Matchings in 4-total restrained domination vertex critical graphs

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Abstract

A graph G with no isolated vertex is total restrained domination vertex critical if for any vertex v of G that is not adjacent to a vertex of degree one, the total restrained domination number of $G - v$ is less than the total restrained domination number of G. We call these graphs γ_{tr}-vertex critical. If such a graph G has total restrained domination number k, we call it $k-\gamma_{tr}$-vertex critical. In this paper, we study matching properties in $4-\gamma_{tr}$-vertex critical graphs of minimum degree at least two.

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1 Introduction

A vertex in a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a dominating set, if each vertex of G is dominated by some vertex of S. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set S is called *total dominating set* if each vertex x of G is dominated by some vertex $y \neq x$ with $y \in S$. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. For references on domination in graphs see [8].

A *leaf* in a graph G is a vertex of degree one, and a *support vertex* is one that is adjacent to a leaf. A (vertex) *cut-set* in a connected graph G, which is different from the complete graph graph, is a subset S of vertices such that $G - S$ is disconnected. The *connectivity* of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has just one vertex. A graph G is *k-connected* if its connectivity is at least k. For a subset S of vertices, we denote by $c(G-S)$ the number of components of $G - S$. We also use $o(G - S)$ for the number of odd components of $G - S$. For graph theory terminology see [12].

A set of pair-wise independent edges in a graph G is called a *matching*. A matching is *perfect* if it is incident with every vertex of G. A graph G is called *factor-critical* if $G - v$ has a perfect matching for every vertex v.

Note that the removal of a vertex in a graph may decrease the domination number. A graph G is called *vertex domination critical* if $\gamma(G-v) < \gamma(G)$, for every vertex v in G. For references on vertex domination critical graphs see [1, 4, 10].

Chen et al. [2] introduced the *total restrained domination*, which was further studied by J. H. Hattingh et al., [2, 3, 7]. A set

$S \subseteq V(G)$ is a *total restrained dominating set* or just a TRDS if every vertex is adjacent to a vertex in S and every vertex in $V(G)\backslash S$ is also adjacent to a vertex in $V(G)\backslash S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of G.

Gera et al. [6] studied vertex and edge critical total restrained domination in graphs. A graph G is *total restrained domination vertex critical* or just γ_{tr}-vertex critical if, for any vertex v of $V(G) \setminus S(G)$, $\gamma_{tr}(G-v) < \gamma_{tr}(G)$, where $S(G)$ is the set of all support vertices of G. Similarly, G is *total restrained domination edge critical* or just γ_{tr}-edge critical if for any $e \notin E(G)$, $\gamma_{tr}(G+e) < \gamma_{tr}(G)$. They characterized all γ_{tr}-vertex critical trees, as well as those $\gamma_{tr}(G)$-vertex critical graphs G for which $\gamma_{tr}(G) - \gamma_{tr}(G-v) = n-2$ for some $v \in V(G)$. A γ_{tr}-vertex critical graph G with $\gamma_{tr}(G) = k$ is called *$k - \gamma_{tr}$-vertex critical.*

Matching properties in $3-\gamma_{tr}$-vertex critical graphs are studied in [5]. In this paper we study matching properties in $4-\gamma_{tr}$ -vertex critical graphs with minimum degree at least two.

All graphs in this paper are connected, and have minimum degree at least two. We call a vertex v, as a *total restrained domination critical* vertex, or just γ_{tr}-vertex critical vertex, if $\gamma_{tr}(G-v) < \gamma_{tr}(G)$. Thus a graph G is γ_{tr}-vertex critical if each vertex v of G is γ_{tr}-vertex critical. For a vertex v in a γ_{tr}-vertex critical graph G, we denote by S_v a minimum TRDS for $G-v$.

2 Some preliminary results

We begin this section with the following known results.

Theorem 1 (Tutte [11]) *A graph G has a perfect matching if and only if $o(G-S) \leq |S|$ for all $S \subseteq V(G)$.*

Theorem 2 (Lovasz and Plummer [9]) *A graph G is factor-critical if and only if $o(G - S) \leq |S| - 1$ for all $S \subseteq V(G)$.*

Lemma 3 (Jafari Rad [5]) *Let G be a γ_{tr}-vertex critical graph and $v \in V(G)$. If $S_v \cap N_G(v) \neq \emptyset$, then $N_G(v) \subseteq S_v$.*

The following is obviously verified.

Observation 4 *Let G be a $4 - \gamma_{tr}$-vertex critical graph, and let S be a cut-set with at least two vertices. Then for any vertex $v \in S$, $S_v \cap S \neq \emptyset$.*

To obtain our main results, we study minimum degree and connectivity in $4 - \gamma_{tr}$-vertex critical graphs.

Lemma 5 *If G is a $4 - \gamma_{tr}$-vertex critical graph, then $\delta(G) \geq 3$.*

Proof. Let G be a $4 - \gamma_{tr}$-vertex critical graph. Assume to the contrary, that $\delta(G) = 2$. Let x be a vertex with $deg(x) = 2$, and let $N(x) = \{y, z\}$. For S_y to dominate x, it follows that $x \in S_y$ or $z \in S_y$. But S_y is a TRDS for $G - y$, and x is a leaf in $G - y$. We deduce that $x \in S_y$. By Lemma 3, $N(y) \subseteq S_y$.

If $|S_y| = 2$, then $deg(y) = 2$ and $N(y) = \{x, z\}$. But then $\{x, y, z\}$ is a TRDS for G, a contradiction. Thus $|S_y| = 3$. Let $S_y = \{x, z, w\}$, where w is adjacent to z, since $G[S_y]$ has no isolated vertex. By Lemma 3, $deg(y) \in \{2, 3\}$. If w is adjacent to y, then $\{w, z\}$ is a TRDS for G, a contradiction. If not, then $deg(y) = 2$ and z is adjacent to y. Again $\{z, w\}$ is a TRDS for G, a contradiction. ■

Theorem 6 *If G is a $4 - \gamma_{tr}$-vertex critical graph, then G is 2-connected.*

Proof. Let G be a $4-\gamma_{tr}$-vertex critical graph. Assume to the contrary, that G has a cut vertex x. Since $G[S_x]$ is connected, (note that S_x is a TRDS with $|S_x| < 4$), and $G-x$ is disconnected, the set S_x doesn't dominate $G-x$. This is a contradiction, and thus G is 2-connected. ∎

We next show that if there is a cut-set S of size 2 in a $4-\gamma_{tr}$-vertex critical graph G, then the number of odd components of $G-S$ is at most 2.

Theorem 7 *Let G be a $4-\gamma_{tr}$-vertex critical graph. If S is a cut-set of size 2, then $o(G-S) \leq 2$.*

Proof. Let G be a $4-\gamma_{tr}$-vertex critical graph, and let $S = \{x, y\}$ be a minimum cut-set. By Lemma 5, any component of $G-S$ has at least two vertices. Let $G_1, G_2, ..., G_k$ be the odd components of $G-S$. Assume to the contrary, that $k \geq 3$. By Observation 4, $x \in S_y$. Since $|S_y| \leq 3$, we observe that at most two odd components of $G-S$ contain vertices of S_y. This implies that x is adjacent to every vertex of at least $k-2$ odd components of $G-S$.

Further, for $1 \leq i \leq k$, $N(x) \cap V(G_i) \neq \emptyset$. By Lemma 5, for $1 \leq i \leq k$, $|V(G_i)| \geq 3$. This shows that $deg(x) \geq 5$. Also $y \in S_x$, and we similarly have

(1) y is adjacent to every vertex of at least $k-2$ components of $G-S$,

(2) for $1 \leq i \leq k$, $N(y) \cap V(G_i) \neq \emptyset$,

(3) $deg(y) \geq 5$.

As an immediate result of Lemma 3, we deduce that y is not adjacent to x. Let G_j be a component such that $V(G_j) \subseteq N(x)$. There is a vertex $z \in V(G_j)$ such that z is adjacent to y. Since $S_z \cap S \neq \emptyset$, by Lemma 3, $S_z = \{x, y, z_1\}$, where $z_1 \in N(z) \cap V(G_j)$. Using Lemma 3, we obtain $deg_G(z) = 3$ and

z_1 is adjacent to y. But then $S_{z_1} \cap S \neq \emptyset$, and by Lemma 3, $deg(z_1) = 3$. This implies that $V(G_j) = \{z, z_1\}$, a contradiction to the assumption that G_j is an odd component of $G - S$. ■

3 Matching properties

In this section we study matching properties in $4 - \gamma_{tr}$-vertex critical graphs. Theorem 1 leads that any $4 - \gamma_{tr}$-vertex critical claw-free graph of even order has a perfect matching.

Theorem 8 *Any $4 - \gamma_{tr}$-vertex critical claw-free graph of odd order is factor-critical.*

Proof. Let G be a $4 - \gamma_{tr}$-vertex critical claw-free graph of odd order. Suppose to the contrary that G is not factor critical. By Theorem 2, there is a subset $S \subseteq V(G)$ such that $o(G-S) \geq |S|$. Since G is of odd order, we conclude that $o(G - S) \geq |S| + 1$. By Theorem 6, $|S| \geq 2$. So $o(G - S) \geq 3$. From Lemma 7, we obtain that $|S| \geq 3$. Then $o(G - S) \geq 4$. Let G_1, G_2, G_3 and G_4 be four odd components of $G - S$. We proceed with Fact 1.

Fact 1. For any vertex $x \in S$, $|S_x| = 3$ and $S_x \subseteq S$.

Proof of Fact 1. Since $o(G - S) \geq 4$, clearly for any $x \in S$, $|S_x \cap S| \geq 2$. Assume to the contrary that there is a vertex $x \in S$ such that $|S_x \cap S| = 2$. Let $S_x \cap S = \{a, b\}$. If a has some neighbor in at least three components of $G - S$, then there is a $K_{1,3}$ with center a, a contradiction. Thus the neighbors of a in $G-S$ are in at most two components. Similarly the neighbors of b in $G-S$ are in at most two components. Thus we may assume without loss of generality that a is adjacent to all vertices of G_1 and G_2, and b is adjacent to all vertices of G_3 and a vertex of G_4. (since maybe $|S_x| = 3$ and the third vertex of S_x be in G_4). Now we have a $K_{1,3}$ with center a and a leaf in G_1, a leaf in G_2 and a leaf in G_3, a contradiction.□

An immediate consequent is that $|S| \geq 4$, and thus $o(G-S) \geq 5$. Since G is claw-free, each of a, b, and c has neighbors in at most two components of $G - S$. In particular, a is adjacent to some vertex in $G - S$. By Fact 1 and Lemma 3, $S_a \cap \{b, c\} = \emptyset$. This implies that $|S| \geq 6$ and thus $o(G - S) \geq 7$. Now we have a $K_{1,3}$ with center a, b, or c, a contradiction. ■

Now we study $K_{1,4}$-free graphs. We first investigate some properties of cut-sets.

Lemma 9 *If S is a cut-set of size 3 in a $4 - \gamma_{tr}$-vertex critical $K_{1,4}$-free graph G, then $o(G - S) < 4$.*

Proof. Let G be a $4-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph and let S be a cut-set of size 3. Suppose to the contrary, that $o(G-S) \geq 4$. Let G_1, G_2, G_3, G_4 be four odd components of $G - S$. Suppose that $S = \{x, y, z\}$. If $|S_x \cap S| = 1$, then $S_x \cap S$ dominates the vertices of at least four components of $G - S$, giving a $K_{1,4}$, a contradiction. So $|S_x \cap S| = 2$, and so $S_x \cap S = \{y, z\}$. Similarly, $S_y \cap S = \{x, z\}$ and $S_z \cap S = \{x, y\}$.

If y is adjacent to z, then we deduce from $S_y \cap S = \{x, z\}$ and Lemma 3, that $N(y) \subseteq S_y$. According to Lemma 5, we have $deg(y) \geq 3$. So $deg(y) = 3$ and $S_y = N(y) = \{x, z, w_1\}$, where $w_1 \notin S$. Similarly, $S_z = N(z) = \{x, y, w_2\}$, where $w_2 \notin S$. Let $w_1, w_2 \in V(G_1) \cup V(G_2)$. We conclude that S_x does not dominate $V(G_3) \cup V(G_4)$, a contradiction.

Thus y is not adjacent to z. Similarly, $x \notin N(y) \cup N(z)$, and therefore S is an independent set. Let $w \in V(G) \setminus S$. We observe that $S_w \cap S \neq \emptyset$, and since $G[S_w]$ is connected, we find that $N(x) \cap N(y) \cap N(z) \neq \emptyset$. Let $a \in N(x) \cap N(y) \cap N(z)$. Since $S_a \cap S \neq \emptyset$, by Lemma 3 we obtain that $S_a = \{x, y, z\}$ and $deg(a) = 3$. This contradicts the fact that S is independent. ■

Lemma 10 *Let S be a cut-set of size 4 in a $4-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph G. If $c(G-S) \geq 5$, then for any vertex $v \in V(G) \setminus S$, $|S_v| = 3$ and $S_v \subseteq S$.*

Proof. Let G be a $4-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph, and let S be a cut-set of size 4. Suppose that $c(G-S) \geq 5$. Since G is $K_{1,4}$-free, for any vertex $x \in V(G)$, $|S_x \cap S| \geq 2$. Assume to the contrary, that there is a vertex $v \in V(G) \setminus S$ such that $|S_v \cap S| = 2$. Let $S = \{x, y, z, w\}$, and let $G_1, G_2, ..., G_5$ be five components of $G - S$. Let $v \in V(G_1)$ and $S_v \cap S = \{x, y\}$.

If $V(G_1) = \{v\}$ then $N(v) \subseteq S$, and so we have $S_v \cap N_G(v) \neq \emptyset$ by Lemma 5. Hence Lemma 3 implies that $N_G(v) \subseteq S_v$, and we obtain the contradiction $|S_v \cap S| = 3$. So $|V(G_1)| > 1$.

Let $G' = G - v$. If $c(G-S) \geq 6$, then $c(G'-S) \geq 6$. In this case G' has an induced $K_{1,4}$, and so G has an induced $K_{1,4}$. This is a contradiction. We deduce that $c(G-S) = 5$.

We show that $|S_v| = 3$. Assume that $|S_v| = 2$. Then $S_v = \{x, y\}$ and x and y are adjacent. Since G is $K_{1,4}$-free, $N(x) \cap ((V(G_1) \cup ... \cup V(G_5)) - \{v\}) \neq \emptyset$ and $N(y) \cap ((V(G_1) \cup ... \cup V(G_5)) - \{v\}) \neq \emptyset$. If $\{z, w\} \subseteq N(x)$, then by Lemma 3, $|S_x| \geq 4$, a contradiction. Thus $\{z, w\} \not\subseteq N(x)$, and similarly $\{z, w\} \not\subseteq N(y)$. Thus we may assume that $z \in N(x)$ and $w \in N(y)$. If $deg(y) \geq 4$, then by Lemma 3, $|S_y| \geq 4$, a contradiction. Thus $deg(y) = 3$ and similarly $deg(x) = 4$. But then $\{x, y\}$ does not dominate $G - v$, a contradiction. Thus $|S_v| = 3$.

We next show that $x \notin N(y)$. Assume that $x \in N(y)$. Since $G-S$ has five components, and $|S_v \cap S| = 2$, we may assume that one of x or y is adjacent to some vertex in three components of $G - S$. Without loss of generality, assume that y is adjacent to some vertex in each of G_3, G_4 and G_5. Then x is adjacent to any vertex in G_2 and $G_1 - v$. If there are $x_1 \in (N(y) \cap V(G_3)) - N(x)$, $x_2 \in (N(y) \cap V(G_4)) - N(x)$ and $x_3 \in (N(y) \cap V(G_5)) - N(x)$, then $G[\{x, y, x_1, x_2, x_3\}]$ is a $K_{1,4}$, a contradiction. Thus we may assume, without loss of generality, that x is adjacent to any vertex of G_5. By Lemma 3, $\{z,w\} \subseteq S_y$, and $S_y \cap S = \{z,w\}$. Let $a \in N(y) \cap V(G_5)$. Since S_y dominates G_5 we may assume that z is adjacent to a. Now $\{x,y,z\} \subseteq N(a)$. Then by Lemma 3, $S_a = \{x,y,z\}$, since G is $K_{1,4}$-free. This implies that $z \in N(x)$. But then $|S_x| \geq 4$, a contradiction.

Thus x is not adjacent to y.

There is a vertex $z_1 \in V(G) \setminus S$ such that $S_v = \{x,y,z_1\}$. So z_1 is adjacent to both x and y. We show that $N(z_1) \cap \{z,w\} = \emptyset$. It is easy to see that $\{z,w\} \not\subseteq N(z_1)$. Suppose to the contrary that $N(z_1) \cap \{z,w\} \neq \emptyset$. Assume that $z \in N(z_1)$. This implies that $S_{z_1} = \{x,y,z\}$. As an immediate result z is adjacent to both x and y. But then $S_z \cap N(z) \neq \emptyset$, and so by Lemma 3, $S_z = \{x,y,z_1\}$. Further, $N(z_1) = \{x,y,z\}$. So w is adjacent to either x or y. Suppose that w is adjacent to x. Then $N(x) \subseteq S_x$. It follows that $|S_x| > 3$, a contradiction. We conclude that $N(z_1) \cap \{z,w\} = \emptyset$. So $\{z,w\} \subseteq N(x) \cup N(y)$.

Since $|S_x| \leq 3$ and $|S_y| \leq 3$, we have $\{z,w\} \not\subseteq N(x)$, and $\{z,w\} \not\subseteq N(y)$. Thus we may assume that $w \in N(x)$ and $z \in N(y)$. It follows that $\{x,w\} \subseteq S_y$, $\{y,z\} \subseteq S_x$. Further, $S_y \cap S = \{x,w\}$, $S_x \cap S = \{y,z\}$. This provides an induced $K_{1,4}$ since $c(G-S) = 5$, a contradiction. ■

Lemma 11 *If S is a cut-set of size 4 in a $4-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph G, then $o(G-S) < 5$.*

Proof. Let G be a $4-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph and let S be a cut-set of size 4. Suppose to the contrary, that $o(G-S) \geq 5$. Let $S = \{x,y,z,w\}$ and let $v \in V(G) \setminus S$. By Lemma 10, $|S_v| = 3$ and $S_v \subseteq S$. Suppose that $S_v = \{x,y,z\}$, where y is adjacent to both x and z. If y is adjacent to w, then by Lemma 3, we conclude that $deg_G(y) = 3$ and $N(y) = \{x,z,w\}$, since $S_y \cap S \neq \emptyset$. But then $S_1 = \{x,z,w\}$ is a cut-set with

$o(G - S_1) \geq 6$, contradicting Lemma 9. So y is not adjacent to w. Since w is dominated by $S_v = \{x, y, z\}$, we may assume, without loss of generality, that z is adjacent to w. Similarly (similar to the case that y is not adjacent to w) we see that z is not adjacent to x. Since G is $K_{1,4}$-free, we observe that $|S_z \cap S| \geq 2$. Now Lemma 3 implies that $N_G(z) \subseteq S_z$. According to Lemma 5, $deg(z) \geq 3$ and hence $deg(z) = 3$. Since z and x are not adjacent, there is a vertex $a \in V(G) - S$ which is adjacent to z. Thus $S_z = \{y, w, a\}$. Since y and w are not adjacent and $G[S_z]$ has no isolated vertex, we see that a is adjacent to y, z, and w. By Lemma 10, $S_a = \{y, z, w\}$. Now since $|S_y \cap S| \geq 2$, by Lemma 3, $S_y = \{x, z, a\}$. In particular, $deg(y) = 3$. But from $S_a = \{y, z, w\}$, we deduce that w is adjacent to all vertices of $G - (S \cup \{a\})$. This produces a $K_{1,4}$, a contradiction. ■

Lemma 12 *Let S be a cut-set of size at least 5 in a $4 - \gamma_{tr}$-vertex critical $K_{1,4}$-free graph G. If $c(G - S) \geq 6$, then for any vertex $v \in V(G)$, $|S_v| = 3$ and $S_v \subseteq S$.*

Proof. Let G be a $4 - \gamma_{tr}$-vertex critical $K_{1,4}$-free graph and let S be a cut-set of size at least 5. Suppose that $c(G - S) \geq 6$. It can be easily seen that for any vertex $x \in V(G)$, $|S_x \cap S| \geq 2$. Assume to the contrary, that there is a vertex $y \in V(G)$ such that $|S_y \cap S| = 2$. It follows that any vertex of $S_y \cap S$ dominates the vertices of at least three components of $G - S$. This yields an induced $K_{1,4}$, a contradiction. ■

Lemma 13 *If S is a cut-set of size 5 in a $4 - \gamma_{tr}$-vertex critical $K_{1,4}$-free graph G, then $o(G - S) < 6$.*

Proof. Let G be a $4 - \gamma_{tr}$-vertex critical $K_{1,4}$-free graph and let S be a cut-set of size 5. Suppose to the contrary, that $o(G - S) \geq 6$. Let $y \in S$ be a vertex such that $deg_{G[S]}(y) = \Delta(G[S])$. If $deg_{G[S]}(y) \geq 4$, then by Lemma 3, $|S_y| \geq 4$, a contradiction.

Thus $\Delta(G[S]) \leq 3$. If $deg_{G[S]}(y) = 3$, then by Lemma 12, $|S_y| = 3$ and $deg_G(y) = 3$. Then $S_1 = S \setminus \{y\}$ is a cut-set with $o(G - S_1) \geq 6$, contradicting Lemma 11. We deduce that $deg_{G[S]}(y) = \Delta(G[S]) \leq 2$. By Lemma 12, $deg_{G[S]}(y) = \Delta(G[S]) = 2$. But then $S_y \not\subseteq S$ contradicting Lemma 12. ■

Now we are ready to give the main results of this paper.

Theorem 14 *Any* $4-\gamma_{tr}$*-vertex critical* $K_{1,4}$*-free graph of even order has a perfect matching.*

Proof. Let G be a $4-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph of even order. Suppose to the contrary that G has no perfect matching. By Theorem 1, there is a subset $S \subseteq V(G)$ such that $o(G-S) \geq |S| + 1$. Since G is of even order, we conclude that $o(G - S) \geq |S|+2$. Since $\kappa(G) \geq 2$, we observe that $|S| \geq 2$. By Theorem 7 and Lemmas 9, 11, 13, we deduce that $|S| \geq 6$. Thus $o(G-S) \geq 8$. Since G is $K_{1,4}$-free, it follows from Lemma 12 that for any vertex $v \in S$, $|S_v| = 3$ and $S_v \subseteq S$. We consider the following cases.

Case 1. Assume that $|S| = 6$. It follows that $o(G - S) \geq 8$. Now let u be a vertex of $G - S$. By Lemma 12, $S_u \subseteq S$ and $|S_u| = 3$. Since $G[S_u]$ is connected, a vertex of S_u is adjacent to the other two vertices of S_u. Now S_u dominates S, and there are three other vertices in S. Thus there is a vertex u_1 in S_u that is adjacent to at least three vertices of S. If $deg_{G[S]}(u_1) \geq 4$, then by Lemma 3, $|S_{u_1}| \geq 4$, a contradiction. Thus $deg_{G[S]}(u_1) = 3$. But $|S_{u_1}| = 3$ and $S_{u_1} \subseteq S$. Thus by Lemma 3, $deg_G(u_1) = 3$. Now $S_1 = S \setminus \{u_1\}$ is a cut-set with $o(G-S_1) \geq 8$, contradicting Lemma 13.

Case 2. Assume that $|S| \geq 7$. It follows that $o(G - S) \geq 9$. Let $G_1, G_2, ..., G_9$ be nine components of $G - S$. Let $v \in V(G_1)$. By Lemma 12, $|S_v| = 3$ and $S_v \subseteq S$. Let $S_v = \{x, y, z\}$, where y is adjacent to both x and z. Since G is $K_{1,4}$-free, we find that

$o(G-S)=9$ and $|S|=7$. If $|V(G_1)|>1$, then $G-(S\cup\{v\})$ has exactly nine components, and any vertex of S_v dominates the vertices of exactly three components of $G-(S\cup\{v\})$. This implies that $(N(x)\cap N(y))\cap(V(G)-(S\cup\{v\}))=(N(y)\cap N(z))\cap(V(G)-(S\cup\{v\}))=\emptyset$. Now we can obtain an induced $K_{1,4}$, for example with center x and leaf y. Thus we assume that $|V(G_1)|=1$. Then $G-(S\cup\{v\})$ has eight components $G_2, G_3, ..., G_9$. It is obvious that each of x, y and z can dominates at most three components. If $(N(x)\cap N(y))\cap(V(G_2)\cup...\cup V(G_9))\neq\emptyset$ and $(N(y)\cap N(z))\cap(V(G_2)\cup...\cup V(G_9))\neq\emptyset$, then S_v dominates at most 7 components of $G-S$, a contradiction. Thus we may assume, without loss of generality, that $(N(x)\cap N(y))\cap(V(G_2)\cup...\cup V(G_9))=\emptyset$. Since $\{x,y\}$ dominates at least five components of $G-S$, there is a $K_{1,4}$ with center x and leaf y or with center y and leaf x, a contradiction. ■

Theorem 15 *Any* $4-\gamma_{tr}$*-vertex critical* $K_{1,4}$*-free graph of odd order is factor-critical.*

Proof. Let G be a $4-\gamma_{tr}$-vertex critical $K_{1,4}$-free graph of odd order. Suppose to the contrary that G is not factor critical. By Theorem 2, there is a subset $S\subseteq V(G)$ such that $o(G-S)\geq|S|$. Since G is of odd order, we conclude that $o(G-S)\geq|S|+1$. By Theorem 6, $|S|\geq 2$. So $o(G-S)\geq 3$. It follows from Theorem 7 that $|S|\geq 3$. But then we use Lemmas 9, 11 and 13 to deduce that $|S|\geq 6$. This implies that $o(G-S)\geq 7$. By Lemma 12, for any vertex $v\in V(G)$, $|S_v|=3$ and $S_v\subseteq S$. We consider the following cases.

Case 1. Assume that $|S|=6$. If there is a vertex $v\in S$ such that v is not adjacent to a vertex in $G-S$, then $S_1=S-\{v\}$ is a vertex cut set with $o(G-S_1)\geq 8$, contradicting Lemma 13. Thus for any vertex $v\in S$, each vertex of S_v is adjacent to some vertex in $G-S$. Now Lemma 3 implies that for any vertex $v\in S$, $S_v\cap N[v]=\emptyset$. It follows that $\Delta(G[S])=2$. Let $S=\{v_1,v_2,...,v_6\}$ and let v_2 be adjacent to v_1 and v_3. Then

$S_{v_2} = \{v_4, v_5, v_6\}$, and we may assume that v_5 is adjacent to v_4 and v_6. Since S_{v_2} dominates $\{v_1, v_3\}$, we may assume that $v_1 \in N(v_6)$ and $v_4 \in N(v_3)$. Thus $G[S]$ is a cycle on six vertices. Let $w \in G - S$. By Lemma 12, $S_w \subseteq S$. But then S_w cannot dominate S, a contradiction.

Case 2. Assume that $|S| \geq 7$. Then $o(G - S) \geq 8$. Let $v \in S$. By Lemma 12, we assume that $S_v = \{x, y, z\} \subseteq S$, where y is adjacent to x and z. Since G is $K_{1,4}$-free, any vertex of S_v dominates at most three components of $G-S$. If $(N(x) \cap N(y)) \cap (V(G) - S) \neq \emptyset$ and $(N(y) \cap N(z)) \cap (V(G) - S) \neq \emptyset$, then S_v dominates at most 7 components of $G-S$, a contradiction. Thus we may assume, without loss of generality, that $(N(x) \cap N(y)) \cap (V(G)-S) = \emptyset$. Since $\{x, y\}$ dominates at least five components of $G - S$, there is a $K_{1,4}$ with center x and leaf y or with center y and leaf x, a contradiction. ■

References

[1] N. Ananchuen and M. D. Plummer, *Matching properties in domination k-critical graphs*, Discrete Math. 277 (2004), 1-13.

[2] X. Chen, D. Xiang and L. Sun, *On total restrained domination in graphs*, Czechoslovak Mathematical Journal, 55 (2005), 165-173.

[3] J. Cyman and J. Raczek, *On the total restrained domination number of a graph*, Australasian Journal of Combinatorics, 36 (2006), 91-100.

[4] O. Favaron, D. Sumner and E. Wojcicka, *The diameter of domination k-critical graphs*, J. Graph Theory 18 (1994), 723-734.

[5] N. Jafari Rad, *Some properties in total restrained domination critical graphs*, Ars Combinatorics, To appear.

[6] R. Gera, J. H. Hattingh, N. Jafari Rad, E. J. Joubert, and Lucas C. van der Merwe, *Vertex and edge critical total restrained domination in graphs*, Bulletin of ICA, 57 (2009), 107-117

[7] J. H. Hattingh, E. Jonck, E. J. Joubert and A. Rimmer, *Total restrained domination in trees*, Discrete Math. 307 (2007), 1643-1650.

[8] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, editors. *Famental of domination in graphs*: Advanced Topics. Marcel Dekker, Inc, New York, NY, 1998.

[9] L. Lovasz, and M.D. Plummer, *Matching Theory*, North-Holland Inc., Amsterdam, 1986.

[10] D. P. Sumner, *Critical concepts in domination*, Discrete Math. 86 (1990), 33-46.

[11] W.T. Tutte, *The factorization of linear graphs*, J. London Math. Soc. 22 (1947), 107-111.

[12] D. B. West, *Introduction to graph theory*, (2nd edition), Prentice all, USA (2001).