfect matching between the two sets joined by a dashed line.

Proposition 3.9 *Let $m \geq 2$ be an integer. Then G_m is 3-γ_t-critical, and $ecc_{G_m}(y_1) = diam(G_m) = 3$.*

Proof. By the construction of G_m, $\text{ecc}_{G_m}(y_1) = \text{diam}(G_m) = d_{G_m}(y_1, y_2) = 3$.

First we prove that $\gamma_t(G_m) = 3$. Since $\{x_1^{(1)}, x_4^{(1)}, x_1^{(2)}\}$ is a total dominating set of G_m, $\gamma_t(G_m) \leq 3$. Suppose that $\gamma_t(G_m) = 2$, and let S be a γ_t-set. Since $d_{G_m}(y_1, y_2) = 3$ and S γ_t-dominates Y, $|S \cap V(C^{(1)})| = |S \cap V(C^{(2)})| = 1$. Write $S \cap V(C^{(1)}) = \{x_{j_1}^{(1)}\}$ and $S \cap V(C^{(2)}) = \{x_{j_2}^{(2)}\}$. Let j_1' and j_2' be integers with $x_{j_1}^{(1)} \in X_{j_1'}^{(1)}$ and $x_{j_2}^{(2)} \in X_{j_2'}^{(2)}$. By the definition of S, $x_{j_1}^{(1)} x_{j_2}^{(2)} \in E(G_m)$, and hence $j_1' \neq j_2'$. If $j_2' \equiv j_1' - 1 (\text{mod } 5)$ or $j_2' \equiv j_1' + 1 (\text{mod } 5)$, then one of the vertices in $N_{C^{(1)}}(x_{j_1}^{(1)})$ is not γ_t-dominated by S; if $j_2' \equiv j_1' - 2 (\text{mod } 5)$ or $j_2' \equiv j_1' + 2 (\text{mod } 5)$, then one of the vertices in $N_{C^{(2)}}(x_{j_2}^{(2)})$ is not γ_t-dominated by S. Consequently S is not a total dominating set of G_m, which is a contradiction. Thus $\gamma_t(G_m) = 3$.

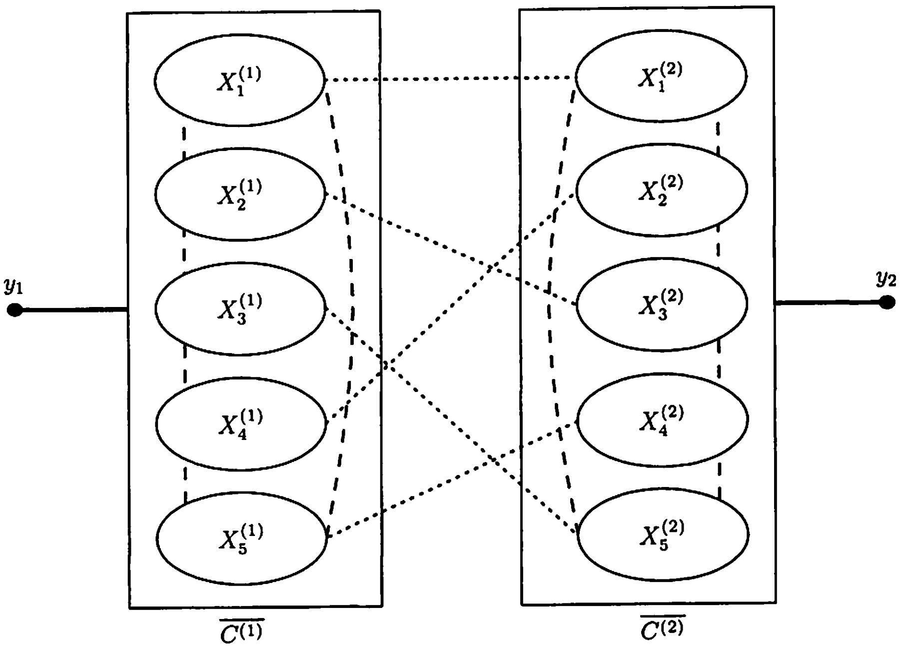

Figure 2: Graph G_m

Next we prove that $\gamma_t(G_m - v) = 2$ for any $v \in V(G_m)$. Let $v \in V(G_m)$.

Case 1: $v \in Y$.

Write $v = y_i$. Then $\{x_1^{(3-i)}, x_4^{(3-i)}\}$ is a total dominating set of $G_m - y_i$. Hence $\gamma_t(G_m - v) = 2$.

Case 2: $v \in (V(C^{(1)}) \cup V(C^{(2)}))$.

Let $i \in \{1, 2\}$ and $j \in \{1, \cdots, 5\}$ be integers with $v \in X_j^{(i)}$. Let $v' \in N_{C^{(i)}}(v)$ and $v'' \in X_j^{(3-i)}$. Then $\{v', v''\}$ is a total dominating set of $G_m - v$. Hence $\gamma_t(G_m - v) = 2$.

Therefore G_m is a 3-γ_t-critical graph. □

Proposition 3.10 *Let $m \geq 2$ be an integer. Then $\gamma_t(G_m - N_{G_m}[y_1]) \geq \gamma_t(G_m) - 1$.*

Proof. We have $\gamma_t(G_m - N_{G_m}[y_1]) \geq 2 = \gamma_t(G_m) - 1$. □

Next we construct 4-γ_t-critical graphs with diameter 4. Let $m \geq 2$ be an integer. For each $1 \leq i \leq 3$, let $X_i = \{x_{j,l}^{(i)} \mid 1 \leq j \leq 2, 1 \leq l \leq m\}$. Let $Y = \{y_1, y_2\}$. Let $E_1 = \bigcup_{i=1,2}\{xy_i \mid x \in X_i\}$, $E_2 = \bigcup_{i=1,2}\{x_{j,l}^{(i)}x_{j',l'}^{(3)} \mid (j, l) \neq (j', l')\}$, $E_3 = \bigcup_{i=1,2}\{x_{1,l}^{(i)}x_{2,l}^{(i)} \mid 1 \leq l \leq m\}$ and $E_4 = \{x_{j,l}^{(3)}x_{j',l'}^{(3)} \mid l \neq l'\}$. Let H_m be the graph defined by

$$V(H_m) = (\bigcup_{1 \leq i \leq 3} X_i) \cup Y$$

and

$$E(H_m) = \bigcup_{1 \leq i \leq 4} E_i$$

(see Figure 3).

Proposition 3.11 *Let $m \geq 2$ be an integer. Then H_m is 4-γ_t-critical, and $ecc_{H_m}(y_1) = diam(H_m) = 4$.*

Proof. By the construction of H_m, $\mathrm{ecc}_{H_m}(y_1) = \mathrm{diam}(H_m) = d_{H_m}(y_1, y_2) = 4$.

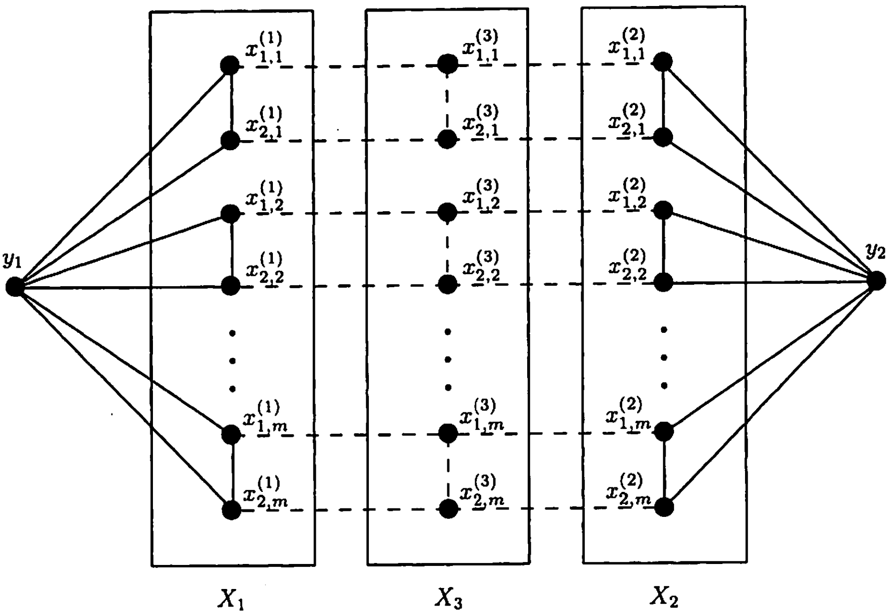

Figure 3: Graph H_m

First we prove that $\gamma_t(H_m) = 4$. Since $\{x_{1,1}^{(1)}, x_{1,1}^{(2)}, x_{1,2}^{(3)}, x_{2,2}^{(3)}\}$ is a total dominating set of H_m, $\gamma_t(H_m) \le 4$. Suppose that $\gamma_t(H_m) \le 3$, and let S be a γ_t-set. Recall that $d_{H_m}(y_1, y_2) = 4$. Since S γ_t-dominates Y, we have $S \cap X_1 \ne \emptyset$ and $S \cap X_2 \ne \emptyset$. Since $|S| \le 3$, there exists a vertex in S which γ_t-dominates both a vertex in $S \cap X_1$ and a vertex in $S \cap X_2$, and hence $|S \cap X_1| = |S \cap X_2| = |S \cap X_3| = 1$. Write $S \cap X_1 = \{x_{j,l}^{(1)}\}$ and $S \cap X_3 = \{x_{j',l'}^{(3)}\}$. Since $x_{j',l'}^{(3)}$ γ_t-dominates $x_{j,l}^{(1)}$, $x_{j,l}^{(1)} x_{j',l'}^{(3)} \in E(H_m)$, and hence $(j, l) \ne (j', l')$. If $l \ne l'$, then $x_{j',l'}^{(1)}$ is not γ_t-dominated by S, a contradiction. Thus $l = l'$, and hence $j' = 3 - j$. By the same argument, we get $S \cap X_2 = \{x_{j,l}^{(2)}\}$. This implies that $x_{j,l}^{(3)}$ is not γ_t-dominated by S, a contradiction. Thus $\gamma_t(H_m) = 4$.

Next we prove that $\gamma_t(H_m - v) \le 3$ for any $v \in V(H_m)$. Let $v \in V(H_m)$.

Case 1: $v \in Y$.

Write $v = y_i$. Then $\{x_{1,1}^{(3-i)}, x_{1,2}^{(3)}, x_{2,2}^{(3)}\}$ is a total dominating set of $H_m - y_i$. Hence $\gamma_t(H_m - v) \le 3$.

Case 2: $v \in X_1 \cup X_2$.

Write $v = x_{j,l}^{(i)}$. Let $l' \in \{1, \cdots, m\} - \{l\}$. Then $\{x_{1,l'}^{(i)}, x_{3-j,l}^{(3-i)}, x_{j,l}^{(3)}\}$ is a total dominating set of $H_m - x_{j,l}^{(i)}$. Hence $\gamma_t(H_m - v) \le 3$.

Case 3: $v \in X_3$.

Write $v = x_{j,l}^{(3)}$. Then $\{x_{j,l}^{(1)}, x_{j,l}^{(2)}, x_{3-j,l}^{(3)}\}$ is a total dominating set of $H_m - x_{j,l}^{(3)}$. Hence $\gamma_t(H_m - v) \le 3$, as desired. □

Proposition 3.12 *Let $m \ge 2$ be an integer. Then $\gamma_t(H_m - N_{H_m}[y_1]) \ge \gamma_t(H_m) - 1$.*

Proof. Suppose that $\gamma_t(H_m - N_{H_m}[y_1]) = 2$. Let S be a γ_t-set of $H_m - N_{H_m}[y_1]$. Since S γ_t-dominates y_2, $S \cap X_2 \ne \emptyset$. Since there is no vertex in X_2 which is adjacent to every vertex in X_3, we see that $S \subseteq X_2 \cup X_3$. Since each vertex in X_2 γ_t-dominates only one vertex in X_2, it follows that $|S \cap X_2| = |S \cap X_3| = 1$. Write $S = \{x_{j,l}^{(2)}, x_{j',l'}^{(3)}\}$. By the definition of S, $x_{j,l}^{(2)} x_{j',l'}^{(3)} \in E(H_m - N_{H_m}[y_1])$, and hence $(j, l) \ne (j', l')$. If $l \ne l'$, then $x_{j',l'}^{(2)}$ is not γ_t-dominated by S, a contradiction. Thus $l = l'$ and hence

$j' = 3 - j$. However, $x_{j,l}^{(3)}$ is not γ_t-dominated by S, a contradiction. Thus $\gamma_t(H_m - N_{H_m}[y_1]) \geq 3 = \gamma_t(H_m) - 1$. □

We construct one more family of candidates L_m for B_1 and B_2. The construction of the graph L_m is based on the following graph R defined in [2].

Let $C = s_1s_2s_3s_4s_1$ be a cycle of order 4, and let $U = \{a_1, a_2, u_1, u_2, v\}$. For each $i = 1, 2$, let $P^{(i)} = t_3^{(i)}t_1^{(i)}t_2^{(i)}t_4^{(i)}$ be a path of order 4. Let $E_1 = \bigcup_{i=1,2}\{s_jt_{j'}^{(i)} \mid j \neq j'\}$, $E_2 = \{tt' \mid t \in V(P^{(1)}), t' \in V(P^{(2)})\}$, $E_3 = \bigcup_{i=1,2}\{a_ix \mid x \in \{s_i, s_3, s_4, t_3^{(1)}, t_4^{(1)}, t_3^{(2)}, t_4^{(2)}\}\}$, $E_4 = \bigcup_{i=1,2}\{u_ix \mid x \in V(P^{(1)}) \cup V(P^{(2)}) \cup \{a_i\}\}$ and $E_5 = \{vs \mid s \in V(C)\}$. Let R be the graph defined by

$$V(R) = V(C) \cup V(P^{(1)}) \cup V(P^{(2)}) \cup U$$

and

$$E(R) = E(C) \cup E(P^{(1)}) \cup E(P^{(2)}) \cup (\bigcup_{1 \leq i \leq 5} E_i)$$

(see Figure 4).

Proposition 3.13 ([2]) *The graph R is 3-γ_t-critical, and $ecc_R(u_1) = diam(R) = d_R(u_1, v) = 3$.*

Proposition 3.14 *We have $\gamma_t(R - N_R[u_1]) \geq \gamma_t(R) - 1$.*

Proof. We have $\gamma_t(R - N_R[u_1]) \geq 2 = \gamma_t(R) - 1$. □

Let $L_m = (R \bullet G_m)(u_1, y_1 : u')$.

Proposition 3.15 *Let $m \geq 2$ be an integer. Then L_m is 5-γ_t-critical, and $ecc_{L_m}(v) = diam(L_m) = 6$.*

Proof. It is easy to see that $\mathrm{ecc}_{L_m}(v) = \mathrm{diam}(L_m) = d_{L_m}(v, y_2) = 6$. By Lemma 3.1(i),(ii) and Propositions 3.9, 3.10, 3.13 and 3.14, $\gamma_t(L_m) = \gamma_t(R) + \gamma_t(G_m) - 1 = 5$. Hence L_m is 5-γ_t-critical by Lemma 3.1(iii) and Propositions 3.9 and 3.13. □

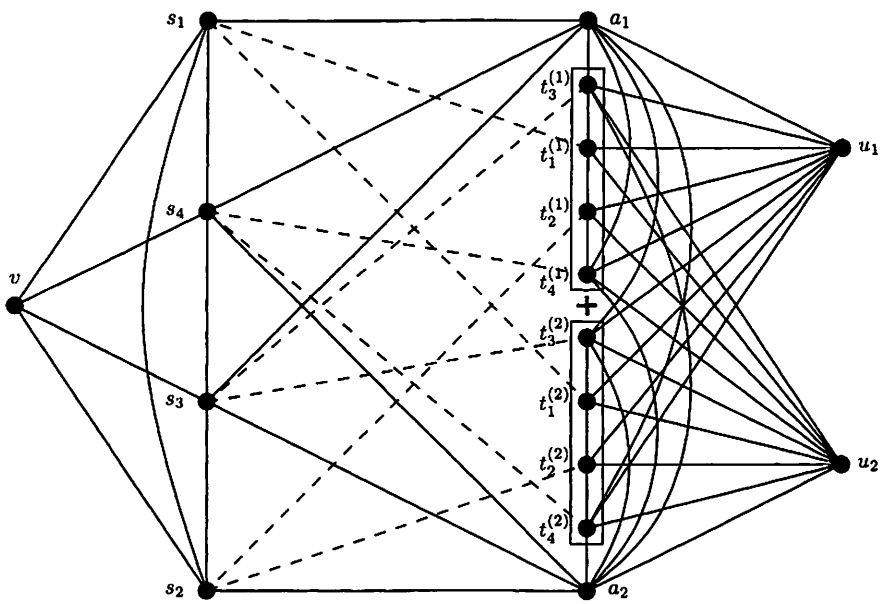

Figure 4: Graph R

Proposition 3.16 *Let $m \geq 2$ be an integer. Then $\gamma_t(L_m - N_{L_m}[v]) \geq \gamma_t(L_m) - 1$.*

Proof. Since the distance between y_2 and u_2 in $L_m - N_{L_m}[v]$ is 5, there is no total dominating set of $L_m - N_{L_m}[v]$ having cardinality at most 3. Hence $\gamma_t(L_m - N_{L_m}[v]) \geq 4 = \gamma_t(L_m) - 1$. □

We are now ready to construct examples which show the sharpness of the bound in Theorem 1.1.

For $m \geq 2$ and $a \geq 0$, let $Z_{3,1}^{(a)}(m) = Z_2^{(a)}(G_m, G_m; y_1, y_1)$, $Z_{3,2}^{(a)}(m) = Z_2^{(a)}(G_m, H_m; y_1, y_1)$ and $Z_{3,3}^{(a)}(m) = Z_2^{(a)}(G_m, L_m; y_1, v)$.

Proposition 3.17 *Let $m \geq 2$ and $a \geq 0$ be integers.*

(i) *The graph $Z_{3,1}^{(a)}(m)$ is a $(3a+5)$-γ_t-critical graph with diameter $\lfloor \frac{5(3a+5)-7}{3} \rfloor$.*

(ii) *The graph $Z_{3,2}^{(a)}(m)$ is a $(3a+6)$-γ_t-critical graph with diameter $\lfloor \frac{5(3a+6)-7}{3} \rfloor$.*

(iii) *The graph $Z_{3,3}^{(a)}(m)$ is a $(3a+7)$-γ_t-critical graph with diameter $\lfloor \frac{5(3a+7)-7}{3} \rfloor$.*

Proof.

(i) By Theorem 3.8 and Propositions 3.9 and 3.10, $Z_{3,1}^{(a)}(m)$ is a $(3a+5)$-γ_t-critical graph with diameter $5a+6 = \lfloor \frac{5(3a+5)-7}{3} \rfloor$.

(ii) By Theorem 3.8 and Propositions 3.9, 3.10, 3.11 and 3.12, $Z_{3,2}^{(a)}(m)$ is a $(3a+6)$-γ_t-critical graph with diameter $5a+7 = \lfloor \frac{5(3a+6)-7}{3} \rfloor$.

(iii) By Theorem 3.8 and Propositions 3.9, 3.10, 3.15 and 3.16, $Z_{3,3}^{(a)}(m)$ is a $(3a+7)$-γ_t-critical graph with diameter $5a+9 = \lfloor \frac{5(3a+7)-7}{3} \rfloor$.

□

Proposition 3.18 *Let $k \geq 3$ be an integer. Then there exist infinitely many connected graphs G such that G is k-γ_t-critical and such that G has diameter 3 if $k = 3$, and has diameter $\lfloor \frac{5k-7}{3} \rfloor$ if $k \geq 4$.*

Proof. Note that Propositions 3.9, 3.11 and 3.17 hold for all integers $m \geq 2$. Therefore if $k = 3$, then graphs G_m are desired graphs by Proposition 3.9. If $k = 4$, then graphs H_m are desired graphs Proposition 3.11. Thus we may assume that $k \geq 5$. If $k \equiv 2 (\mathrm{mod}\ 3)$, then letting $k = 3a + 5$ ($a \geq 0$), we see that graphs $Z_{3,1}^{(a)}(m)$ are desired graphs Proposition 3.17(i); if $k \equiv 0 (\mathrm{mod}\ 3)$, then letting $k = 3a + 6$ ($a \geq 0$), we see that graphs $Z_{3,2}^{(a)}(m)$ are desired graphs by Proposition 3.17(ii); if $k \equiv 1 (\mathrm{mod}\ 3)$, then letting $k = 3a + 7$ ($a \geq 0$), we see that graphs $Z_{3,3}^{(a)}(m)$ are desired graphs by Proposition 3.17(iii). □

References

[1] R. Diestel, "Graph Theory" (4th edition), Graduate Texts in Mathematics **173**, Springer (2010).

[2] W. Goddard, T.W. Haynes, H.A. Henning and L.C. van der Merwe, The diameter of total domination vertex critical graphs, Discrete Math. **286** (2004) 255–261.

[3] M.A. Henning and N.J. Rad, On total domination vertex critical graphs of high connectivity, Discrete Appl. Math. **157** (2009) 1969–1973.

[4] M.A. Henning and A. Yeo, Perfect matchings in total domination critical graphs, Graphs Combin. **27** (2011) 685–701.

[5] C. Wang, Z. Hu and X. Li, A constructive characterization of total domination vertex critical graphs, Discrete Math. **309** (2009) 991–996.