Lattices associated with Hamming graphs

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Abstract

Hamming graph $H(n,k)$ has as vertex set all words of length n with symbols taken from a set of k elements. Suppose L denotes the set $\bigcup_{l=0}^{n+1}\Omega_l$, with $\Omega_l = \{\sum_{i\in I_1} e_i^1 + \sum_{i\in I_2} e_i^2 + \cdots + \sum_{i\in I_k} e_i^k \mid I_j \cap I_{j'} = \emptyset\,(j \neq j'), |\bigcup_{j=1}^k I_j| = l\}$ for $0 \leq l \leq n$ and $\Omega_{n+1} := \{\hat{1}\}$. For any two element $x, y \in L$, define $x \leq y$ if and only if $y = \hat{1}$ or $I_j^x \subseteq I_j^y$ for $1 \leq j \leq k$. Then L is a lattice, denoted by L_O. Reversing the above partial order, we obtain the dual of L_O, denoted by L_R. This article discusses their geometric properties, and computes their characteristic polynomials.

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1 Introduction

Hamming graph $H(n,k)$ has as vertex set all words of length n with symbols taken from a set of k elements. We will take as our set of k elements the set $\{a_1, a_2, \cdots, a_k\}$.

Let e_i^j be the vector with n coordinates that has a a_j in position i and 0 elsewhere. Then, each word in $H(n,k)$ is simply a sum of some $e_i^1, e_i^2, \cdots, e_i^k$ $(i \in I_j \text{ for } 0 \leq j \leq k)$, with the only restrictions on I_j's that $I_j \cap I_{j'} = \emptyset$ $(j \neq j')$, and $\bigcup_{j=1}^k I_j = \{1, 2, \cdots, n\}$.

For $0 \leq l \leq n$ we set $\Omega_l = \{\sum_{i\in I_1} e_i^1 + \sum_{i\in I_2} e_i^2 + \cdots + \sum_{i\in I_k} e_i^k \mid I_j \cap I_{j'} = \emptyset\,(j \neq j'), |\bigcup_{j=1}^k I_j| = l\}$. Given any $x \in \Omega_l$, we represent $x = (I_1^x, I_2^x, \cdots, I_k^x)$ where $x = \sum_{i\in I_1^x} e_i^1 + \sum_{i\in I_2^x} e_i^2 + \cdots + \sum_{i\in I_k^x} e_i^k$, $I_j^x \cap I_{j'}^x = \emptyset\,(j \neq j')$ and $|\bigcup_{j=1}^k I_j^x| = l$.

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Notice that $\Omega_0 := \{(0, 0, \cdots, 0)\}$ and we add a dummy element $\hat{1}$ above all other elements, defining $\Omega_{n+1} := \{\hat{1}\}$, that is $x \leq \hat{1}$, $\forall x \in \bigcup_{l=0}^{n} \Omega_l$.

Suppose L denotes the set $\bigcup_{l=0}^{n+1} \Omega_l$. For any two elements $x, y \in L$, define $x \leq y$ if and only if $y = \hat{1}$ or $I_j^x \subseteq I_j^y$ for $0 \leq j \leq k$. Then L is a finite poset, denoted by L_O. For any two elements $x, y \in L$, define $x \leq y$ if and only if $y = (0, \cdots 0)$ or $I_j^y \subseteq I_j^x$ for $0 \leq j \leq k$. Then L is a finite poset, denoted by L_R.

For any two elements $x, y \in L_O$,

$$x \wedge y = (I_1^x \cap I_1^y, I_2^x \cap I_2^y, \cdots, I_k^x \cap I_k^y),$$

$$x \vee y = \begin{cases} (I_1^x \cup I_1^y, I_2^x \cup I_2^y, \cdots, I_k^x \cup I_k^y), & \text{if } (I_j^x \cup I_j^y) \cap (I_{j'}^x \cup I_{j'}^y) = \emptyset (j \neq j'), \\ \hat{1}, & \text{otherwise.} \end{cases}$$

Similarly, for any two elements $x, y \in L_R$,

$$x \vee y = (I_1^x \cap I_1^y, I_2^x \cap I_2^y, \cdots, I_k^x \cap I_k^y),$$

$$x \wedge y = \begin{cases} (I_1^x \cup I_1^y, I_2^x \cup I_2^y, \cdots, I_k^x \cup I_k^y), & \text{if } (I_j^x \cup I_j^y) \cap (I_{j'}^x \cup I_{j'}^y) = \emptyset (j \neq j'), \\ \hat{1}, & \text{otherwise.} \end{cases}$$

Therefore, both L_O and L_R are finite lattices.

Y. Huo, Y. Liu and Z. Wan ([3, 4, 5, 6, 7]) constructed lattices from orbits of subspaces under finite classical groups. K. Wang and Y. Feng constructed lattices from orbits of flats under affine groups. K. Wang and Z. Li [11] constructed lattices from vector spaces over a finite field. In this paper, we construct two families of lattices from Hamming graphs, compute their characteristic polynomials and discuss their geometric properties.

2 Preliminaries

We recall some terminologies and definitions about finite posets and lattices. For more theory about finite posets and lattices, we would like to refer readers to [1, 9].

Let P be a poset with partial order $\leq$. As usual, we write $a < b$ whenever $a \leq b$ and $a \neq b$. For any two elements $a, b \in P$, we say b *covers* a, denoted by $a < \cdot b$, if $a < b$ and there exists no any $c \in P$ such that $a < c < b$. Let P be a finite poset with the minimum element, denoted by 0. By a rank function on P, we mean a function r from P to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) - 1$ whenever $a < \cdot b$. Observe the rank function of P is unique if it exists. Let P be a finite poset with 0 and 1. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}$$

is called the *characteristic polynomial* of P, where r is the rank function of P.

A poset L is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in L$. Let L be a finite lattice with 0. By an *atom* of L, we mean an element of L covering 0. We say L is *atomic* if any element in $L \setminus \{0\}$ is a union of atoms. A finite atomic lattice L is said to be a *geometric lattice* if L admits a rank function r satisfying

$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b), \forall a,\ b \in L.$$

3 The lattice L_O

The lattice L_O has the minimum element $\hat{0} = (0, 0, \cdots, 0)$, and the maximum element $\hat{1}$. The set of all the atoms of L_O is Ω_1.

Theorem 3.1 *The lattice L_O has the following properties:*

(i) L_O is a finite atomic lattice, that is every element of the lattice is a join of atoms.

(ii) $\forall u, w \in L_O$ such that $u \vee w \neq \hat{1} \Rightarrow r(u \wedge w) + r(u \vee w) = r(u) + r(w)$.

Proof. For any $z \in L_O$, define

$$r(z) = \begin{cases} n+1, & \text{if } z = \hat{1}, \\ |I_1^z| + |I_2^z| + \cdots + |I_k^z|, & \text{otherwise} . \end{cases}$$

Then r is the rank function of L_O.

(i) $\hat{1} = e_1^1 \vee e_1^2$ and an element $z \neq \hat{1}$ of the lattice is of the form

$$z = \sum_{i \in I_1^z} e_i^1 + \sum_{i \in I_2^z} e_i^2 + \cdots + \sum_{i \in I_k^z} e_i^k, I_j^z \cap I_{j'}^z = \emptyset (j \neq j'),$$

so $z = (\bigvee_{i \in I_1^z} e_i^1) \bigvee (\bigvee_{i \in I_2^z} e_i^2) \bigvee \cdots \bigvee (\bigvee_{i \in I_k^z} e_i^k)$ is a join of atoms.

(ii) $r(z) = |I_1^z| + |I_2^z| + \cdots + |I_k^z|$, so the formula is true because of the inclusion-exclusion formula of sets.

Lemma 3.2 *The Möbius function of L_O is*

$$\mu(x, y) = \begin{cases} (-1)^{r(y) - r(x)}, & \text{if } x \leq y \neq \hat{1} \text{ or } x = y = \hat{1}, \\ -(1-k)^{n - r(x)}, & \text{if } x < y = \hat{1}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The Möbius function of L_O is

$$\mu(x,y) = \begin{cases} (-1)^{r(y)-r(x)}, & \text{if } x \leq y \neq \hat{1} \text{ or } x = y = \hat{1}, \\ -\sum_{x \leq z < y} \mu(x,z), & \text{if } x < y = \hat{1}, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} & -\sum_{\hat{0} \leq z < \hat{1}} \mu(\hat{0}, z) \\ = & -\textstyle\sum_{i=0}^{n} k^i C_n^i (-1)^i \\ = & -(1-k)^n \end{aligned}$$

and

$$\begin{aligned} & -\sum_{\hat{0} \neq x \leq z < \hat{1}} \mu(x, z) \\ = & -\textstyle\sum_{i=0}^{n-r(x)} k^i C_{n-r(x)}^i (-1)^i \\ = & -(1-k)^{n-r(x)}. \end{aligned}$$

Hence the desired result follows. □

Theorem 3.3 *The characteristic polynomial of* L_O *is*

$$\chi(L_O, x) = -(1-k)^n + x(x-k)^n.$$

Proof. By Lemma 3.2, we obtain

$$\begin{aligned} & \chi(L_O, x) \\ = & \sum_{\hat{0} \leq y \leq \hat{1}} \mu(\hat{0}, y) x^{n+1-r(y)} \\ = & \mu(\hat{0}, \hat{1}) + \sum_{\hat{0} \leq y < \hat{1}} \mu(\hat{0}, y) x^{n+1-r(y)} \\ = & -(1-k)^n + \sum_{i=0}^{n} k^i C_n^i (-1)^i x^{n+1-i} \\ = & -(1-k)^n + x(x-k)^n, \end{aligned}$$

as desired. □

4 The lattice L_R

The lattice L_R has the minimum element $\hat{1}$, and the maximum element $\hat{0} = (0, 0, \cdots, 0)$. The set of all the atoms of L_R is Ω_n.

Theorem 4.1 *The lattice* L_R *has the following properties:*

(i) L_R *is a finite atomic lattice, that is every element of the lattice is a join of atoms.*

(ii) $\forall u, w \in L_R$ *such that* $u \vee w \neq \hat{0} \Rightarrow r(u \wedge w) + r(u \vee w) = r(u) + r(w)$.

Proof. For any $z \in L_R$, define

$$r(z) = \begin{cases} n+1, & \text{if } z = \hat{0}, \\ n+1-(|I_1^z| + |I_2^z| + \cdots + |I_k^z|), & \text{otherwise}, \\ 0, & \text{if } z = \hat{1}, \end{cases}$$

Then r is the rank function of L_R.

(i) $\hat{0} = e_1^1 \vee e_1^2$ and an element $z \neq \hat{0}$ of the lattice is of the form

$$z = \sum_{i \in I_1^z} e_i^1 + \sum_{i \in I_2^z} e_i^2 + \cdots + \sum_{i \in I_k^z} e_i^k,\ |I_1^z| + |I_2^z| + \cdots + |I_k^z| = l,$$

there are $x, y \in \Omega_n$,

$$x = \sum_{i \in I_1^x} e_i^1 + \sum_{i \in I_2^x} e_i^2 + \cdots + \sum_{i \in I_k^x} e_i^k,\ |I_1^x| + |I_2^x| + \cdots + |I_k^x| = n,$$

$$y = \sum_{i \in I_1^y} e_i^1 + \sum_{i \in I_2^y} e_i^2 + \cdots + \sum_{i \in I_k^y} e_i^k,\ |I_1^y| + |I_2^y| + \cdots + |I_k^y| = n,$$

with

$$I_1^z = I_1^x \cap I_1^y,\ I_2^z = I_2^x \cap I_2^y,\ \cdots,\ I_k^z = I_k^x \cap I_k^y,$$

then $z = x \vee y$ is a join of atoms.

(ii) $r(z) = n+1-(|I_1^z| + |I_2^z| + \cdots + |I_k^z|)$, so the formula is true because of the inclusion-exclusion formula of sets.

Lemma 4.2 *The Möbius function of* L_R *is*

$$\mu(x,y) = \begin{cases} (-1)^{r(x)-r(y)}, & \text{if } \hat{1} \neq x \leq y \text{ or } x = y = \hat{1}, \\ -(1-k)^{r(y)-1}, & \text{if } \hat{1} = x < y, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. The Möbius function of L_R is

$$\mu(x,y) = \begin{cases} (-1)^{r(x)-r(y)}, & \text{if } \hat{1} \neq x \leq y \text{ or } x = y = \hat{1}, \\ -\sum_{x<z\leq y} \mu(z,y), & \text{if } \hat{1} = x < y, \\ 0, & \text{otherwise}. \end{cases}$$

We have

$$\begin{aligned} & -\sum_{\hat{1}<z\leq y} \mu(z,y) \\ = & -\textstyle\sum_{i=0}^{r(y)-1} k^i C_{r(y)-1}^i (-1)^i \\ = & -(1-k)^{r(y)-1}. \end{aligned}$$

Hence the desired result follows. □

Theorem 4.3 *The characteristic polynomial of L_R is*

$$\chi(L_R, x) = x^{n+1} - (1 - k + kx)^n.$$

Proof. By Lemma 4.2, we obtain

$$\begin{aligned} & \chi(L_R, x) \\ = & \sum_{\hat{1} \leq y \leq \hat{0}} \mu(\hat{1}, y) x^{n+1-r(y)} \\ = & \mu(\hat{1}, \hat{1}) x^{n+1} + \sum_{\hat{1} < y \leq \hat{0}} \mu(\hat{1}, y) x^{n+1-r(y)} \\ = & x^{n+1} - \sum_{i=0}^{n} k^i C_n^i (1-k)^{n-i} x^i \\ = & x^{n+1} - (1 - k + kx)^n, \end{aligned}$$

as desired. □

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