A sharp upper bound on the spectral radius of generalized weighted digraphs*

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Abstract

A generalized weighted digraph $G = (V, E)$ is a digraph with n vertices and m arcs without loops and multiarcs, where each arc is assigned with the weight that is a non-negative and symmetric matrix of same order of p. In this paper, we give a sharp upper bound for the spectral radius of generalized weighted digraphs (see Theorem 2.7), which generalizes some other results on the spectral radius of weighted digraphs in [4], [11] and [16].

Keywords: Weighted digraph; Weighted graph; Spectral radius; Bound

1 Introduction

Let $G = (V, E)$ be a *digraph*, without loops and multiarcs, with vertex set $V = \{v_1, v_2, ..., v_n\}$. If each arc $(v_i, v_j) \in E$, where v_i is the initial vertex and v_j the terminal vertex, has been assigned ω_{ij}, a non-negative and symmetric matrix of order p, as the weight on this arc, then G is called *generalized weighted digraph (short for GWD-graph)*. The adjacency matrix $A(G) = (a_{ij})$ of a GWD-graph G is a block matrix, where the matrix block a_{ij} of order p is defined by

$$a_{ij} = \begin{cases} \omega_{ij} & \text{if } (v_i, v_j) \in E; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

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Clearly, $A(G)$ is indeed a non-negative matrix of order np and has a positive number as its largest eigenvalue by Perron-Furobinius theorem. Let $\rho(G)$ denote the largest eigenvalue of $A(G)$ which is also called the spectral radius of the GWD-graph G. Let $N^+(v_i)$ be the out-neighbor set of v_i in G and denote by $\omega_i^+ = \sum_{v_j \in N^+(v_i)} \omega_{ij}$ the the sum of the i-block row of $A(G)$ corresponding to vertex v_i. Clearly, w_i^+ is also non-negative symmetric matrix of order p and denote by $\rho(w_i^+)$ the spectral radius of w_i^+. If V can be decomposed into disjoint union of two nonempty sets V_1 and V_2 such that $\rho(\omega_i^+)$ is a constant for every vertex $v_i \in V_1$ and $\rho(\omega_j^+)$ is a constant for every vertex $v_j \in V_2$, then G will be called a *generalized weight-semiregular bipartite digraph (for SRB-GWD-graph)*. If $\rho(\omega_k^+)$ is a constant for every $v_k \in V$, then G will be called a *generalized weight-regular digraph (short for R-GWD-graph)*. A GWD-graph G can be viewed as *weighted digraph (short for WD-graph)* if w_{ij} is a positive number for $(v_i, v_j) \in E$, and a WD-graph G is a commonly *digraph* if $w_{ij} = 1$ for $(v_i, v_j) \in E$. A GWD-graph G can be viewed as *weighted graph (short for W-graph)* if $A(G)$ is symmetric, and a W-graph is a commonly graph if $w_{ij} = 1$ for $(v_i, v_j) \in E$. The terminology not defined here can be found in ([1]-[2]).

The bounds for the spectral radius of graphs and digraphs have been investigated to a great extent (to see [4]-[9], [10]-[14] for references), but there are a few of results for the weighted graphs and digraphs (to see [15]-[18] for references).

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and d_i be the degree of vertex v_i, where $i = 1, 2, ..., n$. Abraham Berman and Xiao-Dong Zhang in [4] gave a bound of spectral radius for graphs:

$$\rho(G) \leq \max_{v_i \sim v_j} \{\sqrt{d_i d_j}\}. \tag{2}$$

Equality holds if and only if G is a regular or bipartite semiregular graph. (2) is generalized below for directed graph D by Lingsheng Shi in [11]:

$$\begin{aligned} \min\{\sqrt{d_i^+ d_j^+} : (v_i, v_j) \in E(D)\} &\leq \rho(D) \\ &\leq \max\{\sqrt{d_i^+ d_j^+} : (v_i, v_j) \in E(D)\}. \end{aligned} \tag{3}$$

Either of the equalities holds if and only if D is out-regular or out-semiregular graph. (2) is also generalized for weighted graph G by Kinkar Ch.Das in [16] in the following theorem:

Theorem 1.1 ([16]). *Let G be a weighted graph which is simple, connected and let ρ be the largest eigenvalue (in modulus) of G, so that $|\rho|$ is the* *spectral radius of G. Then*

$$|\rho| \leq \max_{i\sim j}\{\sqrt{\sum_{k:k\sim i}\rho_1(\omega_{ik})\sum_{k:k\sim j}\rho_1(\omega_{jk})}\}. \tag{4}$$

where the edge weights w_{ij} are positive definite matrices having the same order, $\rho_1(\omega_{ij})$ is the largest eigenvalue of ω_{ij}. Equality holds if and only if the followings hold:

(i) *G is a weight-regular graph or G is a weight-semiregular bipartite graph;*

(ii) *all the ω_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho_1(\omega_{ij})$.*

When the edge weights w_{ij} are positive numbers, (4) becomes (5) (in Corollary 2.5 in [16])

$$\rho(G) \leq \max_{i\sim j}\sqrt{\omega_i\omega_j}. \tag{5}$$

where $\omega_i = \sum_{k:k\sim i}\omega_{ik}$ is the sum of the weights of the edges incident to vertex i.

In this paper, we will give a generation of Theorem 1.1 for digraphs, from which we obtain the results stated in (2),(3),(4) and (5).

2 Lemmas and results

Lemma 2.1. *Let B be a real symmetric $n\times n$ matrix with $\rho(B)$ as its largest eigenvalue. Then for any $\mathbf{x}\in R^n(\mathbf{x}\neq 0), \mathbf{y}\in R^n(\mathbf{y}\neq 0)$, the spectral radius $\rho(B)$ satisfies,*

$$|\mathbf{x}^TB\mathbf{y}| \leq \rho(B)\sqrt{\mathbf{x}^T\mathbf{x}}\sqrt{\mathbf{y}^T\mathbf{y}}.$$

Equality holds if and only if $\mathbf{x}$ is an eigenvector of B corresponding to $\rho(B)$ and $\mathbf{y}=\alpha\mathbf{x}$ for some $\alpha\in R$.

Let G be a GWD-graph with adjacency matrix $A(G)$ defined in (1). Then $A(G)$ is a non-negative matrix, and by famous Perron-Frobinius theorem has a positive number ρ as its largest eigenvalue which corresponds the Perron-vector $\mathbf{x}=(\mathbf{x}_1^T,\mathbf{x}_2^T,...,\mathbf{x}_n^T)^T$ where $\mathbf{x}_i\geq 0$ is a column vector in

R^p corresponding the vertex v_i of G, that is,

$$
\begin{aligned}
A(G)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdots \\ \mathbf{x}_n \end{pmatrix} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdots \\ \mathbf{x}_n \end{pmatrix} \\
&= \rho\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdots \\ \mathbf{x}_n \end{pmatrix}.
\end{aligned} \tag{6}
$$

Thus (6) can be represented according s-th block row as

$$\sum_{1\le k\le n} a_{sk}\mathbf{x}_k = \sum_{v_k\in N^+(v_s)} w_{sk}\mathbf{x}_k = \rho\mathbf{x}_s. \tag{7}$$

By multiplying $\mathbf{x}_s^T$ to (7) we obtain,

$$\sum_{v_k\in N^+(v_s)} \mathbf{x}_s^T w_{sk}\mathbf{x}_k = \rho\mathbf{x}_s^T\mathbf{x}_s. \tag{8}$$

Let (v_s, v_t) be an arc of G. From (8) and Lemma 2.1 we have two equalities bellow:

$$
\begin{aligned}
\rho\mathbf{x}_s^T\mathbf{x}_s &= \sum_{v_k\in N^+(v_s)} \mathbf{x}_s^T w_{sk}\mathbf{x}_k \\
&\le \sum_{v_k\in N^+(v_s)} |\mathbf{x}_s^T w_{sk}\mathbf{x}_k| \\
&\le \sum_{v_k\in N^+(v_s)} \rho(w_{sk})\sqrt{\mathbf{x}_s^T\mathbf{x}_s}\sqrt{\mathbf{x}_k^T\mathbf{x}_k}.
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\rho\mathbf{x}_t^T\mathbf{x}_t &= \sum_{v_l\in N^+(v_t)} \mathbf{x}_t^T w_{tl}\mathbf{x}_l \\
&\le \sum_{v_l\in N^+(v_t)} |\mathbf{x}_t^T w_{tl}\mathbf{x}_l| \\
&\le \sum_{v_l\in N^+(v_t)} \rho(w_{tl})\sqrt{\mathbf{x}_t^T\mathbf{x}_t}\sqrt{\mathbf{x}_l^T\mathbf{x}_l}.
\end{aligned} \tag{10}
$$

First we give the bound of spectral radius of the GWD-graph.

Lemma 2.2. *Let G be a connected GWD-graph with adjacency matrix $A(G)$ defined in (1), let ρ be the spectral radius of G. Then*

$$\rho \le \max_{(v_i,v_j)\in E}\Big\{\sqrt{\sum_{v_k\in N^+(v_i)} \rho(\omega_{ik}) \sum_{v_k\in N^+(v_j)} \rho(\omega_{jk})}\Big\}. \tag{11}$$

Proof. Let $\mathbf{x} = (\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_n^T)^T$ be an the eigenvector corresponding to the spectral radius ρ of G, where $A(G)\mathbf{x} = \rho\mathbf{x}$ is shown in (6) and $\mathbf{x} \neq 0$ is non-negative and $\rho > 0$ by the Perron-Frobinius theorem.

First let $a = \max_{1\leq k\leq n} \{\mathbf{x}_k^T \mathbf{x}_k\} > 0$ (since $\mathbf{x} \neq 0$). We now chose $\mathbf{x}_{i_0}$, the vector component of $\mathbf{x}$, such that $\mathbf{x}_{i_0}^T \mathbf{x}_{i_0} = a$ and there exists $v_{j_0} \in N^+(v_{i_0})$ satisfying $\mathbf{x}_{j_0}^T \mathbf{x}_{j_0} = \max_{v_k\in N^+(v_{i_0})} \{\mathbf{x}_k^T \mathbf{x}_k\} \geq \max_{v_l\in N^+(v_i)} \{\mathbf{x}_l^T \mathbf{x}_l\}$ whenever $\mathbf{x}_i^T \mathbf{x}_i = a$. Clearly, (v_{i_0}, v_{j_0}) is an arc of G and the corresponding vector components $\mathbf{x}_{i_0}$ and $\mathbf{x}_{j_0}$ are called the *maximum relevant components* of $\mathbf{x}$ and will insist throughout this paper. Taking $s = i_0$ in (9) and by Lemma 2.1 we obtain,

$$\begin{aligned} \rho \mathbf{x}_{i_0}^T \mathbf{x}_{i_0} &= \sum_{v_k\in N^+(v_{i_0})} \mathbf{x}_{i_0}^T \omega_{i_0 k} \mathbf{x}_k \\ &\leq \sum_{v_k\in N^+(v_{i_0})} |\mathbf{x}_{i_0}^T \omega_{i_0 k} \mathbf{x}_k| \\ &\leq \sum_{v_k\in N^+(v_{i_0})} \rho(\omega_{i_0 k}) \sqrt{\mathbf{x}_{i_0}^T \mathbf{x}_{i_0}} \sqrt{\mathbf{x}_k^T \mathbf{x}_k} \\ &\leq \sqrt{\mathbf{x}_{i_0}^T \mathbf{x}_{i_0}} \sqrt{\mathbf{x}_{j_0}^T \mathbf{x}_{j_0}} \sum_{v_k\in N^+(v_{i_0})} \rho(\omega_{i_0 k}). \end{aligned} \tag{12}$$

Similarly, taking $t = j_0$ in (10) we obtain,

$$\begin{aligned} \rho \mathbf{x}_{j_0}^T \mathbf{x}_{j_0} &= \sum_{v_l\in N^+(v_{j_0})} \mathbf{x}_{j_0}^T \omega_{j_0 l} \mathbf{x}_l \\ &\leq \sum_{v_l\in N^+(v_{j_0})} |\mathbf{x}_{j_0}^T \omega_{j_0 l} \mathbf{x}_l| \\ &\leq \sum_{v_l\in N^+(v_{j_0})} \rho(\omega_{j_0 l}) \sqrt{\mathbf{x}_{j_0}^T \mathbf{x}_{j_0}} \sqrt{\mathbf{x}_l^T \mathbf{x}_l} \\ &\leq \sqrt{\mathbf{x}_{j_0}^T \mathbf{x}_{j_0}} \sqrt{\mathbf{x}_{i_0}^T \mathbf{x}_{i_0}} \sum_{v_l\in N^+(v_{j_0})} \rho(\omega_{j_0 l}). \end{aligned} \tag{13}$$

We claim that $\mathbf{x}_{j_0} \neq \bar{0}$, since otherwise $\mathbf{x}_k = \bar{0}$ for all k with $v_k \in N^+(v_{i_0})$ by the choice of j_0. Then from the first equity of (12), we get $\rho \mathbf{x}_{i_0}^T \mathbf{x}_{i_0} = 0$, and so $\rho = 0$ since $\mathbf{x}_{i_0}^T \mathbf{x}_{i_0} \neq 0$, which is impossible. Thus, by multiplying the two sides of (12) and (13), we get

$$\rho \leq \sqrt{\sum_{v_k\in N^+(v_{i_0})} \rho_1(\omega_{i_0 k}) \sum_{v_l\in N^+(v_{j_0})} \rho_1(\omega_{j_0 l})},$$

which leads to (11) since (v_{i_0}, v_{j_0}) is an arc of G. □

Corollary 2.3. *Under the assumption of Lemma 2.2, if (11) is an equity, then (12) and (13) are equities.*

Proof. Otherwise, by multiplying the two sides of (12) and (13) we obtain

$$\rho < \sqrt{\sum_{v_k \in N^+(v_{i_0})} \rho_1(\omega_{i_0k}) \sum_{v_l \in N^+(v_{j_0})} \rho_1(\omega_{j_0l})} \leq \rho.$$

It is a contradiction. □

Lemma 2.4. *Under the assumption of Lemma 2.2, if (11) is an equity then we have*

(a) $\mathbf{x}_{i_0}$ *is a common eigenvector corresponding to the largest eigenvalue of* w_{i_0k} *for all* k *with* $v_k \in N^+(v_{i_0})$ *and* $\mathbf{x}_k = b\mathbf{x}_{i_0}$ *(particularly,* $\mathbf{x}_{j_0} = b\mathbf{x}_{i_0}$ *), where* $b > 0$ *is a constant.*

(b) *For any* $v_j \in N^+(v_{i_0})$ *and any* $v_l \in N^+(v_j)$, *we have* $\mathbf{x}_l = \mathbf{x}_{i_0}$.

(c) $\rho = b \sum_{v_k \in N^+(v_{i_0})} \rho(\omega_{i_0k})$ *and* $\rho = b^{-1} \sum_{v_l \in N^+(v_j)} \rho(\omega_{jl})$ *for* $v_j \in N^+(i_0)$, *where* b *is defined in (a).*

Proof. Since (11) is an equity, (12) and the (13) must be equity by Corollary 2.3. From the equity (12) we obtain the following for any $\mathbf{x}_k$ with $v_k \in N^+(v_{i_0})$,

$$\begin{aligned} \mathbf{x}_{i_0}^T \omega_{i_0k} \mathbf{x}_k &= |\mathbf{x}_{i_0}^T \omega_{i_0k} \mathbf{x}_k| \\ &= \rho(\omega_{i_0k}) \sqrt{\mathbf{x}_{i_0}^T \mathbf{x}_{i_0}} \sqrt{\mathbf{x}_k^T \mathbf{x}_k} \\ &= \sqrt{\mathbf{x}_{i_0}^T \mathbf{x}_{i_0}} \sqrt{\mathbf{x}_{j_0}^T \mathbf{x}_{j_0}} \rho(\omega_{i_0k}). \end{aligned} \tag{14}$$

By Lemma 2.1, the second equity of (14) gives that $\mathbf{x}_k = b_{i_0k}\mathbf{x}_{i_0}$ and $\mathbf{x}_{i_0}$ is the common eigenvectors of ω_{i_0k} corresponding to the largest eigenvalue $\rho(\omega_{i_0k})$. The last equity of (14) gives that $\mathbf{x}_k^T \mathbf{x}_k = \mathbf{x}_{j_0}^T \mathbf{x}_{j_0}$, and hence for $(v_{i_0}, v_i) \in E$ and $(v_{i_0}, v_j) \in E$ we have

$$b_{i_0i}^2 \mathbf{x}_{i_0}^T \mathbf{x}_{i_0} = \mathbf{x}_i^T \mathbf{x}_i = \mathbf{x}_j^T \mathbf{x}_j = b_{i_0j}^2 \mathbf{x}_{i_0}^T \mathbf{x}_{i_0},$$

which gives $b_{i_0i}^2 = b_{i_0j}^2$. Additionally, $\mathbf{x}_{i_0}^T \omega_{i_0k} \mathbf{x}_{i_0} > 0$ since $\rho(\omega_{i_0k}) > 0$, and $b_{i_0k} \mathbf{x}_{i_0}^T \omega_{i_0k} \mathbf{x}_{i_0} = \mathbf{x}_{i_0}^T \omega_{i_0k} \mathbf{x}_k = |\mathbf{x}_{i_0}^T \omega_{i_0k} \mathbf{x}_k| > 0$. Hence $b_{i_0k} > 0$ for any $v_k \in N^+(v_{i_0})$. It follows that $b_{i_0i} = b_{i_0j} = b$. Consequently, $\mathbf{x}_k = b\mathbf{x}_{i_0}$ for all k with $v_k \in N^+(v_{i_0})$ and (a) follows.

For any $v_j \in N^+(v_{i_0})$, $\mathbf{x}_j = b\mathbf{x}_{i_0} = \mathbf{x}_{j_0}$ by (a). By replacing j_0 with j in the equity (13), we have for all $\mathbf{x}_l$ with $v_l \in N^+(v_j)$,

$$\begin{aligned} \mathbf{x}_j^T \omega_{jl} \mathbf{x}_l &= |\mathbf{x}_j^T \omega_{jl} \mathbf{x}_l| \\ &= \rho(\omega_{jl}) \sqrt{\mathbf{x}_j^T x_j} \sqrt{\mathbf{x}_l^T \mathbf{x}_l} \\ &= \sqrt{\mathbf{x}_j^T \mathbf{x}_j} \sqrt{\mathbf{x}_{i_0}^T \mathbf{x}_{i_0}} \rho(\omega_{jl}). \end{aligned} \tag{15}$$

By Lemma 2.1, the second equity of (15) gives that $\mathbf{x}_l = c_{jl}\mathbf{x}_j$ and $\mathbf{x}_j$ is the common eigenvectors of ω_{jl} corresponding to the largest eigenvalue $\rho(\omega_{jl})$. The last equity of (15) gives that $\mathbf{x}_l^T\mathbf{x}_l = \mathbf{x}_{i_0}^T\mathbf{x}_{i_0}$. We have $c_{jl}^2 b^2 \mathbf{x}_{i_0}^T\mathbf{x}_{i_0} = \mathbf{x}_{i_0}^T\mathbf{x}_{i_0}$ and so $(c_{jl}b)^2 = 1$. Additionally, from the first equity of (15), we have $c_{jl}\mathbf{x}_j^T\omega_{jl}\mathbf{x}_j = \mathbf{x}_j^T\omega_{jl}\mathbf{x}_l = |\mathbf{x}_j^T\omega_{jl}\mathbf{x}_l| > 0$. Note $\mathbf{x}_j^T\omega_{jl}\mathbf{x}_j > 0$ since $\rho(\omega_{jl}) > 0$, we have $c_{jl} > 0$ and thus $c_{jl} = b^{-1}$. Therefore, $\mathbf{x}_l = b^{-1}\mathbf{x}_j = b^{-1}b\mathbf{x}_{i_0} = \mathbf{x}_{i_0}$. It follows (b).

From the equities (12) and (13) we have

$$\begin{cases} \rho\mathbf{x}_{i_0}^T\mathbf{x}_{i_0} = \sqrt{\mathbf{x}_{i_0}^T\mathbf{x}_{i_0}}\sqrt{\mathbf{x}_{j_0}^T\mathbf{x}_{j_0}} \sum\limits_{v_k \in N^+(v_{i_0})} \rho(\omega_{i_0k}); \\ \rho\mathbf{x}_{j_0}^T\mathbf{x}_{j_0} = \sqrt{\mathbf{x}_{i_0}^T\mathbf{x}_{i_0}}\sqrt{\mathbf{x}_{j_0}^T\mathbf{x}_{j_0}} \sum\limits_{v_l \in N^+(v_{j_0})} \rho(\omega_{j_0l}). \end{cases} \tag{16}$$

According to (a), $\mathbf{x}_{j_0} = b\mathbf{x}_{i_0}$, which put into (16) we obtain (c). □

Lemma 2.5. *Under the assumption of Lemma 2.2, if (11) is an equity then we have*

(i) *Let $P = v_{i_0}v_{i_1}v_{i_2},\dots,v_{i_r}$ be a directed path in G. If $0 \le t \le r$ is even then $\mathbf{x}_{i_t} = \mathbf{x}_{i_0}$ and $\rho = b \sum\limits_{v_l \in N^+(v_{i_t})} \rho(\omega_{i_tl})$; if t is odd then $\mathbf{x}_{i_t} = b\mathbf{x}_{i_0}$ and $\rho = b^{-1} \sum\limits_{v_l \in N^+(v_{i_t})} \rho(\omega_{i_tl})$, where b is defined in Lemma 2.4.*

(ii) *$\mathbf{x}_{i_0}$ is a common eigenvector of ω_{i_tl} corresponding to the largest eigenvalue $\rho(\omega_{i_tl})$ for all $1 \le t \le r$ and l with $v_l \in N^+(v_{i_t})$.*

Proof. Firstly we know that v_{i_0} corresponds $\mathbf{x}_{i_0}$ by the assumption. Then (i) holds for $t = 0, 1$ by Lemma 2.4(a) and (c). Now let $t = 2$, we have known that $\mathbf{x}_{i_2} = \mathbf{x}_{i_0}$ from Lemma 2.4(b), and next need to show that $\rho = b \sum\limits_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2l})$.

Let $\mathbf{x}_q^T\mathbf{x}_q = \max\limits_{v_l \in N^+(v_{i_2})} \{\mathbf{x}_l^T\mathbf{x}_l\}$. Since $(v_{i_2}, v_q) \in E$ and $\mathbf{x}_{i_2} = \mathbf{x}_0$, $\mathbf{x}_q^T\mathbf{x}_q \le \mathbf{x}_{j_0}^T\mathbf{x}_{j_0}$ by the choice of j_0 in the proof of Lemma 2.2. As similar as (12) we have

$$\begin{aligned} \rho x_{i_2}^T x_{i_2} &= \sum_{v_l \in N^+(v_{i_2})} x_{i_2}^T \omega_{i_2l} x_l \\ &\le \sum_{v_l \in N^+(v_{i_2})} |x_{i_2}^T \omega_{i_2l} x_l| \\ &\le \sum_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2l}) \sqrt{x_{i_2}^T x_{i_2}} \sqrt{x_l^T x_l} \\ &\le \sum_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2l}) \sqrt{x_{i_2}^T x_{i_2}} \sqrt{x_q^T x_q} \\ &\le \sum_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2l}) \sqrt{x_{i_2}^T x_{i_2}} \sqrt{x_{j_0}^T x_{j_0}}. \end{aligned} \tag{17}$$

Note that $\mathbf{x}_{i_1} = \mathbf{x}_{j_0}$. By replacing $\mathbf{x}_{j_0}$ with $\mathbf{x}_{i_1}$ in (13) and combining (17) we get,

$$\begin{aligned} \rho &\leq \sqrt{\sum_{v_l \in N^+(v_{i_1})} \rho(\omega_{i_1 l})} \sqrt{\sum_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2 l})} \\ &\leq \max_{(v_i, v_j) \in E} \{ \sqrt{\sum_{v_k \in N^+(v_i)} \rho(\omega_{ik}) \sum_{v_k \in N^+(v_j)} \rho(\omega_{jk})} \} \\ &= \rho. \end{aligned}$$

Therefore,

$$b^{-1} \sum_{v_l \in N^+(v_{i_1})} \rho(\omega_{i_1 l}) = \rho = \sqrt{\sum_{v_l \in N^+(v_{i_1})} \rho(\omega_{i_1 l})} \sqrt{\sum_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2 l})}.$$

Hence $b^{-1} \sqrt{\sum_{v_l \in N^+(v_{i_1})} \rho(\omega_{i_1 l})} = \sqrt{\sum_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2 l})}$, and so

$$\rho = b^{-1} \sum_{v_l \in N^+(v_{i_1})} \rho(\omega_{i_1 l}) = b \sum_{v_l \in N^+(v_{i_2})} \rho(\omega_{i_2 l}).$$

Furthermore, (17) must be an equity by the argument of Corollary 2.3. Since $\mathbf{x}_{i_1} = \mathbf{x}_{j_0}$, from the equity (17) we have

$$\mathbf{x}_l^T \mathbf{x}_l = \mathbf{x}_{j_0}^T \mathbf{x}_{j_0} = \mathbf{x}_{i_1}^T \mathbf{x}_{i_1}, \tag{18}$$

for all x_l with $v_l \in N^+(v_{i_2})$. Again the equity (17) implies that $\mathbf{x}_l = b_l \mathbf{x}_{i_2} = b_l \mathbf{x}_{i_0}$ by Lemma 2.1, where $b_l > 0$ similarly as in the proof of "$b_{i_0 k} > 0$" in Lemma 2.4(a). Thus (18) gives that $b_l^2 \mathbf{x}_{i_0}^T \mathbf{x}_{i_0} = b^2 \mathbf{x}_{i_0}^T \mathbf{x}_{i_0}$, and so $b_l = b$. Hence $\mathbf{x}_{i_3} = \mathbf{x}_l = b\mathbf{x}_{i_0} = \mathbf{x}_{j_0}$.

Notice that $\mathbf{x}_{i_2} = \mathbf{x}_{i_0}$ and now $\mathbf{x}_{i_3} = \mathbf{x}_{j_0}$. Regarding v_{i_2} as v_{i_0} and repeating the above process, we will get (i) by induction.

At last, the corresponding (17) for any v_{i_t} is an equity from the proof of (i), we claim that $|\mathbf{x}_{i_t}^T \omega_{i_t l} \mathbf{x}_l| = \rho(\omega_{i_t l}) \sqrt{\mathbf{x}_{i_t}^T \mathbf{x}_{i_t}} \sqrt{\mathbf{x}_l^T \mathbf{x}_l}$ for all $\mathbf{x}_l$ with $v_l \in N^+(v_{i_t})$ and so $\mathbf{x}_{i_0}$ ($\mathbf{x}_{i_t} = \mathbf{x}_{i_0}$, or $b\mathbf{x}_{i_0}$) is a common eigenvector of $\omega_{i_t l}$ corresponding to the largest eigenvalue $\rho(\omega_{i_t l})$ by Lemma 2.1. Thus (ii) follows. □

Lemma 2.6. *Under the assumption of Lemma 2.2, if $\mathbf{x}_{i_0}$ is a common eigenvector of ω_{ij} corresponding to the largest eigenvalue $\rho(\omega_{ij})$ for all $1 \leq i, j \leq n$, then $\rho(w_i^+) = \sum_{v_j \in N^+(v_i)} \rho(\omega_{ij})$, where $w_i^+ = \sum_{v_j \in N^+(v_i)} \omega_{ij}$.*

Proof. Suppose that $\rho(w_i^+)\mathbf{y}_i = w_i^+ \mathbf{y}_i$, where $\mathbf{y}_i \neq 0$. By Lemma 2.1 we have

$$\begin{aligned}
\mathbf{y}_i^T \rho(w_i^+)\mathbf{y}_i &= \mathbf{y}_i^T w_i^+ \mathbf{y}_i \\
&\leq |\mathbf{y}_i^T w_i^+ \mathbf{y}_i| \\
&\leq \sum_{v_j \in N^+(v_i)} |\mathbf{y}_i^T \omega_{ij} \mathbf{y}_i| \\
&\leq \sum_{v_j \in N^+(v_i)} \rho(\omega_{ij}) \mathbf{y}_i^T \mathbf{y}_i .
\end{aligned}$$

Hence, $\rho(w_i^+) \leq \sum_{v_j \in N^+(v_i)} \rho(\omega_{ij})$. On the other hand, we have $w_i^+ \mathbf{x}_{i_0} = \sum_{v_j \in N^+(v_i)} \omega_{ij} \mathbf{x}_{i_0} = \sum_{v_j \in N^+(v_i)} \rho(\omega_{ij}) \mathbf{x}_{i_0}$, which implies that

$$\rho(w_i^+) \geq \sum_{v_j \in N^+(v_i)} \rho(\omega_{ij}).$$

It follows our result. □

Now we come to the stage to prove our main result.

Theorem 2.7. *Let G be a connected GWD-graph with adjacency matrix $A(G)$ defined in (1), and let ρ be the spectral radius of G. Then*

$$\rho \leq \max_{(v_i,v_j) \in E} \{ \sqrt{\sum_{v_k \in N^+(v_i)} \rho(\omega_{ik}) \sum_{v_k \in N^+(v_j)} \rho(\omega_{jk})} \}. \tag{19}$$

Moreover, if G is strongly connected, equality holds if and only if the one of the followings are satisfied:

(i) *G is a R-GWD-graph and ω_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho(\omega_{ij})$ for every arc (v_i, v_j) in G.*

(ii) *G is a SRB-GWD-graph and ω_{ij} have a common eigenvector corresponding to the largest eigenvalue $\rho(\omega_{ij})$ for every arc (v_i, v_j) in G.*

Proof. (19) is proved in Lemma 2.2. Next we suppose that (19) is an equality. Then all inequalities in (12) and (13) must be equalities by Corollary 2.3. Let $V_0 = \{v_i \in V \mid \mathbf{x}_i = \mathbf{x}_{i_0}\}$ and $V_1 = \{v_j \in V \mid \mathbf{x}_j = b\mathbf{x}_{i_0}\}$, where $b > 0$ is determined in Lemma 2.4. Since G is strongly connected, by Lemma 2.5(i) V must be the disjoint union of V_0 and V_1 if $b \neq 1$, and $V = V_0$ if $b = 1$. Now we distinguish two cases bellow.

Case 1. $b = 1$;

In this case, any $\mathbf{x}_i = \mathbf{x}_{i_0}$ for all $1 \leq i \leq n$, we have $\rho = \sum_{v_j \in N^+(v_i)} \rho(\omega_{ij})$ by Lemma 2.5(i). Thus, by Lemma 2.6, $\rho = \rho(w_i^+)$ for any $1 \leq i \leq n$ and so G is a R-GWD-graph.

Case 2. $b \neq 1$;

For $v_i \in V_0$, let (v_i, v_j) be any arc of G. Then $\mathbf{x}_i = \mathbf{x}_{i_0}$ and $\mathbf{x}_j = b\mathbf{x}_{i_0}$ by Lemma 2.5(i), and so $v_j \in V_1$. Thus V_0 is independent. For $v_i \in V_1$, let (v_i, v_j) be any arc of G. Since G is strongly connected, there exists a shortest directed path $P = v_{i_0} \cdots v_i$ staring from v_{i_0} and ending at v_i. Since $\mathbf{x}_i = b\mathbf{x}_{i_0}$ by assumption, P contains exactly even number of vertices by Lemma 2.5(i). Then $P' = P + v_j = v_{i_0} \cdots v_i v_j$ is a directed path having odd number of vertices and so $v_j \in V_0$ by Lemma 2.5(i). If $v_j \in P = v_{i_0} \cdots v_i$, then there are even number of vertices in P before v_j and so $v_j \in V_0$. Thus V_1 is independent. Therefore, G is a bipartite digraph with the partition (V_0, V_1). Additionally, for $v_s \in V_0$ we have $\mathbf{x}_s = \mathbf{x}_{i_0}$ and so $\rho(w_s^+) = \sum_{v_h \in N^+(v_s)} \rho(\omega_{sh}) = \rho b^{-1}$ by Lemma 2.6 and Lemma 2.5(i). Similarly, we have $\rho(w_t^+) = \sum_{v_h \in N^+(v_t)} \rho(\omega_{th}) = \rho b$ for $v_t \in V_1$. Thus G is a SRB-GWD-graph.

Conversely, let $\mathbf{x}_{i_0}$ be a common eigenvector of ω_{ij} corresponding to the largest eigenvalue $\rho(\omega_{ij})$ for all $1 \le i, j \le n$. Then $\rho(\omega_i^+) = \sum_{1 \le j \le n} \rho(\omega_{ij})$ by Lemma 2.6. First we suppose that G is a SRB-GWD-graph with bipartite $V = U \cup W$ such that $\rho(\omega_i^+) = \alpha$ for $v_i \in U$ and $\rho(\omega_j^+) = \beta$ for $v_j \in W$. The following equation can be easily verified:

$$\sqrt{\alpha\beta}\begin{pmatrix} \mathbf{x}_{i_0} \\ \mathbf{x}_{i_0} \\ \cdots \\ \cdots \\ \cdots \\ \mathbf{x}_{i_0} \\ \sqrt{\frac{\beta}{\alpha}}\mathbf{x}_{i_0} \\ \cdots\cdots \\ \cdots \\ \sqrt{\frac{\beta}{\alpha}}\mathbf{x}_{i_0} \end{pmatrix} = A \begin{pmatrix} \mathbf{x}_{i_0} \\ \mathbf{x}_{i_0} \\ \cdots \\ \cdots \\ \cdots \\ \mathbf{x}_{i_0} \\ \sqrt{\frac{\beta}{\alpha}}\mathbf{x}_{i_0} \\ \cdots \\ \cdots \\ \cdots \\ \sqrt{\frac{\beta}{\alpha}}\mathbf{x}_{i_0} \end{pmatrix},$$

$$\text{where } A = \begin{pmatrix} & U & & & W & \\ 0 & \cdots & 0 & a_{1,k+1} & \cdots & a_{1,n} \\ 0 & \cdots & 0 & a_{2,k+1} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{k,k+1} & \cdots & a_{k,n} \\ a_{k+1,1} & \cdots & a_{k+1,k} & 0 & \cdots & 0 \\ a_{k+2,1} & \cdots & a_{k+2,k} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n,1} & \cdots & a_{n,k} & 0 & \cdots & 0 \end{pmatrix}.$$

Thus $\sqrt{\alpha\beta}$ is an eigenvalue of $A(G)$, and so by (19)

$$
\begin{aligned}
\rho(G) &\le \max_{(v_i,v_j)\in E}\{\sqrt{\sum_{v_k\in N^+(v_i)}\rho(\omega_{ik})\sum_{v_k\in N^+(v_j)}\rho(\omega_{jk})}\} \\
&= \sqrt{\alpha\beta} \\
&\le \rho(G)
\end{aligned}
$$

which implies that (19) is an equity. At last, for R-GWD-graph G, we can easily see that $\rho(G) = \rho(\omega_i^+)$ for any $v_i \in V$, and (19) is certainly an equity. We complete this proof. □

If positive definite matrix w_{ij} is positive number for $(v_i, v_j) \in E$ then our generalized weighted digraph G will be the *weighted digraph*, and if $w_{ij} = 1$ for $(v_i, v_j) \in E$ then our weighted digraph G will be the commonly digraph. In the first case, $\rho(w_{ij}) = w_{ij}$ for $(v_i, v_j) \in E$ and so $\sum_{v_k\in N^+(v_i)}\rho(\omega_{ik}) = \sum_{v_k\in N^+(v_i)}\omega_{ik} = w_i^+$; in the latter case, $\rho(w_{ij}) = 1$ for $(v_i, v_j) \in E$ and so $\sum_{v_k\in N^+(v_i)}\rho(\omega_{ik}) = d_i^+(= d^+(v_i))$.

It immediately follows the results from Theorem 2.7.

Corollary 2.8. *Let $G = (V, E)$ be a commonly connected weighted digraph in which the arc (v_i, v_j) weighted with positive numbers w_{ij} and $w_i^+ = \sum_{v_k\in N^+(v_i)} w_{ik}$, then,*

$$\rho(G) \le \max_{(v_i,v_j)\in E}\sqrt{\omega_i^+\omega_j^+}.$$

Moreover, if G is strongly connected, then equality holds if and only if G is a weigh-regular (i.e., w_i^+ is a constant for $v_i \in V$) or G is a weigh-semiregular bipartite digraph (i.e., G has a partition (U, W) such that w_i^+ and w_j^+ are constants for $v_i \in U$ and $v_j \in W$).

Corollary 2.9 ([11]). *Let $G = (V, E)$ be a simple and connected digraph. Then*

$$\rho(G) \le \max_{(v_i,v_j)\in E}\sqrt{d_i^+d_j^+}.$$

where d_i^+ is the outdegree of v_i. Moreover, if G is strongly connected, then equality holds if and only if G is a out-regular digraph or G is a out-semiregular bipartite digraph.

If the adjacency matrix $A(G)$ of GWD-graph G defined in (1) is symmetric, then our generalized weighted digraph G can be viewed as *weighted graph*, and if $w_{ij} = 1$ for $(v_i, v_j) \in E$ then our weighted graph G will be the commonly graph. In weighted graph, if the edge weights w_{ij} are positive definite matrices, then we know that $\rho(\omega_{ij}) > 0$, and the results of Lemma 2.1 and Lemma 2.6 hold. Then the spectral radius of weighted graph G is the largest eigenvalue (in modulus) of G. Consequently, we obtain the results stated in (2), (4) and (5).

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