On Holey Schröder Designs of Type $2^n u^1$

HANTAO ZHANG, STANLEY ZIEWACZ
Computer Science Department
The University of Iowa
Iowa City, IA 52242 U.S.A.

Abstract

It has been shown by Bennett et al. in 1998 that a holey Schröder design with n holes of size 2 and one hole of size u, i.e., of type $2^n u^1$, exists if $1 \leq u \leq 4$ and $n \geq u+1$ with the exception of $(n,u) \in \{(2,1),(3,1),(3,2)\}$, or $u \geq 16$ and $n \geq \lceil 5u/4 \rceil + 14$. In this paper, we extend this result by showing that, for $1 \leq u \leq 16$, a holey Schröder design of type $2^n u^1$ exists if and only if $n \geq u+1$, with the exception of $(n,u) \in \{(2,1),(3,1),(3,2)\}$ and with the possible exception of $(n,u) \in \{(7,5),(7,6),(11,9),(11,10)\}$. For general u, we prove that there exists an HSD($2^n u^1$) for all $u \geq 17$ and $n \geq \lceil 5u/4 \rceil + 4$. Moreover, if $u \geq 35$, then an HSD($2^n u^1$) exists for all $n \geq \lceil 5u/4 \rceil + 1$; if $u \geq 95$, then an HSD($2^n u^1$) exists for all $n \geq \lceil 5u/4 \rceil - 2$. We also improve a well-known result on the existence of holey Schröder designs of type h^n by removing the remaining possible exception of type 6^4.

1 Introduction

In algebra, every Latin square is equivalent to a *quasigroup* $(Q,*)$ where $* : Q \times Q \rightarrow Q$ is defined by the Latin square. The *order* of the quasigroup is $|Q|$. If $(Q,*)$ satisfies $x*x = x$ and $(x*y)*(y*x) = x$ for all $x, y \in Q$, it is called an *idempotent Schröder quasigroup*. Two quasigroups $(Q,*)$ and $(Q,\cdot)$ are *orthogonal* to each other if $|\{(x*y, x\cdot y) : x,y \in Q\}| = |Q|^2$. $(Q,*)$ is *self-orthogonal* if $(Q,*)$ is orthogonal to its transpose.

Idempotent Schröder quasigroups, or ISQs, are associated with other combinatorial configurations such as a class of edge-colored block designs with block size 4, triple tournaments and self-orthogonal Latin squares with the Weisner property (see [8], [2], [12] and [13]). A pair of orthogonal Latin squares, say $(Q,*)$ and $(Q,\cdot)$, are said to have the *Weisner property* if $x*y = z$ and $x\cdot y = w$ whenever $z*w = x$ and $z\cdot w = y$ for all $x,y,z,w \in Q$. Let $(Q,\cdot)$ be the transpose of $(Q,*)$, i.e., $z\cdot w = w*z$. If $(Q,*)$ is an ISQ, then from $z * w = x$ and $z \cdot w = y$, we have $x * y = (z * w) * (w * z) = z$. Similarly, we also have $x \cdot y = w$. The following theorem gives a complete solution of the existence of ISQ.

Theorem 1.1 ([8], [5]) *An idempotent Schröder quasigroup of order v exists if and only if $v \equiv 0, 1$ (mod 4) and $v \neq 5, 9$.*

An ISQ(v) is equivalent to an edge-colored design CBD$[G_6; v]$ which is investigated in [8]. An *edge-colored design* CBD$[G_6; v]$ on a v-set Q is a partition of the colored edges of a triplicate complete graph $3K_v$, each K_v receives one color for its edges from three different colors, into blocks (a, b, c, d), each containing edges $\{a, b\}, \{c, d\}$ colored with color 1, edges $\{a, c\}, \{b, d\}$ with color 2, and edges $\{a, d\}, \{b, c\}$ with color 3. If we define a binary operation $(\cdot)$ as $a \cdot b = c, b \cdot a = d, c \cdot d = a$ and $d \cdot c = b$ from the block (a, b, c, d) and define $x \cdot x = x$ for every $x \in Q$, an ISQ(v) is obtained on set Q. On the other hand, suppose Q is an ISQ. If $a \cdot b = c$, $b \cdot a = d$, then we must have $c \cdot d = (a \cdot b) \cdot (b \cdot a) = a$ and $d \cdot c = (b \cdot a) \cdot (a \cdot b) = b$. So the block (a, b, c, d) is determined and a CBD$[G_6; v]$ can be obtained in this way.

The concept of edge-colored design can be generalized to that of *holey Schröder design* which is a triple $(X, \mathcal{H}, \mathcal{B})$ satisfying the following properties:

1. $\mathcal{H}$ is a partition of X into subsets called *holes*,
2. $\mathcal{B}$ is a family of 4-tuples of X (called *blocks*) such that a hole and a block contain at most one common point,
3. the pairs of points in a block (a, b, c, d) are colored as $\{a, b\}$ and $\{c, d\}$ with color 1, $\{a, c\}$ and $\{b, d\}$ with color 2, and $\{a, d\}$ and $\{b, c\}$ with color 3,
4. every pair of points from distinct holes occurs in three blocks with different colors.

The *type* of the HSD is the multiset $\{|H| : H \in \mathcal{H}\}$ which is denoted by an exponential notation: $s_1^{n_1} s_2^{n_2} \cdots s_t^{n_t}$ means we have n_i occurrences of $s_i = |H|$ in $\{|H| : H \in \mathcal{H}\}$.

Each HSD is equivalent to a holey ISQ, called *frame Schröder quasigroup* (FSQ), which is equivalent to a frame self-orthogonal Latin square (FSOLS) with the Weisner property [13]. For the existence of FSOLS of type $2^n u^1$, we have the following theorem [15].

Theorem 1.2 *There exists FSOLS*($2^n u^1$) *if and only if* $n \geq 1 + u$ *and* $u \geq 2$, *or* $n \geq 4$ *and* $u = 0, 1$.

For HSDs, however, it is found that the type $2^3 2^1$ does not exist by exhaustive computer search. This means that some types of FSOLS cannot have the Weisner property.

Another class of designs related to HSDs is *group divisible design* (GDD). A GDD is a 4-tuple $(X, \mathcal{G}, \mathcal{B}, \lambda)$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*,
2. $\mathcal{B}$ is a family of subsets of X (called *blocks*) such that a group and a block contain at most one common point,
3. every pair of points from distinct groups occurs in exactly λ blocks.

The *type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$ and we will also use an "exponential" notation for the type of GDD. We also use the notation GDD$(K, M; \lambda)$ to denote the GDD when its block sizes belong to K and group sizes belong to M. In particular, a GDD$(\{k\}, \{2, u\}, 1)$, where there is only one group of size u, is denoted by k-GDD of type $2^n u^1$.

Theorem 1.3 [9, 10] *There exists a* 4*-GDD of type* $2^n u^1$ *for each* $n \geq 6$, $n \equiv 0 \pmod 3$ *and* $u \equiv 2 \pmod 3$ *with* $2 \leq u \leq n - 1$, *except for* $(n, u) = (6, 5)$ *and possibly excepting* $(n, u) \in \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}$.

If $M = \{1\}$, then the GDD becomes a *pairwise balanced design* (*PBD*) [14]. If $K = \{k\}, M = \{n\}$ and the type is n^k, then the GDD becomes a *transversal design*, TD(k, n). The following results are well known (see [1] and [4], for example).

Theorem 1.4 (*a*) *There exists a TD*$(4, m)$ *for any positive integer* $m, m \notin \{2, 6\}$.

(*b*) *There exists a TD*$(5, m)$ *for every positive integer* $m \notin \{2, 3, 6, 10\}$.

(*c*) *There exists a TD*$(6, m)$ *for* $m \geq 5$ *and* $m \notin \{6, 10, 14, 18, 22\}$,

(*d*) *There exists a TD*$(7, m)$ *for* $m \geq 7$ *and* $m \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$.

It is well known that the existence of a TD(k, n) is equivalent to the existence of $k - 2$ MOLS(n). It is easy to see that if we erase the colors in the blocks, the HSD becomes a GDD with block size 4 and $\lambda = 3$. But the converse may be not true. It is proved in [7] that a 4-GDD with $\lambda = 3$ and of type h^u exists if and only if $h^2u(u-1) \equiv 0 \ (mod\ 4)$.

Theorem 1.5 [3] *An HSD(h^u) exists if and only if $h^2u(u-1) \equiv 0 \ (mod\ 4)$ with the exception of $(h, u) \in \{(1,5), (1,9), (2,4)\}$ and the possible exception of $(h, u) = (6, 4)$.*

In this paper, we improve the above theorem by removing the only possible exception. In the appendix of this paper, we provide for the first time an HSD(6^4) in the form of a quasigroup. Thus, we have the following theorem.

Theorem 1.6 *An HSD(h^u) exists if and only if $h^2u(u-1) \equiv 0 \ (mod\ 4)$ with the exception of $(h, u) \in \{(1,5), (1,9), (2,4)\}$.*

The following results are obtained in [4]:

Theorem 1.7 *(a) For $1 \leq u \leq 4$, an HSD(2^nu^1) exists if and only if $n \geq u+1$ with the exception of $(n, u) \in \{(2,1), (3,1), (3,2)\}$.*

(b) There exists an HSD(2^nu^1) for $1 \leq u \leq 64$ and $n \geq 85$.

(c) There exists an HSD(2^nu^1) for $u \geq 16$ and $n \geq \lceil 5u/4 \rceil + 14$.

There are two gaps left out by the above theorem regarding the existence of HSD(2^nu^1): (1) $5 \leq u \leq 15$ and $n \leq 85$; and (2) $u \geq 16$ and $u+1 \leq n \leq \lceil 5u/4 \rceil + 13$. In this paper, for (1) we establish the existence of HSDs of type 2^nu^1 for $5 \leq u \leq 16$ and every $n \geq u+1$. For (2), we improve the existing result by increasing the range of n by at least 10 values for each u. The main result of this paper is the following theorem.

Theorem 1.8 *(a) For $5 \leq u \leq 16$, an HSD(2^nu^1) exists if and only if $n \geq u+1$, with the possible exception of $(n, u) \in \{(7,5), (7,6), (11,9), (11,10)\}$.*

(b) For $u \geq 17$, an HSD(2^nu^1) exists if $n \geq \lceil 5u/4 \rceil + 4$. Moreover, if $u \geq 35$, then an HSD(2^nu^1) exists for all $n \geq \lceil 5u/4 \rceil + 1$; if $u \geq 95$, then an HSD(2^nu^1) exists for all $n \geq \lceil 5u/4 \rceil - 2$.

2 Construction Tools

To construct HSDs directly, sometimes we use *starter blocks.* Suppose the block set $\mathcal{B}$ of an HSD is closed under the action of some Abelian group G, then we are able to list only part of the blocks (starter or base blocks) which determines the structure of the HSD. We can also attach some infinite points to an Abelian group G. When the group acts on the blocks, the infinite points remain fixed. Formally, let $\mathcal{B}$ be the block set of an HSD over the point set $S = G \cup X$, where $(G, +)$ is a group, X is a set of *infinite points*, $G \cap X = \emptyset$. The addition $(+)$ is extended over X as follows: $g+x = x+g = x$ for any $g \in G$ and $x \in X$. A set $\mathcal{A} \subset \mathcal{B}$ is called *starter blocks* of $\mathcal{B}$ if $\mathcal{A}$ is a minimum subset of $\mathcal{B}$ satisfying the property that for any $\mathbf{a} \in \mathcal{A}$ and any $g \in G$, $\mathbf{a} + g \in \mathcal{B}$, and for any $\mathbf{b} \in \mathcal{B}$, there exist $\mathbf{a} \in \mathcal{A}$ and $g \in G$ such that $\mathbf{b} = \mathbf{a} + g$, where $\mathbf{a} + g = (a_1 + g, a_2 + g, a_3 + g, a_4 + g)$ when $\mathbf{a} = (a_1, a_2, a_3, a_4)$. In the following example $x_1, x_2, ...,$ are infinite points.

Example 2.1 *An HSD*$(2^8 5^1)$

points: $Z_{16} \cup \{x_1, x_2, ..., x_5\}$

holes: $\{\{i, i+8\} : 0 \le i \le 7\} \cup \{x_1, x_2, ..., x_5\}$

starter blocks: $(0, 1, 2, x_5)$, $(0, 2, 4, x_4)$, $(0, 3, 15, 6)$, $(0, 10, 7, x_2)$, $(0, 11, 6, x_1)$, $(0, 12, 5, x_3)$.

In this example, the entire set of blocks is developed from the starter blocks by adding $a \in Z_{16}$ to the starter blocks.

To check the starter blocks, we need only calculate whether the differences $\pm(x - y)$ from all pairs $\{x, y\}$ with color i in the starter blocks are precisely $G \setminus S$ for $1 \le i \le 3$, where S is the set of the differences of the holes. For the above example, for color 1, the set of differences from the six blocks is $\{\pm1, \pm2, \pm3, \pm9, \pm10, \pm11, \pm12\}$, which is exactly $Z_{16} \setminus \{0, 8\}$. This is also true for colors 2 and 3.

We have pointed out in the previous section that there is an equivalence between an FSQ and an HSD. That is, for all distinct $a, b, c, d \in Q$, $a*b = c$, $b * a = d$, $c * d = a$, $d * c = b$ in the FSQ if and only if (a, b, c, d) is a block of the HSD. So we are free to use either form. In fact, all the designs found by computer in this paper are in the form of Schröder quasigroups. To allow the existence of starter blocks with a group G, for quasigroup $(Q, *)$, we require that $Q = G \cup X$ and for all $x, y, z \in Q$, $x * y = z$ if and only if $(x + g) * (y + g) = (z + g)$ for any $g \in G$ [16, 17]. Since HSDs have a more compact form than quasigroups, we will present them as HSDs in this paper.

The above idea of starter blocks can be also generalized: Instead of adding 1 to each point of the starter blocks, we may add k, where $k > 1$, to develop the block set; we refer to this as the $+k$ method. In this case, for a set $\mathcal{A}$ to be starter blocks, we require that for any $\mathbf{a} \in \mathcal{A}$ and any $g \in G$, a + $kg \in \mathcal{B}$. For quasigroups, we require that for all $x, y, z \in Q$, $x * y = z$ if and only if $(x + kg) * (y + kg) = (z + kg)$ for any $g \in G$ [16, 17].

Example 2.2 *An HSD*($2^6 5^1$)

points: $Z_{12} \cup \{x, y, z, v, w\}$

holes: $\{\{i, i+6\} : 0 \le i \le 5\} \cup \{x, y, z, v, w\}$

starter blocks: $(0,3,1,x), (0,7,3,w), (0,8,4,v), (0,9,10,z), (0,10,2,y)$, $(1,0,11,y)$, $(1,3,8,z), (1,5,4,x), (1,8,5,w), (1,11,6,v), (2,0,5,z)$, $(2,1,10,w)$, $(2,5,3,v)$, $(2,9,1,y)$, $(2,10,7,x), (3,2,4,w), (3,4,6,x)$, $(3,6,1,v)$, $(3,10,11,z), (3,11,8,y)$.

By adding 4 (mod 12) to the 20 starter blocks, we obtain a set of 60 blocks.

Next, we state several recursive constructions of HSDs, which are commonly used in other block designs [4]. The following construction comes from the weighting construction of GDDs [14].

Construction 2.3 (*Weighting*) *Suppose* $(X, \mathcal{H}, \mathcal{B})$ *is a GDD with* $\lambda = 1$ *and let* $w : X \mapsto Z^+ \cup \{0\}$. *Suppose there exist HSDs of type* $\{w(x) : x \in B\}$ *for every* $B \in \mathcal{B}$. *Then there exists an HSD of type* $\{\sum_{x \in H} w(x) : H \in \mathcal{H}\}$.

Lemma 2.4 *There exists an HSD*($2^n u^1$) *for each* $n \ge 6$, $n \equiv 0 \pmod 3$ *and* $u \equiv 2 \pmod 3$ *with* $2 \le u \le n-1$, *except for* $(n, u) = (6, 5)$ *and possibly excepting* $(n, u) \in \{(21,17), (33,23), (33,29), (39,35), (57,44)\}$.

Proof From Theorem 1.3, there exist 4-GDDs of the same type. So we can give all points of this GDD weight one to get the desired HSD($2^n u^1$). □

Using Theorem 1.4(a), if we give every point of an HSD weight m and input TD($4, m$) to each block of the HSD, we can obtain the following construction.

Construction 2.5 *Suppose there exists an HSD*($h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$), *then there exists an HSD*($(mh_1)^{n_1} (mh_2)^{n_2} \cdots (mh_k)^{n_k}$), *where* $m \ne 2, 6$.

The next construction may be called "filling in holes". It is used commonly in constructing designs.

Construction 2.6 *Suppose there exist an HSD of type* $\{s_i : 1 \le i \le k\}$ *and HSDs of type* $\{ h_{i_j} : 1 \le j \le n_i\} \cup \{a\}$, *where* $\sum_{j=1}^{n_i} h_{i_j} = s_i$ *and* $1 \le i \le k-1$, *then there exists an HSD of type* $\{h_{i_j} : 1 \le j \le n_i, 1 \le i \le k-1\} \cup \{s_k + a\}$.

The next construction comes from [8].

Construction 2.7 *Suppose there exists an FSOLS*($h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$), *then there exists an HSD*($(4h_1)^{n_1}(4h_2)^{n_2} \cdots (4h_k)^{n_k}$).

Lemma 2.8 *If there exists an HSD*($2^m k^1$), *there exists an HSD*($2^{3m}(2m+k)^1$).

Proof Because there exists an HSD($2^m k^1$), $m \ge 4$. By Theorem 1.6, there exists an HSD($(2m)^4$). We adjoin k points to this HSD($(2m)^4$), and fill three holes of size $2m$ with an HSD($2^m k^1$), leaving one hole of size $2m+k$. The result is the desired HSD($2^{3m}(2m+k)^1$). □

The following lemma is an extension of Lemma 6.1 in [4].

Lemma 2.9 *For* $m \ge 4$, $u \not\equiv 0 \pmod 4$, *and* $u < 4m$, *there exists an HSD*($2^{4m}u^1$).

Proof From Theorem 1.2, there exists an FSOLS($2^m s^1$) for $m \ge 4$ and $0 \le s < m-1$. Applying Construction 2.7 to this FSOLS, we obtain an HSD of type $8^m(4s)^1$. Adjoin $k = 1, 2, 3$ points to this HSD and fill the holes of size 8 with an HSD($2^4 k^1$), we obtain an HSD of type $2^{4m}(4s+k)^1$, where $m \ge 4$, $0 \le s < m-1$ and $1 \le k \le 3$. □

Lemma 2.10 *If an HSD*($2^m s^1$) *exists, then an HSD*($2^{5m}t^1$) *exists for* $5s \le t \le 5s+4$.

Proof Applying Construction 2.5 (with $m = 5$) to the HSD($2^m s^1$), we obtain an HSD of type $10^m(5s)^1$. To this HSD we adjoin t points, where $0 \le t \le 4$, and fill the holes of size 10 with an HSD($2^5 t^1$), we obtain an HSD($2^{5m}(5s+t)^1$). □

Lemma 2.11 *If there exists a TD*$(5, m)$, *then there exists an HSD*($(2m)^4 s^1$), *where* $m \le s \le 3m$.

Proof Give weight 2 to each point of first four groups of a TD$(5, m)$. Give weight 1, 2 or 3 to the points of the fifth group. Since there exist HSDs of type 2^4t^1, $t = 1, 2, 3$ from Theorem 1.7(a), we obtain the desired HSD by Lemma 2.3. □

Lemma 2.12 ([4], Lemma 6.2) *If there exists a TD*$(6, m)$*, then there exist:*

(a) *HSD*$(2^{4m+k}s^1)$ *for* $k = 0, 1, 5, 6, ..., m$ *and* $m \le s \le 3m$.

(b) *HSD*$(2^{4m+k}s^1)$ *for* $k = 0, 4, 5, 6, ..., m$ *and* $m + 1 \le s \le 3m + 3$.

(c) *HSD*$((2m)^4(2k)^1s^1)$*, where* $0 \le k \le m$ *and* $m \le s \le 3m$.

Note that the HSD in (c) of the above lemma was constructed implicitly in the proof of (a) in [4].

Lemma 2.13 *If there exist a TD*$(6, m)$*, an HSD*(2^mt^1)*, and an HSD*(2^kt^1)*, where* $4 \le k \le m$*, then there exists an HSD*$(2^{4m+k}s^1)$ *for* $m + t \le s \le 3m + t$.

Proof Take the HSD$((2m)^4(2k)^1u^1)$ from Lemma 2.12(c), where $0 \le k \le m$ and $m \le u \le 3m$, we first adjoin t points, to the HSD$((2m)^4(2k)^1u^1)$ and then fill the holes of sizes $2m$ and $2k$ with an HSD(2^mt^1) and an HSD(2^kt^1), we obtain an HSD$(2^{4m+k}(u + t)^1)$ where $m + t \le u + t \le 3m + t$. □

Lemma 2.14 ([4], Lemma 6.3) *If there exists a TD*$(7, m)$*, then there exist*

(a) *HSD*$(2^{5m+k}s^1)$ *for* $k = 0, 1, 5, 6, ..., m$ *and* $0 \le s \le 4m$.

(b) *HSD*$(2^{5m+k}s^1)$ *for* $k = 4, 5, ..., m$ *and* $1 \le s \le 4m + 3$.

(c) *HSD*$((2m)^5(2k)^1s^1)$ *for* $0 \le k \le m$ *and* $0 \le s \le 4m$.

Note that the HSD in (c) of the above lemma was constructed implicitly in the proof of (a) in [4].

Lemma 2.15 *If there exist a TD*$(7, m)$*, an HSD*(2^mt^1)*, and an HSD*(2^kt^1)*, where* $4 \le k \le m$*, then there exists HSD*$(2^{5m+k}s^1)$ *for* $t \le s \le 4m + t$.

Proof We adjoin t points to the HSD$((2m)^5(2k)^1u^1)$ from Lemma 2.14(c), where $0 \le u \le 4m$, and fill the holes of sizes $2m$ and $2k$ with an HSD(2^mt^1) and an HSD(2^kt^1), we obtain an HSD$(2^{5m+k}(u + t)^1)$, where $t \le u + t \le 4m + t$. □

3 HSD($2^n u^1$) for some specific n

From the results in the previous section, we may show the following results.

Lemma 3.1 *An HSD*($2^{17}u^1$) *exists for* $5 \le u \le 16$.

Proof For $5 \le u \le 8$ please see Appendix A1-A4. For $9 \le u \le 11$, we first get an HSD($8^4 10^1$) by applying Lemma 2.11 with $m = 4, u = 10$. Adjoin k points, where $1 \le k \le 3$, to this HSD, and fill three holes of size 8 with an HSD($2^4 k^1$) and the hole of size 10 with an HSD($2^5 k^1$), we obtain an HSD($2^{12+5}(8+k)^1$) for $1 \le k \le 3$. Besides an HSD($2^{17}14^1$) is given in [4], for $12 \le u \le 16$, please see Appendix A7 and A8. □

Lemma 3.2 *An HSD*($2^{18}u^1$) *exists for* $5 \le u \le 17$.

Proof For $u = 5, 8$, we apply Lemma 2.4. For $u = 6, 7$, please see Appendix A2 and A3.

For $9 \le u \le 11$, we first get an HSD($8^4 12^1$) by applying Lemma 2.11 with $m = 4, u = 12$. Adjoin k points, where $1 \le k \le 3$, to this HSD, and fill three holes of size 8 with an HSD($2^4 k^1$) and the hole of size 12 with an HSD($2^6 k^1$), we obtain an HSD($2^{12+6}(8+k)^1$) for $1 \le k \le 3$.

For $12 \le u \le 17$, we obtain an HSD($2^n u^1$) by Lemma 2.8 with $m = 6$, because an HSD($2^6 k^1$) exists for $k = 0, 1, 2, 3, 4, 5$. □

Lemma 3.3 *An HSD*($2^{19}u^1$) *exists for* $5 \le u \le 16$.

Proof For $5 \le u \le 10$, please see Appendix A1-A6. For $11 \le u \le 13$, we first get an HSD($10^4 8^1$) by applying Lemma 2.11 with $m = 5, u = 8$. Adjoin k points, where $1 \le k \le 3$, to the HSD, and fill three holes of size 10 with an HSD($2^5 k^1$) and the hole of size 8 with an HSD($2^4 k^1$), we obtain an HSD($2^{19}(10+k)^1$) for $1 \le k \le 3$. For $14 \le u \le 16$, please see Appendix A8. □

Lemma 3.4 *An HSD*($2^{21}u^1$) *exists for* $5 \le u \le 20$.

Proof For $5 \le u \le 15$, from Lemma 2.12(a) with $m = 5, k = 1$, we have HSD($2^n u^1$). For $16 \le u \le 18$, we adjoin k points, where $2 \le k \le 4$, to an HSD(14^4), and fill three holes of size 14 with an HSD($2^7 k^1$) from Theorem 1.7(a), leaving one hole of size $14 + k$ where $2 \le k \le 4$. For $u = 19$, please see Appendix A8. For $u = 20$, we apply Lemma 2.4. □

Lemma 3.5 *An HSD*(12^4t^1) *exists for* $0 \le t \le 16$.

Proof We start with a TD(5, 4). In the first four groups of the TD we give all the points weight 3. In the last group we give the points a weight of 0, 1, 2, 3, or 4 for a total weight of t. We need HSDs of type (3^4k^1), where $0 \le k \le 4$. For $k = 0, 3$, the designs are given in Theorem 1.6; for $k = 1, 2, 4$, they are given in Appendix A9. The resulting design is an HSD(12^4t^1). □

Lemma 3.6 *An HSD*($2^{22}u^1$) *exists for* $5 \le u \le 16$.

Proof For $5 \le u \le 8$, please see Appendix A1-A4.

For $u = 9$, we generate an HSD($8^5 12^1$) from an HSD($2^5 3^1$) by Construction 2.5 with $m = 4$. To this HSD we adjoin one point, fill four holes of size 8 by an HSD($2^4 1^1$) and one hole of size 12 by an HSD($2^6 1^1$), we obtain an HSD($2^{22}(8+1)^1$).

For $10 \le u \le 14$, we first get an HSD($10^4 14^1$) by applying Lemma 2.11 with $m = 5, u = 14$. Adjoin k points, where $0 \le k \le 4$, to this HSD, and fill three holes of size 10 with an HSD(2^5k^1) and the hole of size 14 with an HSD(2^7k^1), we obtain an HSD($2^{22}(10+k)^1$) for $0 \le k \le 4$.

For $u = 15$, we take an HSD($12^4 8^1$) from Lemma 3.5. Adjoin three points to this HSD, and fill three holes of size 12 with an HSD($2^6 3^1$) and one hole of size 8 with an HSD($2^4 3^1$), we obtain an HSD($2^{22}(12+3)^1$).

Finally for $u = 16$, please see Appendix A8. □

Lemma 3.7 *An HSD*($2^{23}u^1$) *exists for* $5 \le u \le 16$.

Proof For $5 \le u \le 8$, please see Appendix A1-A4.

For $9 \le u \le 11$, we take the HSD($8^5 14^1$) from Appendix A9, adjoin k points, $1 \le k \le 3$, to the HSD, and fill four holes of size 8 with an HSD(2^4k^1) and one hole of size 14 with an HSD(2^7k^1), we obtain an HSD($2^{23}(8+k)^1$).

For $12 \le u \le 16$, we take an HSD($12^4 10^1$) from Lemma 3.5. Adjoin k points, where $0 \le k \le 4$, to this HSD, and fill three holes of size 12 with an HSD(2^6k^1) and the hole of size 10 with an HSD(2^5k^1), we obtain an HSD($2^{23}(12+k)^1$) for $0 \le k \le 4$. □

Lemma 3.8 *An HSD*($2^{26}u^1$) *exists for* $0 \le u \le 18$.

Proof For $0 \leq u \leq 14$, we first apply Lemma 2.12(c) with $m = 5$ and $s = 12$ to get an HSD($10^4(2k)^1 12^1$), where $0 \leq k \leq 5$. Adjoin t points, where $0 \leq t \leq 4$, to this HSD, and fill the holes of sizes 10 and 12 with an HSD($2^5 t^1$) and an HSD($2^6 t^1$), respectively. The result is an HSD($2^{26}(2k+t)^1$), where $0 \leq 2k+t \leq 14$.

For $14 \leq u \leq 18$, we first get an HSD($14^4 10^1$) by applying Lemma 2.11 with $m = 7, u = 10$. Adjoin k points, where $0 \leq k \leq 4$, to this HSD, and fill three holes of size 14 with an HSD($2^7 k^1$) and the hole of size 10 with an HSD($2^5 k^1$), we obtain an HSD($2^{26}(14+k)^1$) for $0 \leq k \leq 4$. □

Lemma 3.9 *An HSD($2^{27}u^1$) exists for* $0 \leq u \leq 26$.

Proof For $0 \leq u \leq 14$, we first apply Lemma 2.12(c) with $m = 5$ and $s = 14$ to get an HSD($10^4(2k)^1 14^1$), where $0 \leq k \leq 5$, then adjoin t points, where $0 \leq t \leq 4$, to this HSD, and fill the holes of sizes 10 and 14 with an HSD($2^5 t^1$) and an HSD($2^7 t^1$), respectively. The result is an HSD($2^{27}(2k+t)^1$), where $0 \leq 2k+t \leq 14$.

For $14 \leq u \leq 18$, we first get an HSD($14^4 12^1$) by applying Lemma 2.11 with $m = 7, u = 12$. Adjoin k points, where $0 \leq k \leq 4$, to this HSD, and fill three holes of size 14 with an HSD($2^7 k^1$) and the hole of size 12 with an HSD($2^6 k^1$), we obtain an HSD($2^{27}(14+k)^1$) for $0 \leq k \leq 4$.

For $18 \leq u \leq 26$, we start with an HSD(18^4), adjoin k points to this HSD, and fill in the first three holes of size 18, where $0 \leq k \leq 8$, with an HSD($2^9 k^1$). The resulting design is an HSD of type $2^{27}(18+k)^1$, where $18 \leq 18+k \leq 26$, and this completes the proof of the lemma. □

Lemma 3.10 *An HSD($2^{29}u^1$) exists for* $5 \leq u \leq 21$.

Proof For $u = 5$, we take an HSD($2^{23}17^1$) from Appendix A8 and fill the hole of size 17 with an HSD($2^6 5^1$). For $u = 6$, please see Appendix A2. For $7 \leq u \leq 21$, we can get the designs from Lemma 2.12(a) with $m = 7$ and $k = 1$. □

Lemma 3.11 *An HSD($2^{31}u^1$) exists for* $5 \leq u \leq 21$.

Proof For $u = 5$, we can get an HSD($2^{25}17^1$) from Lemma 2.10 with $m = 5$, $s = 3$ and $t = 17$. Fill the hole of 17 with an HSD($2^6 5^1$), we have an HSD($2^{31}5^1$).

For $6 \leq u \leq 18$, we form a $\{6,7\}$-GDD of type 6^7 by deleting one block from a TD(7,7). In the first five groups of this GDD, we give all of the points weight 2. In the fifth group, we give one point of weight 2 and the other points weight 0. In the last group we give the points a weight of 1, 2, or 3 for a total weight of u where $6 \leq u \leq 18$. Since there are HSDs of types 2^n for $n = 5, 6, 7$ and $2^n k^1$ for $n = 4, 5, 6$ and $k = 1, 2, 3$ by Theorem 1.7(a), we get an HSD of type $12^5 2^1 u^1$ for $6 \leq u \leq 18$. To this HSD we fill the holes of size 12 with an HSD(2^6) and obtain an HSD($2^{31}u^1$) for $6 \leq u \leq 18$.

For $7 \leq u \leq 21$, we use a TD(8,7): In the first four groups of the TD(8,7) we give all of the points a weight of two. In the fifth, sixth and seventh groups, we give one point weight two and the other points weight zero. In the last group, we give the points a weight of 1, 2, or 3, for a total weight of u. Since we have HSDs of types $2^n k^1$ for $n = 4, 5, 6, 7$ and $k = 1, 2, 3$, we get an HSD of type $14^4 2^3 u^1$ for $7 \leq u \leq 21$. By filling in the holes of size 14 with an HSD(2^7), the resulting design is an HSD($2^{31}u^1$) for $7 \leq u \leq 21$. □

Lemma 3.12 *An HSD*($2^{37}u^1$) *exists for* $0 \leq u \leq 28$.

Proof We will use a TD(8,7): In the first five groups of the TD(8,7) we give all of the points a weight of two. In the sixth and seventh groups, we give one point weight two and the other points weight zero. In the last group, we give the points a weight of 0, 1, 2, 3, or 4, for a total weight of u. Since we have HSDs of types $2^n k^1$ for $n = 5, 6, 7$ and $0 \leq k \leq 4$, we get an HSD of type $14^5 2^2 u^1$ for $0 \leq u \leq 28$. By filling in the holes of size 14 with an HSD(2^7), the resulting design is an HSD($2^{37}u^1$) for $0 \leq u \leq 28$. □

4 HSD($2^n u^1$) for $5 \leq u \leq 16$

Lemma 4.1 *An HSD*($2^n u^1$) *exists for* $5 \leq u \leq 16$ *and* $u + 1 \leq n \leq 25$, *except possibly* $(n, u) \in \{(7,5), (7,6), (11,9), (11,10)\}$.

Proof For $n = 6$ and $u = 5$, please see Example 2.2. For $n = 8$ and $5 \leq u \leq 7$, please see Example 2.1 and Appendix A2 and A3.

Now let us consider $9 \leq n \leq 15$. For $n = 9, 12, 15$, $u < n$ and $u = 5, 8, 11, 14$, we apply Lemma 2.4. For $n = 12$ and $9 \leq u \leq 11$, we obtain an HSD($2^n u^1$) by Lemma 2.8 with $m = 4$, because an HSD($2^4 k^1$) exists for $k = 1, 2, 3$. For $n = 15$ and $10 \leq u \leq 14$, we obtain an HSD($2^n u^1$) by Lemma 2.8 with $m = 5$, because an HSD($2^5 k^1$) exists for $k = 0, 1, 2, 3, 4$. Besides an HSD($2^n 11^1$) for $n = 13, 14$ can be found in [4], the other designs for $9 \leq n \leq 15$ are given in Appendix A1-A8, except $n = 11$ and $u = 9, 10$.

For $n = 16, 24$, $5 \le u \le 15$, $u < n$, and $u \ne 8, 12$, by Lemma 2.9, we have an HSD($2^n u^1$). For $n = 16$ and $u = 8, 12$, let $t = u - 2$, we obtain first an HSD($8^4 t^1$) by Lemma 2.11 with $m = 4$ and $u = t$. To this HSD, we adjoin 2 points and fill the holes of size 8 with an HSD($2^4 2^1$), to obtain an HSD($2^{16} u^1$) for $u = 8, 12$. For $n = 24$ and $u = 8, 12, 16$, we get them from Lemma 2.12(b) with $m = 5, k = 4$.

The cases of $n = 17, 18, 19, 21, 22, 23$ are covered by Lemmas 3.1, 3.2, 3.3, 3.4, 3.6, 3.7, respectively.

For $n = 20, 25$ and $5 \le u \le 19$, we can get an HSD($2^n u^1$) from Lemma 2.10 with $m = 4, 5$, $1 \le s \le 3$ and $t = u$. □

Lemma 4.2 *An HSD*($2^n u^1$) *exists for* $5 \le u \le 18$ *and* $26 \le n \le 32$.

Proof For $n = 26, 27, 29$, and 31, we have Lemmas 3.8, 3.9, 3.10, and 3.11 to cover these cases, respectively.

For $n = 28, 32$ and $u \ne 8, 12, 16$, the designs are provided by Lemma 2.9 with $m = 7, 8$. For $n = 28, 32$ and $u = 8, 12, 16$, we can get them from Lemma 2.12(a) with $m = 7, 8$ and $k = 0$.

For $n = 30$ we apply Lemma 2.10 with $m = 6$, $1 \le s \le 4$, and $t = u$. □

Lemma 4.3 *An HSD*($2^n u^1$) *exists for* $5 \le u \le 24$ *and* $33 \le n \le 40$.

Proof For $n = 33, 34$ and $u = 5, 6$, we obtain at first an HSD($2^{25} t^1$) by Lemma 2.10 with $m = 5$, $s = 4$, and $t = u + 16$, $u + 18$. Fill an HSD($2^8 u^1$) and an HSD($2^9 u^1$) into the holes of sizes $u + 16$ and $u + 18$, we obtain an HSD($2^{33} u^1$) and an HSD($2^{34} u^1$), respectively. For $n = 33, 34$ and $7 \le u \le 24$, an HSD($2^n u^1$) exists by Lemma 2.12(b) with $m = 7$ and $k = 5, 6$.

For $n = 35, 36, 40$, the designs come from Lemma 2.14(a) with $m = 7, k = 0, 1, 5$. For $n = 37$, the designs are provided by Lemma 3.12.

For $n = 38$ and $5 \le u \le 7$, we first apply Lemma 2.10 with $m = 6$, $s = 4$, and $t = u + 16$, to get an HSD($2^{30}(u + 16)^1$). Fill the hole of size $u + 16$ by an HSD($2^8 u^1$), we obtain an HSD($2^{38} u^1$). For $n = 38$ and $8 \le u \le 24$, we have an HSD($2^n u^1$) by Lemma 2.12(a) with $m = 8$ and $k = 6$.

Finally for $n = 39$, apply Lemma 2.14(b) with $m = 7$, $k = 4$. □

Lemma 4.4 *An HSD*($2^n u^1$) *exists for* $5 \le u \le 30$ *and* $41 \le n \le 49$.

Proof For $n = 41, 42, 44$ and $1 \leq u \leq 31$, an HSD($2^n u^1$) exists by Lemma 2.14(b) with $m = 7$ and $k = 6, 7$, and $m = 8$ and $k = 4$.

For $n = 43$ and $5 \leq u \leq 7$, there exists an HSD($2^{35}(u+16)^1$ by Lemma 2.10 with $m = 7, 0 \leq s \leq 6$, and $t = u+16$. Fill an HSD($2^8 u^1$) into the holes of size $u+16$, we obtain an HSD($2^{43} u^1$). For $n = 43$ and $u = 8$, because we have an HSD($2^8 7^1$), by Construction 2.5, we have an HSD($8^8 28^1$). Add two points to this design and fill an HSD($2^4 2^1$) into the holes of size 8 and an HSD($2^{11} 8^1$) into the hole of size 28, we obtain an HSD($2^{43} 8^1$). For $n = 43$ and $u = 9$, an HSD($2^n u^1$) exists by Lemma 2.12(a) with $m = 9$ and $k = 7$. For $n = 43$ and $10 \leq u \leq 30$, an HSD($2^n u^1$) exists by Lemma 2.12(b) with $m = 9$ and $k = 7$.

For $n = 45, 46, 47, 48$ and $1 \leq u \leq 35$, the designs come from Lemma 2.14(b) with $m = 8$ and $k = 5, 6, 7, 8$.

For $n = 49$ and $0 \leq u \leq 32$, because an HSD($2^7 t^1$) exists for $0 \leq t \leq 4$, we get an HSD($14^7 (7t)^1$) from Construction 2.5 with $m = 7$. Add k points to this design, $0 \leq k \leq 4$, and fill an HSD($2^7 k^1$) into the holes of size 14, we obtain an HSD($2^{49}(7t+k)^1$) for $0 \leq 7t + k \leq 32$. □

Lemma 4.5 *An HSD($2^n u^1$) exists for $5 \leq u \leq 39$ and $50 \leq n \leq 66$.*

Proof For $50 \leq n \leq 54$ and $5 \leq u \leq 39$, we apply Lemma 2.14(b) with $m = 9$ and $5 \leq k \leq 9$. For $n = 55, 56$ and $5 \leq u \leq 44$, we apply Lemma 2.14(a) with $m = 11$ and $k = 0, 1$.

For $n = 57, 58$ and $5 \leq u \leq 11$, by Lemma 2.14(a) with $m = 9$ and $k = 0, 1$, there exist HSD($2^s(u+24)^1$) for $s = 45, 46$ and $5 \leq u \leq 11$. Fill in the hole of size $u + 24$ with an HSD($2^{12} u^1$), we obtain an HSD($2^{s+12} u^1$) for $s = 45, 46$. For $n = 57, 58$ and $12 \leq u \leq 39$, an HSD($2^n u^1$) exists from Lemma 2.12(b) with $m = 12$ and $k = 9, 10$.

For $59 \leq n \leq 66$ and $5 \leq u \leq 47$, an HSD($2^n u^1$) exists by by Lemma 2.14(b) with $m = 11$, $4 \leq k \leq 11$. □

Lemma 4.6 *An HSD($2^n u^1$) exists for $5 \leq u \leq 51$ and $67 \leq n \leq 84$.*

Proof For $67 \leq n \leq 78$ and $5 \leq u \leq 51$, an HSD($2^n u^1$) exists by by Lemma 2.14(b) with $m = 12$ and $7 \leq k \leq 12$, and $m = 13$ and $8 \leq k \leq 13$.

For $n \in \{79, 82, 83\}$ and $5 \leq u \leq 15$, by Lemma 2.14(a) with $m = 11$ and $k = 8$, and $m = 12$ and $k = 6, 7$, there exist HSD($2^s(u+32)^1$) for $s = 63, 66, 67$. Fill in the hole of size $u + 32$ with HSD($2^{16} u^1$) in HSD($2^s(u+32)^1$), we obtain an HSD($2^n u^1$) for $n \in \{79, 82, 83\}$. For $n = 79$

and $16 \le u \le 67$, we apply Lemma 2.12(b) with $m = 16$ and $k = 15$. For $n = 82, 83$ and $u = 16$, we apply Lemma 2.12(c) with $m = 18$, $k = 8$ and $s = 2t$, where $t = 10, 11$, to obtain an HSD($36^4 16^1 s^1$). Fill the holes of sizes 36 and $2t$ with an HSD of types 2^{18}, 2^t, respectively, we obtain an HSD($2^{72+t} 16^1$) for $t = 10, 11$. For $n = 82, 83$ and $17 \le u \le 54$, we apply Lemma 2.12(b) with $m = 17$ and $k = 14, 15$.

For $80 \le n \le 81$ and $5 \le u \le 64$, we apply Lemma 2.14(a) with $m = 16$ and $k = 0, 1$. Finally for $n = 84$ and $5 \le u \le 67$, we apply Lemma 2.14(b) with $m = 16$, $k = 4$. □

As a summary of Theorem 1.7 and Lemmas 4.1–4.6, we have the following result.

Theorem 4.7 *An HSD($2^n u^1$) exists for $1 \le u \le 16$ and any $n \ge u + 1$, except possibly $(n, u) \in \{(7,5), (7,6), (11,9), (11,10)\}$.*

5 HSD($2^n u^1$) for general u

In [4], a general result for the existence of HSD($2^n u^1$) for any $u \ge 16$ was established with $n \ge \lceil 5u/4 \rceil + 14$. In this section, we will improve this result by increasing the range of n. At first, we present two lemmas that are needed later.

Lemma 5.1 *An HSD($2^n u^1$) exists for $5 \le u \le 20$ and $n \in \{21, 25, 29, 33, 37, 45, 59\}$.*

Proof The cases of $n = 21, 29, 37$ are covered by Lemmas 3.4, 3.10, and 3.12, respectively. The cases of $n = 33, 45, 59$ are covered by Lemmas 4.3, 4.4, and 4.5, respectively. For $n = 25$, we apply Lemma 2.10 with $m = 5$ and $1 \le s \le 4$ because HSDs of type $2^5 s^1$ exist for all $1 \le s \le 4$. □

Lemma 5.2 *An HSD($2^{79} u^1$) exists for $0 \le u \le 62$.*

Proof For $u \le 16$, the designs are provided by Theorem 4.7. For $17 \le u \le 62$, we obtain first an HSD($32^4 30^1 t^1$) from Lemma 2.12(c) with $m = 16, k = 15$ and $s = t$, where $17 \le t \le 48$. Adjoin k points to this HSD, where $0 \le k \le 14$, and fill the holes of sizes 32 and 30 with HSDs of types $2^{16} k^1$, $2^{15} k^1$, respectively, we obtain an HSD($2^{79} (t+k)^1$) for $17 \le t+k \le 62$. □

Lemma 5.3 *There exists an HSD($2^n u^1$) for $u \geq 17$ and $\lceil 5u/4 \rceil + 4 \leq n \leq 44$.*

Proof Since $u \geq 17$, it must be the case that $n \geq 26$ because $n \geq \lceil 5u/4 \rceil + 4$. For the same reason, we have $u \leq \lfloor 4n/5 \rfloor - 3$. Table 1 shows the existence of an HSD($2^n u^1$) for $26 \leq n \leq 44$ and $17 \leq u \leq \lfloor 4n/5 \rfloor - 3$. The required HSDs of type ($2^k t^1$) in Table 1 for the application of Lemmas 2.10, 2.13 and 2.15 are provided by Theorem 4.7.

n	u	Justification
26	17-18	Lemma 3.8
27	17-26	Lemma 3.9
28,29	17-21	Lemma 2.12(a) with $m = 7$ and $k = 0, 1$
30	17-21	Lemma 2.10 with $m = 6$, $s = 3, 4$, and $t = 17 - 21$
31	17-21	Lemma 3.11
32,33	17-24	Lemma 2.12(a) with $m = 8$, $k = 0, 1$
34	17-26	Lemma 2.13 with $m = 7$, $k = 6$, and $1 \leq t \leq 5$
35,36	17-28	Lemma 2.14(a) with $m = 7$, $k = 0, 1$
37,38	17-28	Lemma 2.13 with $m = 8$, $k = 5, 6$, and $1 \leq t \leq k - 1$
39-41	17-31	Lemma 2.15 with $m = 7$, $k = 4, 5, 6$, and $1 \leq t \leq k - 1$
42,44	17-32	Lemma 2.13 with $m = 9$, $k = 6, 8$, and $1 \leq t \leq k - 1$
43	17-31	Lemma 2.13 with $m = 9$, $k = 7$, and $1 \leq t \leq k - 3$

Table 1: Existence proof of HSD($2^n u^1$) for Lemma 5.3

Lemma 5.4 *An HSD($2^n u^1$) exists for $45 \leq n \leq 87$ and $17 \leq u \leq \lfloor 4n/5 \rfloor - 1$.*

Proof Let $\mu(n) = \lfloor 4n/5 \rfloor - 1$.

For $45 \leq n \leq 48$, $\mu(48) = 37$, we apply Lemma 2.15 with $m = 8$, $5 \leq k \leq 8$, and $1 \leq t \leq k - 3$ to obtain an HSD($2^{k+40} u^1$) where $1 \leq u \leq k + 29$ and $5 \leq k \leq 8$.

For $49 \leq n \leq 54$, $\mu(54) = 42$ and we use Lemma 2.15 with $m = 9$, $4 \leq k \leq 9$, and $1 \leq t \leq k - 3$ to obtain an HSD($2^{k+45} u^1$) where $1 \leq u \leq k + 33$ and $4 \leq k \leq 9$.

For $55 \leq n \leq 56$, $\mu(56) = 43$ and we apply Lemma 2.14(a) with $m = 11$ and $k = 0, 1$ to obtain an HSD($2^n u^1$) where $0 \leq u \leq 44$.

For $n = 57, 58$, $\mu(58) = 45$ and we apply Lemma 2.13 with $m = 12$, $k = 9, 10$, and $0 \leq t \leq k - 1$, to obtain an HSD($2^{48+k} u^1$), where $k = 9, 10$ and $12 \leq u \leq 35 + k$.

For $59 \le n \le 66$, $\mu(66) = 51$ and an HSD($2^n u^1$) exists by Lemma 2.15 with $m = 11$, $4 \le k \le 11$ and $1 \le t \le k-3$. to obtain an HSD($2^{k+55}u^1$) where $4 \le k \le 11$ and $1 \le u \le k+41$.

For $67 \le n \le 72$, $\mu(72) = 56$ and an HSD($2^n u^1$) exists by Lemma 2.15 with $m = 12$, $7 \le k \le 12$ and $1 \le t \le k-3$, to obtain an HSD($2^{k+60}u^1$) where $7 \le k \le 12$ and $1 \le u \le k+45$.

For $73 \le n \le 78$, $\mu(78) = 61$ and an HSD($2^n u^1$) exists by Lemma 2.14(b) with $m = 13$, $8 \le k \le 13$ and $1 \le t \le k-3$, to obtain an HSD($2^{k+65}u^1$) where $8 \le k \le 13$ and $1 \le u \le k+49$.

For $n = 79$, the case is covered by Lemma 5.2. For $n = 80, 81$, $\mu(81) = 63$, we apply Lemma 2.14(a) with $m = 16$ and $k = 0, 1$, to obtain an HSD($2^n u^1$) where $0 \le u \le 64$.

For $n = 82, 83$, $\mu(83) = 65$ and we apply Lemma 2.13 with $m = 17, k = 14, 15$, $1 \le t \le k-1$, to obtain an HSD($2^{68+k}u^1$) where $k = 14, 15$ and $17 \le u \le k+50$.

For $84 \le n \le 87$, $\mu(87) = 68$ and we apply Lemma 2.15 with $m = 16$, $4 \le k \le 7$, $1 \le t \le k-3$, to obtain an HSD($2^{k+80}u^1$) where $4 \le k \le 7$, $1 \le u \le k+61$.

In the above proof, the required HSDs of type $2^k t^1$, where $1 \le t \le k-3$, for the applications of Lemmas 2.13 and 2.15 come from Theorem 4.7. The reason for letting $t \le k-3$ instead of $t \le k-1$ because we do not know the existence of HSD($2^n u^1$) for $(n, u) = (7,5), (7,6), (11,9), (11,10)$. □

Lemma 5.5 *An HSD($2^n u^1$) exists for* $88 \le n \le 116$ *and* $17 \le u \le \lfloor 4n/5 \rfloor$.

Proof Let $\mu(n) = \lfloor 4n/5 \rfloor$.

For $88 \le n \le 92$, $\mu(92) = 73$, let $n = 5s + k$, where $s = 16$ and $8 \le k \le 12$. Because an TD(7, 16) exists, using Lemma 2.15 with $m = 16$, $8 \le k \le 12$, $1 \le t \le k-3$, we obtain an HSD($2^n u^1$) for $1 \le u \le k+61$ and $8 \le k \le 12$.

For $93 \le n \le 102$, $\mu(102) = 81$, let $n = 5s + k$, where $s = 17$ and $8 \le k \le 17$. using Lemma 2.15 with $m = 17$, $8 \le k \le 17$, $1 \le t \le k-3$, we obtain an HSD($2^n u^1$) for $1 \le u \le k+65$ and $8 \le k \le 17$.

For $103 \le n \le 114$, $\mu(114) = 91$, let $n = 5s + k$, where $s = 19$ and $8 \le k \le 19$. using Lemma 2.15 with $m = 19$, $8 \le k \le 19$, $1 \le t \le k-3$, we obtain an HSD($2^n u^1$) for $1 \le u \le k+73$ and $8 \le k \le 19$.

For $115 \le n \le 116$, $\mu(116) = 92$, we apply Lemma 2.14(a) with $m = 23$ and $k = 0, 1$ to obtain an HSD($2^n u^1$) for $1 \le u \le 92$.

The required HSDs of type $2^k t^1$ for the application of Lemma 2.15 in the above proof come from Theorem 4.7. □

Lemma 5.6 *An HSD($2^n u^1$) exists for $n \geq 117$ and $1 \leq u \leq \lfloor 4n/5 \rfloor + 2$.*

Proof Let $n = 5s + j$, where $12 \leq j \leq 16$. Since $n \geq 117$, we have $s \geq 21$. If there exists a TD$(7, s)$, then using Lemma 2.15 with $m = s$, $k = j$, $1 \leq t \leq k-1$, we obtain an HSD($2^n u^1$) for $1 \leq u \leq 4s + k - 1$. Because $k = j = n - 5s$ and $s \geq (n-16)/5$, we have $u \leq \lfloor 4n/5 \rfloor + 2$. The required HSDs of type $2^k t^1$ in the above proof come from Theorem 4.7.

For those $s \in M_7 = \{22, 26, 30, 34, 38, 46, 60\}$, we do not have a TD$(7, s)$. However, since a TD$(7, s-1)$ exists, we may use Lemma 2.15 with $m = s-1, 17 \leq k \leq 21, 1 \leq t \leq k-1$, where $k = j+5$, to obtain an HSD($2^n u^1$) for $1 \leq u \leq 4(s-1)+k-1 = 4s+j$. Because $j = n-5s$ and $s \geq (n-21)/5$, $u \leq \lfloor 4n/5 \rfloor + 3$. The required HSDs of type $2^k t^1$ in the above proof come from Lemma 5.1. □

Theorem 5.7 *There exists an HSD($2^n u^1$) for $u \geq 17$ and $n \geq \lceil 5u/4 \rceil + 4$.*

Proof Since $u \geq 17$, it must be the case that $n \geq 26$ because $n \geq \lceil 5u/4 \rceil + 4$. For $26 \leq n \leq 44$, the theorem holds by Lemma 5.3. For the case of $n \geq 45$, the theorem holds by Lemmas 5.4, 5.5, and 5.6, because $u \leq \lfloor 4n/5 \rfloor - 3$ implies $n \geq \lceil 5u/4 \rceil + 4$. □

Actually, Lemmas 4.6, 5.5 and 5.6 can be used to prove a stronger result when $u \geq 35$.

Theorem 5.8 *There exists an HSD($2^n u^1$) if $u \geq 35$ and $n \geq \lceil 5u/4 \rceil + 1$ or $u \geq 95$ and $n \geq \lceil 5u/4 \rceil - 2$.*

Proof When $u \geq 35$, $n \geq \lceil 5u/4 \rceil + 1 \geq 45$, so Lemmas 4.6, 5.5, and 5.6 apply. From $u \leq \lfloor 4n/5 \rfloor - 1$ we obtain $n \geq \lceil 5u/4 \rceil + 1$. When $u \geq 95$, $n \geq \lceil 5u/4 \rceil - 2 \geq 117$, so Lemma 5.6 applies. From $u \leq \lfloor 4n/5 \rfloor + 2$ we obtain $n \geq \lceil 5u/4 \rceil - 2$. □

6 Conclusions

We have investigated the existence of HSD($2^n u^1$) for $5 \leq u \leq 16$. We also improved the general result for $u \geq 17$ by decreasing the lower bound of n from $\lceil 5u/4 \rceil + 14$ to $\lceil 5u/4 \rceil + 4$. Most recursive constructions used in this paper are standard in combinatorial designs and many of the direct constructions of HSDs in this paper are carried out by computer. The main result of this paper can be summarized in the following theorem:

Theorem 6.1 *(a) For* $1 \le u \le 16$, *an HSD*($2^n u^1$) *exists if and only if* $n \ge u+1$ *with the exception of* $(n,u) \in \{(2,1),(3,1),(3,2)\}$, *and with the possible exception of* $(n,u) \in \{(7,5),(7,6),(11,9),(11,10)\}$.

(b) For $u \ge 17$, *an HSD*($2^n u^1$) *exists if* $n \ge \lceil 5u/4 \rceil + 4$. *Moreover, if* $u \ge 35$, *then there exists an HSD*($2^n u^1$) *for* $n \ge \lceil 5u/4 \rceil + 1$; *if* $u \ge 95$, *then there exists an HSD*($2^n u^1$) *for* $n \ge \lceil 5u/4 \rceil - 2$.

Proof (*a*) is a combination of Theorems 1.7 and 4.7. (*b*) is a combination of Theorems 5.7 and 5.8. □

Besides the four possible exceptions in the part *a*) of the above theorem, from the part (*b*) of the above theorem, it is clear that the existence problem of HSD($2^n u^1$) remains open for $17 \le u \le 34$ and $u+1 \le n \le \lceil 5u/4 \rceil + 3$, or $35 \le u \le 94$ and $u+1 \le n \le \lceil 5u/4 \rceil$, or $u \ge 95$ and $u+1 \le n \le \lceil 5u/4 \rceil - 3$.

Acknowledgments

The first author would like to acknowledge the support of National Science Foundation under Grant CCR-0541070. The authors sincerely thank an anonymous referee whose comments helped to improve this paper substantially.

References

[1] R.J. Abel, A.E. Brouwer, C.J. Colbourn and J.H. Dinitz, *Mutually orthogonal Latin squares (MOLS)* , in: The CRC Handbook of Combinatorial Designs (2nd ed., C.J. Colbourn and J.H. Dinitz eds), CRC Press, Inc., 2006

[2] R.D. Baker, *Quasigroups and tactical systems*, Aequationes Math. **18**(1978), 296-303.

[3] F.E. Bennett and R. Wei, *The existence of Schröder designs with equal-sized holes*, Discrete Math. **170**(1997), 15-28.

[4] F.E. Bennett, R. Wei, and H. Zhang *Holey Schröder designs of type* $2^n u^1$, J. Combin. Designs **6**(1998), 131-150.

[5] F.E. Bennett, R. Wei and L. Zhu, *Incomplete idempotent Schröder quasigroups and related packing designs*, Aequationes Math. **51**(1996), 100-114.

[6] P.K. Brayton, D. Coppersmith and A.J. Hoffman, *Self - orthogonal Latin squares of all order* $n \neq 2, 3, 6$, Bull. Amer. Math Soc. Vol 80, (1), 1974, 116-118

[7] A.E. Brouwer, A. Schrijver and H. Hanani, *Group divisible designs with block size 4*, Discrete Math., **20**(1977), 1-10.

[8] C.J. Colbourn and D.R. Stinson, *Edge-colored designs with block size four,* Aequationes Math., **36**(1988), 230-245.

[9] G. Ge, A.C.H. Ling, *Group divisible designs with block size four and group type* $g^u m^1$ *for small* g, Discrete Math. **285** (2004) 97-120

[10] G. Ge, *Group divisible designs*, in: The CRC Handbook of Combinatorial Designs (2nd ed., C.J. Colbourn and J.H. Dinitz eds), CRC Press, Inc., 2006

[11] H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math., **11**(1975), 255-369.

[12] C.C. Lindner, N.S. Mendelsohn and S.R. Sun, *On the construction of Schröder quasigroups*, Discrete Math., **32**(1980), 271-280.

[13] N.S. Mendelsohn, *Self-orthogonal Weisner designs*, Ann. NY Acad. Sci., **319**(1979), 391-398.

[14] R.M. Wilson, *Constructions and uses of pairwise balanced designs*, Math. Centre Tracts, **55**(1974), 18-41.

[15] Xu Yunqing, H. Zhang, and Zhu Lie, *Existence of frame SOLS of type* $a^n b^1$, Discrete Math. **250**(2002), 211-230.

[16] H. Zhang, *Specifying Latin squares in propositional logic*, in R. Veroff (ed.): Automated Reasoning and Its Applications, Essays in honor of Larry Wos, MIT Press, 1997.

[17] H. Zhang, *Combinatorial designs by SAT solvers*, (eds. Toby Walsh) Handbook of Satisfiability. IOS Press, 2008.

Appendix

An FSQ(6^4)

*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0		20	22		18	7		17	13		11	21		23	10		19	4	1	2	8	16	14	5
1	11		0	18		21	17		15	22		20	5		6	19		23	12	14	9	3	2	8
2	1	18		4	23		19	0		7	20		22	12		21	9		3	13	6	10	15	16
3		2	20		14	18		21	19		22	13		17	23		11	1	8	4	10	5	16	7
4	14		9	23		15	21		3	5		18	20		22	11		19	17	0	2	8	12	6
5	22	3		10	20		13	23		21	6		19	18		7	0		15	16	12	4	9	1
6		19	7		8	22		18	21		2	4		20	16		23	10	13	5	14	1	11	17
7	20		19	17		3	11		18	8		22	21		15	23		9	2	6	5	14	0	12
8	18	22		16	19		4	3		1	12		23	0		20	21		10	15	13	6	7	9
9		5	18		21	16		22	23		8	19		14	1		17	20	11	10	7	2	13	4
10	21		3	14		19	20		22	23		9	8		18	2		15	5	12	17	11	6	0
11	10	12		20	9		23	6		13	19		18	21		16	22		7	3	1	0	4	15
12		23	4		5	10		14	7		18	1		19	21		20	22	16	17	11	13	8	2
13	17		23	22		6	8		20	19		12	2		9	18		21	0	11	3	15	5	14
14	23	21		13	0		18	19		20	15		7	22		1	6		4	9	16	12	10	3
15		8	13		22	20		11	16		21	23		5	19		2	18	14	7	4	17	1	10
16	5		21	19		23	2		0	18		3	14		20	22		12	6	8	15	9	17	11
17	19	6		21	12		22	20		10	23		16	15		4	18		9	1	0	7	3	13
18	2	11	1	7	15	13	5	9	12	4	3	6	17	8	0	10	14	16						
19	7	15	12	8	17	0	10	2	1	11	5	16	4	9	13	14	3	6						
20	13	14	10	2	3	4	16	15	6	17	9	0	1	11	12	5	8	7						
21	8	17	15	1	6	9	14	12	4	16	0	10	11	2	7	13	5	3						
22	4	0	16	11	2	1	7	5	9	14	17	15	10	6	3	8	12	13						
23	16	9	6	5	11	12	1	8	10	2	14	7	13	3	4	17	15	0						

In the following we list some HSDs which are used in the previous sections. Most of them are obtained by computer. In the following list, the point set of an HSD($2^n u^1$) consists of Z_{2n} and u infinite points which are denoted by alphabet. For simplicity, we only list the starter blocks or the corresponding Latin square. We also use the $+k$ method to develop blocks, which means that we add 2 or more (mod $2n$) to each point of the starter blocks to obtain all blocks.

A1 HSD($2^n 5^1$) for $n \leq 23$

$n = 10$ ($+1$ *mod* 20) :
$(0,1,2,x_5), (0,2,4,x_4), (0,4,19,11), (0,5,9,x_1), (0,6,3,14), (0,13,6,x_2), (0,17,5,x_3)$

$n = 11$ ($+2$ *mod* 22) :
$(0,2,3,17), (0,4,1,16), (0,5,18,13), (0,6,16,x_5), (0,8,7,x_2), (0,9,2,15), (0,12,8,x_1),$
$(0,15,9,5), (0,19,5,x_4), (0,21,19,x_3), (1,0,2,x_3), (1,3,21,x_5), (1,13,10,x_2),$
$(1,17,5,x_1), (1,20,6,x_4)$

$n = 13$ $(+2 \; mod \; 26)$:
$(0,1,21,2), (0,2,18,14), (0,3,19,23), (0,5,17,19), (0,6,9,21), (0,8,2,x_2),$
$(0,9,7,x_1), (0,10,1,4), (0,11,22,x_5), (0,12,11,x_3), (0,15,23,7), (0,25,16,x_4),$
$(1,6,24,x_1), (1,7,22,x_3), (1,10,11,x_4), (1,19,23,x_2), (1,20,25,x_5)$

$n = 14$ $(+1 \; mod \; 28)$:
$(0,1,2,x_5), (0,2,4,x_4), (0,4,7,23), (0,5,16,22), (0,8,27,18), (0,10,22,7),$
$(0,11,15,x_2), (0,21,8,x_1), (0,25,5,x_3)$

$n = 17$ $(+2 \; mod \; 34)$:
$(0,1,10,23), (0,2,9,x_1), (0,3,27,12), (0,7,29,20), (0,8,32,31), (0,10,8,30),$
$(0,11,18,3), (0,14,30,x_4), (0,16,22,19), (0,21,19,5), (0,23,28,21), (0,25,11,15),$
$(0,28,7,x_5), (0,29,1,x_2), (0,30,31,x_3), (1,3,21,9), (1,7,30,x_3), (1,11,2,x_5),$
$(1,19,20,x_1), (1,27,31,x_4), (1,30,22,x_2)$

$n = 19$ $(+2 \; mod \; 38)$:
$(0,1,32,12), (0,2,12,37), (0,3,30,36), (0,4,25,3), (0,5,18,27), (0,7,23,6),$
$(0,8,13,x_5), (0,10,14,x_3), (0,11,34,18), (0,13,11,9), (0,14,22,17), (0,15,7,35),$
$(0,21,28,x_1), (0,26,2,x_4), (0,27,33,1), (0,29,17,x_2), (0,35,21,29), (0,37,9,23),$
$(1,5,10,x_5), (1,8,24,x_2), (1,13,9,x_3), (1,16,29,x_1), (1,19,37,x_4)$

$n = 22$ $(+1 \; mod \; 44)$:
$(0,1,2,x_5), (0,2,4,x_4), (0,4,27,32), (0,6,21,39), (0,7,24,37), (0,8,26,16), (0,9,12,x_2),$
$(0,11,15,x_1), (0,12,37,9), (0,14,34,13), (0,15,25,6), (0,17,6,30), (0,41,5,x_3)$

$n = 23$ $(+2 \; mod \; 46)$:
$(0,1,11,9), (0,2,1,35), (0,5,8,39), (0,6,43,13), (0,7,29,11), (0,8,36,7),$
$(0,9,4,25), (0,10,37,29), (0,11,9,x_3), (0,12,15,1), (0,13,39,x_1), (0,14,6,31),$
$(0,15,20,x_4), (0,16,3,14), (0,18,30,4), (0,19,27,40), (0,22,24,17), (0,37,18,15),$
$(0,42,12,x_2), (0,45,31,x_5), (1,5,33,7), (1,6,26,x_1), (1,7,11,35), (1,11,27,x_2),$
$(1,18,42,x_3), (1,20,41,x_4), (1,44,34,x_5)$

A2 HSD($2^n 6^1$) for $n \leq 29$

$n = 8$ $(+2 \; mod \; 16)$:
$(0,1,3,x_6), (0,2,5,x_5), (0,4,10,x_1), (0,5,9,x_4), (0,6,7,x_2), (0,7,2,13), (0,13,4,x_3),$
$(1,2,4,x_6), (1,5,6,x_5), (1,8,15,x_3), (1,11,0,x_2), (1,14,2,x_4), (1,15,5,x_1)$

$n = 9$ $(+1 \; mod \; 18)$:
$(0,1,2,x_6), (0,2,4,x_5), (0,4,8,x_1), (0,6,3,13), (0,7,1,x_2), (0,13,6,x_3), (0,15,5,x_4)$

$n = 10$ $(+2 \; mod \; 20)$:
$(0,1,8,3), (0,2,19,8), (0,3,2,x_1), (0,5,17,x_3), (0,7,9,15), (0,8,7,x_6), (0,14,16,x_4),$
$(0,16,1,x_5), (0,19,3,x_2), (1,3,16,x_5), (1,4,10,x_3), (1,8,17,x_1), (1,9,15,x_4),$
$(1,12,8,x_2), (1,17,6,x_6)$

$n = 11$ $(+1 \; mod \; 22)$:
$(0,1,2,x_6), (0,2,4,x_5), (0,5,10,x_1), (0,6,3,13), (0,7,1,15), (0,9,13,x_2), (0,18,6,x_3),$
$(0,19,5,x_4)$

$n = 12$ $(+2 \; mod \; 24)$:
$(0,1,5,11), (0,2,10,x_2), (0,3,20,6), (0,4,17,10), (0,5,15,x_6), (0,8,11,x_3),$
$(0,9,1,x_5), (0,11,9,5), (0,13,16,15), (0,18,3,x_1), (0,19,2,x_4), (1,3,2,x_1),$
$(1,4,6,x_5), (1,8,12,x_6), (1,10,5,x_4), (1,11,17,x_2), (1,17,18,x_3)$

$n = 13$ $(+1 \; mod \; 26)$:
$(0,1,2,x_6), (0,2,4,x_5), (0,6,15,22), (0,8,23,7), (0,9,14,x_2), (0,11,17,3),$
$(0,21,7,x_1), (0,22,6,x_3), (0,23,5,x_4)$

$n = 14$ $(+2 \; mod \; 28)$:
$(0,1,5,18), (0,3,22,21), (0,4,7,27), (0,5,25,x_5), (0,6,17,x_1), (0,7,16,4),$
$(0,9,8,x_4), (0,10,15,20), (0,19,2,23), (0,20,27,x_6), (0,25,9,x_3), (0,26,24,x_2),$
$(1,3,16,x_6), (1,5,8,x_1), (1,7,9,19), (1,12,0,x_3), (1,13,7,x_2), (1,16,10,x_5),$
$(1,18,3,x_4)$

$n = 15$ $(+1 \; mod \; 30)$:
$(0,1,2,x_6), (0,2,4,x_5), (0,4,8,x_2), (0,5,19,25), (0,7,16,24), (0,9,12,x_3),$
$(0,10,3,17), (0,12,1,18), (0,19,9,x_1), (0,27,5,x_4)$

$n = 17$ $(+1 \; mod \; 34)$:
$(0,1,2,x_6), (0,2,4,x_5), (0,4,11,23), (0,5,25,12), (0,6,31,16), (0,8,13,x_2),$
$(0,9,12,x_3), (0,10,14,28), (0,11,26,10), (0,27,6,x_1), (0,31,5,x_4)$

$n = 18$ $(+2 \; mod \; 36)$:
$(0,1,6,33), (0,2,23,9), (0,3,33,29), (0,4,19,x_4), (0,7,27,8), (0,8,21,22),$
$(0,9,5,30), (0,12,2,23), (0,13,8,25), (0,14,17,2), (0,20,32,x_2), (0,23,13,x_3),$
$(0,26,10,x_5), (0,29,1,7), (0,30,31,x_1), (0,31,20,x_6), (0,33,9,17), (1,3,12,x_4),$
$(1,12,8,x_3), (1,17,34,x_1), (1,25,3,x_2), (1,27,29,x_5), (1,32,31,x_6)$

$n = 19$ $(+1 \; mod \; 38)$:
$(0,1,2,x_6), (0,2,4,x_5), (0,4,22,27), (0,6,31,15), (0,8,13,x_2), (0,9,12,x_3),$
$(0,10,32,9), (0,11,17,31), (0,12,8,26), (0,13,3,24), (0,31,10,x_1), (0,35,5,x_4)$

$n = 22$ $(+2 \; mod \; 44)$:
$(0,1,33,4), (0,2,32,23), (0,5,10,x_6), (0,6,4,34), (0,7,25,41), (0,8,2,28),$
$(0,9,6,1), (0,11,7,18), (0,12,3,43), (0,13,29,31), (0,15,5,x_5), (0,16,35,11),$
$(0,20,9,21), (0,21,23,5), (0,23,15,29), (0,25,19,27), (0,27,13,x_2), (0,31,14,7),$
$(0,34,27,x_3), (0,40,8,x_4), (0,43,18,x_1), (1,4,28,x_5), (1,7,31,x_4), (1,11,2,x_3),$
$(1,26,34,x_2), (1,28,41,x_6), (1,42,13,x_1)$

$n = 23$ $(+1 \; mod \; 46)$:
$(0,1,5,2), (0,2,3,x_3), (0,4,21,43), (0,6,34,21), (0,8,28,19), (0,10,2,14),$
$(0,11,26,41), (0,14,24,6), (0,16,27,x_4), (0,17,10,30), (0,19,6,x_6), (0,25,37,x_5),$
$(0,39,17,x_2), (0,41,32,x_1)$

$n = 29$ $(+1 \; mod \; 58)$:
$(0,2,57,6), (0,3,23,12), (0,4,17,40), (0,5,15,47), (0,6,21,1), (0,8,40,27),$
$(0,9,44,34), (0,12,8,25), (0,14,31,16), (0,16,46,19), (0,18,32,8), (0,19,24,x_3),$
$(0,25,47,x_6), (0,28,7,x_4), (0,36,38,x_5), (0,37,28,x_2), (0,57,6,x_1)$

A3 HSD($2^n 7^1$) for $n \leq 23$

$n = 8$ ($+1$ *mod* 16) :
$(0,1,2,x_7),(0,2,4,x_6),(0,3,6,x_4),(0,9,3,x_2),(0,10,15,x_1),(0,11,7,x_3),(0,12,5,x_5)$

$n = 9$ ($+2$ *mod* 18) :
$(0,5,1,2),(0,6,5,x_6),(0,10,11,x_4),(0,11,6,x_2),(0,13,15,x_5),(0,14,4,x_1),$
$(0,15,8,x_7),(0,16,2,x_3),(1,0,6,x_5),(1,7,17,x_3),(1,11,5,x_1),(1,12,7,x_7),$
$(1,15,12,x_4),(1,16,9,x_2),(1,17,2,x_6)$

$n = 10$ ($+1$ *mod* 20) :
$(0,1,2,x_7),(0,2,4,x_6),(0,6,1,14),(0,8,11,x_1),(0,9,13,x_3),(0,15,6,x_4),(0,16,3,x_2),$
$(0,17,5,x_5)$

$n = 11$ ($+2$ *mod* 22) :
$(0,3,1,x_5),(0,6,16,8),(0,7,14,x_2),(0,9,4,x_6),(0,10,12,x_1),(0,15,3,7),$
$(0,18,15,x_4),(0,19,5,13),(0,20,21,x_7),(0,21,17,x_3),(1,0,21,x_6),(1,3,19,x_1),$
$(1,6,10,x_5),(1,10,16,x_3),(1,13,4,x_4),(1,17,14,x_7),(1,18,13,x_2)$

$n = 12$ ($+1$ *mod* 24) :
$(0,5,15,8),(0,8,19,9),(0,13,7,x_2),(0,15,10,x_3),(0,18,11,x_1),(0,20,16,x_4),$
$(0,21,18,x_5),(0,22,20,x_6),(0,23,22,x_7)$

$n = 13$ ($+2$ *mod* 26) :
$(0,1,25,21),(0,2,12,11),(0,3,2,14),(0,4,21,x_4),(0,5,23,x_1),(0,6,4,x_2),(0,7,19,25),$
$(0,8,17,x_7),(0,11,18,x_6),(0,16,20,x_5),(0,17,10,5),(0,23,1,x_3),(1,8,11,x_6),$
$(1,11,22,x_4),(1,12,20,x_3),(1,13,16,x_7),(1,18,24,x_1),(1,19,3,x_2),(1,25,5,x_5)$

$n = 14$ ($+1$ *mod* 28) :
$(0,1,26,2),(0,2,21,24),(0,7,19,11),(0,12,25,x_5),(0,15,23,x_6),(0,17,10,x_1),$
$(0,18,12,x_2),(0,19,20,x_4),(0,22,17,x_7),(0,23,13,x_3)$

$n = 15$ ($+2$ *mod* 30) :
$(0,1,3,x_7),(0,2,5,x_6),(0,3,2,x_5),(0,5,14,x_1),(0,7,13,19),(0,8,19,26),$
$(0,9,23,13),(0,10,7,x_3),(0,11,24,12),(0,14,8,25),(0,19,1,10),(0,24,10,x_2),$
$(0,25,17,9),(0,26,4,x_4),(1,2,4,x_7),(1,3,10,x_3),(1,4,9,x_5),(1,5,6,x_6),$
$(1,17,21,x_4),(1,18,25,x_1),(1,19,29,x_2)$

$n = 17$ ($+2$ *mod* 34) :
$(0,2,1,29),(0,3,5,12),(0,5,31,13),(0,7,10,9),(0,8,28,10),(0,9,4,28),$
$(0,12,3,x_6),(0,13,20,x_7),(0,14,22,3),(0,21,33,x_5),(0,25,19,x_4),(0,28,29,x_2),$
$(0,29,18,x_3),(0,30,26,x_1),(0,31,21,7),(1,0,23,x_3),(1,3,21,31),(1,9,24,x_2),$
$(1,12,28,x_5),(1,13,26,x_6),(1,16,31,x_7),(1,24,22,x_4),(1,31,17,x_1)$

$n = 18$ ($+1$ *mod* 36) :
$(0,1,14,24),(0,2,7,x_7),(0,4,2,x_2),(0,5,6,x_1),(0,7,27,19),(0,11,3,15),$
$(0,13,16,32),(0,14,21,6),(0,19,10,x_6),(0,27,1,x_3),(0,30,5,x_4),(0,33,11,x_5)$

$n = 19$ ($+2$ *mod* 38) :
$(0,1,15,x_4),(0,2,17,32),(0,4,9,x_3),(0,5,26,x_2),(0,6,20,28),(0,7,24,13),(1,2,4,x_4),$
$(0,9,29,35),(0,10,6,20),(0,12,5,23),(0,16,3,1),(0,17,1,x_5),(0,20,4,x_7),$
$(0,21,33,37),(0,25,27,11),(0,29,2,x_6),(0,31,21,29),(0,33,25,x_1),(0,35,31,7),$
$(1,13,19,x_7),(1,24,32,x_5),(1,26,35,x_2),(1,28,2,x_1),(1,29,26,x_3),(1,36,31,x_6)$

$n = 22$ (+1 *mod* 44) :
$(0,1,21,3), (0,2,39,x_6), (0,3,4,19), (0,4,12,x_5), (0,5,36,12), (0,6,34,9), (0,7,1,x_2),$
$(0,9,11,34), (0,11,26,40), (0,12,38,21), (0,13,24,x_4), (0,16,30,x_1), (0,34,17,x_3),$
$(0,36,31,x_7)$

$n = 23$ (+2 *mod* 46) :
$(0,1,19,12), (0,2,29,33), (0,3,8,39), (0,5,34,13), (0,6,13,5), (0,9,17,44),$
$(0,10,21,1), (0,11,15,24), (0,12,43,21), (0,13,10,37), (0,14,20,9), (0,16,32,14),$
$(0,20,24,x_1), (0,21,1,x_3), (0,22,5,18), (0,38,27,x_4), (0,39,40,x_2), (0,41,25,x_7),$
$(0,42,16,x_6), (0,43,18,x_5), (1,2,40,x_7), (1,3,35,x_6), (1,7,41,23), (1,11,5,37),$
$(1,13,44,x_4), (1,17,15,x_1), (1,18,8,x_3), (1,30,3,x_5), (1,32,29,x_2)$

A4 HSD($2^n 8^1$) for $n \leq 23$

$n = 10$ (+4 *mod* 20) :
$(0,1,2,x_6), (0,3,16,x_5), (0,4,3,x_2), (0,8,14,x_3), (0,9,1,x_7), (0,11,7,x_4),$
$(0,13,15,16), (0,17,12,x_8), (0,18,13,x_1), (1,2,5,x_2), (1,3,14,10), (1,6,12,x_6),$
$(1,7,6,x_1), (1,13,16,x_3), (1,14,10,x_5), (1,15,18,x_8), (1,16,7,x_7), (1,17,13,x_4),$
$(2,0,1,x_8), (2,3,11,x_6), (2,5,3,x_5), (2,8,16,x_1), (2,13,6,x_2), (2,14,19,x_3),$
$(2,16,14,x_4), (2,19,4,x_7), (3,5,19,x_1), (3,7,4,x_2), (3,8,1,x_5), (3,10,18,x_7),$
$(3,14,12,x_4), (3,15,9,x_3), (3,16,5,x_6), (3,18,11,x_8)$

$n = 11$ (+1 *mod* 22) :
$(0,1,2,x_8), (0,2,4,x_7), (0,3,6,x_6), (0,6,1,14), (0,8,12,x_1), (0,10,3,x_3), (0,15,9,x_2),$
$(0,17,7,x_5), (0,18,5,x_4)$

$n = 13$ (+1 *mod* 26) :
$(0,1,2,x_8), (0,2,4,x_7), (0,3,6,x_6), (0,7,25,16), (0,8,15,5), (0,11,5,x_2), (0,12,16,x_3),$
$(0,20,8,x_4), (0,21,12,x_1), (0,22,7,x_5)$

$n = 14$ (+2 *mod* 28) :
$(0,2,5,17), (0,4,12,x_2), (0,6,27,x_7), (0,7,3,x_3), (0,8,20,10), (0,9,1,x_8),$
$(0,11,13,15), (0,13,19,x_6), (0,16,21,x_5), (0,17,6,27), (0,19,24,x_1), (0,25,18,x_4),$
$(0,27,17,25), (1,0,22,x_6), (1,5,17,x_2), (1,6,4,x_3), (1,7,6,x_5), (1,11,20,x_7),$
$(1,14,18,x_8), (1,24,9,x_4), (1,26,7,x_1)$

$n = 17$ (+1 *mod* 34) :
$(0,1,2,x_8), (0,2,4,x_7), (0,3,6,x_6), (0,7,31,15), (0,8,20,29), (0,10,1,21), (0,11,19,6),$
$(0,12,16,x_4), (0,15,22,x_1), (0,28,10,x_2), (0,29,9,x_3), (0,30,7,x_5),$

$n = 19$ (+1 *mod* 38) :
$(0,1,2,x_8), (0,2,4,x_7), (0,3,6,x_6), (0,6,11,x_2), (0,8,17,32), (0,9,22,12), (0,11,1,23),$
$(0,12,30,17), (0,14,18,x_4), (0,17,10,30), (0,31,15,x_1), (0,33,9,x_3), (0,34,7,x_5)$

$n = 22$ (+2 *mod* 44) :
$(0,1,42,6), (0,2,29,x_8), (0,3,9,33), (0,4,21,30), (0,11,10,31), (0,12,31,36),$
$(0,13,17,x_2), (0,14,13,39), (0,15,7,9), (0,16,28,10), (0,17,1,26), (0,23,11,24),$
$(0,24,37,x_6), (0,27,3,11), (0,29,32,25), (0,33,40,x_1), (0,34,41,x_7), (0,38,36,x_3),$

$(0,41,23,x_4),(0,43,14,x_5),(1,7,30,x_6),(1,11,13,x_3),(1,13,3,31),(1,20,11,x_1),$
$(1,31,26,x_8),(1,36,40,x_2),(1,38,9,x_5),(1,40,12,x_4),(1,41,18,x_7)$

$n=23$ ($+1$ *mod* 46) :
$(0,1,2,x_8),(0,2,4,x_7),(0,3,6,x_6),(0,5,25,32),(0,6,33,9),(0,8,24,39),(0,9,37,21),$
$(0,10,20,x_1),(0,11,17,x_3),(0,12,16,x_4),(0,13,18,x_2),(0,14,38,13),(0,17,5,31),$
$(0,18,35,8),(0,42,7,x_5)$

A5 HSD($2^n 9^1$) for $n \leq 19$

$n=10$ ($+4$ *mod* 20) :
$(0,2,3,x_6),(0,3,7,x_5),(0,4,11,x_2),(0,5,18,x_7),(0,8,4,x_4),(0,9,6,x_1),$
$(0,13,8,x_8),(0,14,5,x_3),(0,15,1,x_9),(1,0,9,x_5),(1,3,12,x_1),(1,4,10,x_6),$
$(1,6,4,x_9),(1,7,8,x_3),(1,10,19,x_7),(1,13,5,x_4),(1,17,2,x_2),(1,19,7,x_8),$
$(2,1,5,x_6),(2,4,19,x_9),(2,5,7,x_3),(2,9,8,x_5),(2,14,0,x_2),(2,15,17,x_7),$
$(2,16,9,x_1),(2,18,10,x_8),(2,19,6,x_4),(3,2,7,x_4),(3,4,2,x_3),(3,7,1,x_2),$
$(3,9,10,x_9),(3,11,8,x_6),(3,12,9,x_8),(3,14,11,x_1),(3,16,4,x_7),(3,18,14,x_5),$

$n=13$ ($+2$ *mod* 26) :
$(0,1,6,x_2),(0,2,17,5),(0,3,23,24),(0,7,21,x_7),(0,8,22,x_3),(0,10,11,17),$
$(0,11,9,x_1),(0,12,16,x_4),(0,20,14,x_9),(0,21,18,x_8),(0,22,15,x_5),(0,23,24,x_6),$
$(1,3,7,x_4),(1,8,0,x_7),(1,10,17,x_2),(1,11,2,x_5),(1,12,23,x_6),(1,18,8,x_1),$
$(1,19,3,x_3),(1,22,19,x_8),(1,23,15,x_9)$

$n=14$ ($+1$ *mod* 28) :
$(0,1,2,x_9),(0,2,4,x_8),(0,3,6,x_7),(0,6,13,x_1),(0,7,23,6),(0,8,3,18),(0,12,16,x_4),$
$(0,18,9,x_3),(0,19,11,x_2),(0,23,8,x_5),(0,24,7,x_6)$

$n=15$ ($+2$ *mod* 30) :
$(0,2,5,12),(0,4,2,27),(0,5,14,x_4),(0,6,1,x_9),(0,8,21,x_6),(0,9,8,x_8),$
$(0,10,29,17),(0,12,26,x_2),(0,14,24,x_7),(0,17,12,9),(0,19,13,x_3),(0,21,9,x_1),$
$(0,29,25,x_5),(1,0,29,x_8),(1,3,5,x_7),(1,5,25,17),(1,7,15,x_2),(1,15,4,x_9),$
$(1,18,12,x_3),(1,20,28,x_1),(1,21,14,x_6),(1,24,20,x_5),(1,28,21,x_4)$

$n=19$ ($+2$ *mod* 38) :
$(0,3,7,10),(0,4,30,8),(0,5,12,x_6),(0,7,22,x_3),(0,8,24,x_7),(0,9,2,29),$
$(0,10,27,33),(0,11,10,x_5),(0,13,5,14),(0,15,17,18),(0,17,29,15),(0,20,25,x_1),$
$(0,23,33,2),(0,24,20,x_4),(0,26,32,x_9),(0,32,3,x_8),(0,33,9,11),(0,36,11,x_2),$
$(1,0,27,x_6),(1,9,31,x_4),(1,14,11,x_3),(1,18,5,x_5),(1,21,0,x_1),(1,23,24,x_2),$
$(1,27,33,x_9),(1,29,26,x_8),(1,35,15,x_7)$

A6 HSD($2^n 10^1$) for $n \leq 19$

$n=13$ ($+1$ *mod* 26) :
$(0,1,18,x_2),(0,2,4,22),(0,3,2,x_9),(0,4,1,x_8),(0,7,23,x_1),(0,14,7,x_4),(0,15,9,x_5),$
$(0,16,11,y_0),(0,17,5,x_6),(0,20,12,x_3),(0,21,10,x_7)$

$n = 14$ $(+2 \bmod 28)$:
$(0,3,1,x_8),(0,8,12,x_4),(0,9,22,x_1),(0,10,2,18),(0,15,25,9),(0,19,3,11),$
$(0,21,4,x_5),(0,22,15,x_3),(0,23,18,x_9),(0,24,21,x_7),(0,25,17,x_2),(0,26,27,y_0),$
$(0,27,23,x_6),(1,0,27,x_9),(1,3,25,x_4),(1,12,24,x_6),(1,16,22,x_2),(1,18,20,x_8),$
$(1,19,16,x_3),(1,22,13,x_1),(1,23,10,x_7),(1,24,19,x_5),(1,25,18,y_0)$

$n = 19$ $(+1 \bmod 38)$:
$(0,1,7,x_3),(0,3,32,12),(0,5,4,x_6),(0,6,14,7),(0,9,12,x_2),(0,10,20,5),$
$(0,12,8,22),(0,16,27,y_0),(0,17,35,x_5),(0,25,2,x_1),(0,27,25,x_8),(0,30,16,x_4),$
$(0,34,17,x_7),(0,36,23,x_9)$

A7 HSD(2^n12^1) for $n \leq 17$

$n = 13$ $(+1 \bmod 26)$:
$(0,1,2,y_2),(0,2,4,y_1),(0,3,6,y_0),(0,4,8,x_9),(0,5,10,x_8),(0,14,21,x_1),$
$(0,15,7,x_4),(0,16,1,x_3),(0,17,3,x_2),(0,18,12,x_6),(0,19,9,x_5),(0,20,11,x_7)$

$n = 14$ $(+4 \bmod 28)$:
$(0,1,8,x_4),(0,2,27,x_8),(0,3,19,2),(0,6,16,x_9),(0,12,11,x_3),(0,13,24,y_2),$
$(0,15,9,x_2),(0,18,3,x_7),(0,19,23,x_6),(0,20,5,x_5),(0,22,25,1),(0,23,18,y_0),$
$(0,24,6,y_1),(0,26,13,x_1),(1,2,14,x_2),(1,4,8,x_8),(1,7,5,x_7),(1,8,3,x_4),$
$(1,12,23,x_1),(1,17,13,x_9),(1,18,9,y_1),(1,20,12,y_0),(1,21,19,x_5),(1,22,0,x_6),$
$(1,23,2,x_3),(1,24,27,y_2),(2,5,13,x_3),(2,6,26,x_5),(2,7,0,y_1),(2,10,5,y_0),$
$(2,11,21,x_4),(2,15,14,x_8),(2,17,8,x_1),(2,18,25,y_2),(2,20,4,x_7),(2,21,22,x_6),$
$(2,25,20,x_2),(2,27,7,x_9),(3,1,11,y_0),(3,2,24,x_3),(3,4,19,x_2),(3,5,27,y_1),$
$(3,7,16,x_5),(3,10,14,x_4),(3,11,22,y_2),(3,13,25,x_8),(3,15,6,x_1),(3,20,18,x_9),$
$(3,21,10,x_7),(3,24,21,x_6)$

$n = 17$ $(+1 \bmod 34)$:
$(0,11,31,18),(0,14,29,13),(0,19,9,x_3),(0,22,11,x_2),(0,24,15,x_4),(0,25,13,x_1),$
$(0,26,18,x_5),(0,27,20,x_6),(0,28,22,x_7),(0,29,24,x_8),(0,30,26,x_9),(0,31,28,y_0),$
$(0,32,30,y_1),(0,33,32,y_2)$

A8 Miscellaneous HSD(2^nu^1)

$n = 14, u = 13$ $(+4 \bmod 28)$:
$(0,1,24,x_5),(0,2,20,y_2),(0,4,21,y_0),(0,6,5,y_1),(0,8,10,x_1),(0,9,15,y_3),$
$(0,13,9,x_7),(0,15,16,x_6),(0,16,3,x_4),(0,17,25,x_2),(0,23,13,x_3),(0,26,11,x_8),$
$(0,27,23,x_9),(1,2,3,x_5),(1,4,26,x_7),(1,5,27,y_1),(1,8,9,x_8),(1,13,6,x_4),$
$(1,14,18,x_9),(1,18,2,y_0),(1,21,12,x_3),(1,22,5,x_6),(1,23,11,x_1),(1,24,0,x_2),$
$(1,26,17,y_3),(1,27,7,y_2),(2,3,22,y_1),(2,6,14,x_3),(2,8,21,y_2),(2,11,8,x_7),$
$(2,12,0,x_9),(2,18,11,x_2),(2,19,17,x_5),(2,20,26,x_6),(2,21,24,x_8),(2,22,9,x_1),$
$(2,23,12,x_4),(2,25,15,y_0),(2,27,4,y_3),(3,0,7,x_3),(3,1,26,y_2),(3,11,6,x_2),$
$(3,12,10,x_5),(3,13,11,x_6),(3,15,24,y_0),(3,18,13,x_4),(3,20,12,y_1),(3,21,9,x_9),$
$(3,24,14,y_3),(3,25,4,x_1),(3,26,15,x_7),(3,27,2,x_8)$

$n = 17, u = 13$ $(+2 \ mod \ 34)$:
$(0,1,14,y_2),(0,2,4,28),(0,3,28,x_3),(0,4,5,27),(0,5,7,y_0),(0,6,19,x_9),$
$(0,9,18,x_1),(0,12,22,x_8),(0,13,29,x_7),(0,15,25,y_1),(0,18,21,x_6),(0,20,2,x_5),$
$(0,25,10,y_3),(0,26,11,x_2),(0,33,3,x_4),(1,4,11,y_2),(1,6,26,x_4),(1,8,7,x_3),$
$(1,9,4,x_9),(1,11,31,13),(1,12,9,y_3),(1,14,2,y_0),(1,15,21,x_8),(1,16,8,x_7),$
$(1,24,20,y_1),(1,28,17,x_1),(1,29,0,x_6),(1,31,23,x_5),(1,33,22,x_2)$

$n = 17, u = 15$ $(+2 \ mod \ 34)$:
$(0,2,32,y_3),(0,4,3,x_6),(0,7,30,y_1),(0,8,20,y_2),(0,9,18,y_4),(0,10,28,x_7),$
$(0,11,10,y_5),(0,13,1,x_4),(0,14,8,x_9),(0,15,22,y_0),(0,16,13,5),(0,21,5,x_5),$
$(0,22,7,x_1),(0,27,31,x_8),(0,28,15,x_2),(0,29,27,x_3),(1,0,15,y_0),(1,2,9,y_1),$
$(1,3,31,y_3),(1,4,2,x_3),(1,7,17,x_7),(1,10,33,y_5),(1,11,25,y_2),(1,12,7,y_4),$
$(1,15,6,x_1),(1,16,26,x_8),(1,17,14,x_6),(1,23,10,x_2),(1,30,16,x_5),(1,31,23,x_9),$
$(1,32,24,x_4)$

$n = 17, u = 16$ $(+2 \ mod \ 34)$:
$(0,2,25,y_5),(0,3,8,x_2),(0,4,6,y_1),(0,5,23,x_9),(0,6,20,x_1),(0,7,4,y_3),$
$(0,8,18,y_4),(0,10,3,x_7),(0,12,24,y_0),(0,14,29,y_2),(0,15,2,y_6),(0,16,13,x_5),$
$(0,21,27,x_3),(0,23,21,x_4),(0,29,5,x_6),(0,33,12,x_8),(1,0,33,x_8),(1,4,9,x_2),$
$(1,5,25,y_4),(1,8,16,x_3),(1,9,2,x_7),(1,10,21,y_3),(1,11,26,y_2),(1,13,5,y_1),$
$(1,16,0,x_4),(1,17,29,y_0),(1,21,17,x_1),(1,22,23,y_6),(1,24,28,x_6),(1,26,20,x_9),$
$(1,29,4,x_5),(1,33,24,y_5)$

$n = 19, u = 14$ $(+1 \ mod \ 38)$:
$(0,1,7,18),(0,5,15,x_5),(0,6,28,x_7),(0,7,22,x_6),(0,8,33,y_1),(0,9,18,y_3),$
$(0,12,8,x_2),(0,13,6,y_0),(0,14,3,x_4),(0,15,1,17),(0,17,14,x_9),(0,20,25,x_1),$
$(0,28,27,x_3),(0,34,26,y_2),(0,35,9,y_4),(0,36,34,x_8)$

$n = 19, u = 15$ $(+2 \ mod \ 38)$:
$(0,1,17,7),(0,2,14,x_1),(0,3,30,x_6),(0,4,27,x_2),(0,6,28,10),(0,8,12,y_1),$
$(0,9,37,21),(0,10,23,x_8),(0,11,18,y_2),(0,13,7,x_9),(0,22,36,x_7),(0,24,33,y_4),$
$(0,25,22,y_3),(0,26,15,y_0),(0,29,32,x_3),(0,31,11,y_5),(0,33,35,x_4),(0,35,21,x_5),$
$(1,2,8,x_9),(1,5,9,y_1),(1,9,10,x_8),(1,12,37,y_2),(1,15,0,y_4),(1,16,34,x_4),$
$(1,18,35,y_3),(1,19,14,x_2),(1,22,23,x_6),(1,24,19,x_3),(1,27,36,y_0),(1,32,30,x_5),$
$(1,33,25,x_1),(1,34,26,y_5),(1,37,11,x_7)$

$n = 19, u = 16$ $(+1 \ mod \ 38)$:
$(0,1,27,11),(0,3,20,x_3),(0,6,12,x_2),(0,8,3,y_5),(0,10,25,y_0),(0,15,37,x_6),$
$(0,17,15,x_1),(0,18,9,y_3),(0,24,31,y_1),(0,25,17,x_4),(0,26,30,x_5),(0,27,2,x_7),$
$(0,29,5,x_8),(0,31,34,y_4),(0,33,32,x_9),(0,34,24,y_2),(0,36,16,y_6)$

$n = 21, u = 19$ $(+2 \ mod \ 42)$:
$(0,2,24,x_1),(0,3,38,y_2),(0,4,41,y_0),(0,5,9,x_7),(0,8,20,x_2),(0,9,3,y_3),$
$(0,10,19,y_9),(0,11,39,4),(0,14,12,y_6),(0,15,13,x_3),(0,22,33,y_5),(0,23,1,x_8),$
$(0,24,14,x_6),(0,26,15,y_8),(0,29,10,y_1),(0,30,27,x_4),(0,31,16,y_7),(0,33,6,y_4),$
$(0,36,2,x_5),(0,37,8,x_9),(1,0,6,x_3),(1,2,26,x_7),(1,3,13,x_2),(1,4,21,y_2),$
$(1,7,15,x_6),(1,9,18,y_5),(1,11,35,x_1),(1,13,8,x_4),(1,15,12,y_0),(1,16,41,y_4),$
$(1,18,32,y_3),(1,19,7,x_5),(1,23,39,y_6),(1,24,17,y_1),(1,26,25,x_9),(1,27,14,y_8),$
$(1,30,11,y_7),(1,36,20,x_8),(1,39,38,y_9)$

$n = 22, u = 16$ ($+2$ *mod* 44) :
$(0,1,10,x_3), (0,4,15,43), (0,5,19,y_2), (0,6,32,15), (0,7,1,y_1), (0,8,41,y_4), (0,9,4,36),$
$(0,15,2,y_6), (0,19,3,y_3), (0,20,43,9), (0,21,14,x_4), (0,23,5,x_8), (1,42,19,y_6)$
$(0,25,27,x_9), (0,26,21,x_5), (0,28,16,x_2), (0,30,26,x_6), (0,31,6,x_1), (0,34,37,x_7),$
$(0,37,20,y_5), (0,42,36,y_0), (1,2,33,x_4), (1,3,39,33), (1,4,38,x_9), (1,6,20,y_3),$
$(1,9,16,x_7), (1,10,37,x_1), (1,12,41,x_3), (1,13,17,x_2), (1,15,14,x_5), (1,16,32,y_2),$
$(1,19,43,y_0), (1,21,11,x_6), (1,28,25,y_5), (1,32,34,x_8), (1,34,10,y_1), (1,41,22,y_4),$

$n = 23, u = 17$ ($+2$ *mod* 46) :
$(0,2,29,24), (0,3,31,x_9), (0,10,3,y_3), (0,11,19,32), (0,12,33,3), (0,14,39,y_7),$
$(0,15,13,19), (0,16,10,x_4), (0,17,1,y_2), (0,21,35,x_6), (0,22,6,x_2), (0,26,30,x_5),$
$(0,27,32,15), (0,28,2,x_7), (0,37,28,x_8), (0,38,20,y_1), (0,39,42,y_5), (0,40,7,x_1),$
$(0,41,5,y_6), (0,42,43,y_4), (0,43,12,y_0), (0,45,8,x_3), (1,0,11,x_3), (1,5,9,x_5),$
$(1,9,26,y_7), (1,12,17,x_8), (1,15,21,x_2), (1,16,33,y_5), (1,22,30,y_6), (1,23,45,x_7),$
$(1,27,20,y_3), (1,28,38,x_9), (1,29,41,y_1), (1,34,32,x_6), (1,35,36,x_1), (1,37,2,y_4),$
$(1,38,25,y_0), (1,40,6,y_2), (1,45,19,x_4)$

A9 Miscellaneous HSD($h^n u^1$)

Here we present HSD($h^n u^1$) for $(h, n, u) = (3,4,1), (3,4,2), (3,4,4), (8,5,14)$.

$h = 3, n = 4, u = 1$ ($+6$ *mod* 12)
$(0,2,7,9), (0,3,9,6), (0,5,3,2), (0,6,11,5), (0,7,6,1), (0,9,10,x_1), (0,10,5,3),$
$(0,11,1,10), (1,0,11,x_1), (1,2,4,3), (1,4,10,7), (1,7,8,2), (1,11,2,4), (2,5,11,8),$
$(2,7,0,x_1), (2,9,8,3), (3,9,4,10), (3,10,1,x_1), (4,5,2,x_1), (4,11,10,5), (5,8,3,x_1)$

$h = 3, n = 4, u = 2$ ($+6$ *mod* 12)
$(0,1,6,7), (0,2,5,3), (0,3,10,x_2), (0,5,7,10), (0,6,11,5), (0,7,9,x_1), (0,9,3,6),$
$(1,2,0,x_2), (1,3,2,4), (1,7,8,2), (1,8,10,x_1), (1,10,4,7), (2,3,8,9), (2,5,11,8),$
$(2,9,0,x_1), (2,11,9,x_2), (3,4,5,x_1), (3,9,10,4), (3,10,1,x_2), (4,5,10,11), (4,6,5,x_2),$
$(4,11,2,x_1), (5,6,7,x_1), (5,7,2,x_2)$

$h = 3, n = 4, u = 4$ ($+4$ *mod* 12)
$(0,2,3,x_2), (0,3,9,x_3), (0,6,5,x_1), (0,7,10,x_4), (0,11,1,6), (1,0,3,x_4), (1,3,10,x_2),$
$(1,4,6,x_3), (1,8,7,x_1), (2,1,3,x_3), (2,3,8,x_1), (2,4,1,x_2), (2,5,0,x_4), (3,5,4,x_2),$
$(3,6,8,x_3), (3,9,6,x_1), (3,10,5,x_4)$

$h = 8, n = 5, u = 14$ ($+1$ *mod* 40) :
$(0,3,24,37), (0,7,29,y_3), (0,8,4,y_2), (0,11,19,x_1), (0,12,26,x_8), (0,14,12,x_3),$
$(0,17,1,y_1), (0,19,7,x_5), (0,22,9,x_9), (0,24,13,x_2), (0,31,22,x_7), (0,34,17,x_4),$
$(0,36,2,y_4), (0,38,37,y_0), (0,39,32,x_6)$