# Tournaments whose indecomposability graph admits a vertex cover of size 2

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**Abstract.** Given a tournament $T = (V, A)$, a subset $X$ of $V$ is an interval of $T$ provided that for any $a, b \in X$ and $x \in V - X$, $(a, x) \in A$ if and only if $(b, x) \in A$. For example, $\emptyset$, $\{x\}$ $(x \in V)$ and $V$ are intervals of $T$, called trivial intervals. A tournament whose intervals are trivial is indecomposable; otherwise, it is decomposable. With each indecomposable tournament $T$, we associate its indecomposability graph $\mathbb{I}(T)$ defined as follows: the vertices of $\mathbb{I}(T)$ are those of $T$ and its edges are the unordered pairs of distinct vertices $\{x, y\}$ such that $T - \{x, y\}$ is indecomposable. We characterize the indecomposable tournaments $T$ whose $\mathbb{I}(T)$ admits a vertex cover of size 2.
**Keywords:** Indecomposability graph, Interval, Indecomposable tournament.

## 1 Introduction

A *graph* $G$ is defined by a finite vertex set $V(G)$ and an edge set $E(G)$, where an edge is an unordered pair of distinct vertices. Such a graph is denoted by $(V(G), E(G))$ or simply $(V, E)$. Given a graph $G$, a subset $X$ of $V(G)$ is called a *vertex cover* of $G$ if for each edge $a \in E(G)$, $a \cap X \neq \emptyset$. We say also that $G$ is *X-covered.* Given a graph $G$, consider an integer $k \geq 1$. We say that $G$ is *k-covered* if it is X-covered by a subset $X$ of $V(G)$ with $\mid X \mid = k$.

A *tournament* $T = (V(T), A(T))$ or simply $(V, A)$ consists of a finite *vertex* set $V$ with an *arc* set $A$ of ordered pairs of distinct vertices satisfying: for $x, y \in V$, with $x \neq y$, $(x, y) \in A$ if and only if $(y, x) \notin A$. The *cardinality* of T, denoted by $v(T)$, is that of $V(T)$. Let $T = (V, A)$ be a tournament. For any (distinct) vertices $x$, $y$ of $V$, the notation $x \longrightarrow y$ signifies that $(x, y) \in A$. For any disjoint subsets $I$ and $J$ of $V$, we denote by $I \longrightarrow J$ whenever for each $(x, y) \in I \times J$, $x \longrightarrow y$. Similarly, for each $x \in V$ and for each $Y \subseteq V - \{x\}$, $x \longrightarrow Y$ (resp. $Y \longrightarrow x$) signifies that $x \longrightarrow y$ (resp. $y \longrightarrow x$) for each $y \in Y$. Furthermore $x \sim Y$ means $x \longrightarrow Y$ or $Y \longrightarrow x$. The negation is denoted by $x \not\sim Y$. For each subset $X$ of $V$, set $N_T^-(X) = \{y \in V : y \longrightarrow X\}$ and $N_T^+(X) = \{y \in V : X \longrightarrow y\}$. For convenience, given $x \in V$, $N_T^-(\{x\})$ is denoted by $N_T^-(x)$, and $N_T^+(\{x\})$ is denoted by $N_T^+(x)$. A *transitive* tournament or a *total order* is a tournament $T$ such that for $x$, $y$, $z \in V(T)$, if $x \longrightarrow y$ and $y \longrightarrow z$, then $x \longrightarrow z$. For $m \geq 1$, set $\mathbb{N}_m = \{1, \ldots, m\}$.

The notions of isomorphism, subtournament and embedding are defined in the following manner. First, let $T = (V, A)$ and $T' = (V', A')$ be two tournaments. A one-to-one correspondence $f$ from $V$ onto $V'$ is an *isomorphism* from $T$ onto $T'$ provided that for $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. The tournaments $T$ and $T'$ are then said to be *isomorphic*, if there is an isomorphism from one onto the other, which is denoted by $T \simeq T'$. Second, given a tournament $T = (V, A)$, with each subset $X$ of $V$ is associated the *subtournament* $T[X] = (X, A \cap (X \times X))$ of $T$ *induced* by $X$. For $X \subseteq V$ (resp. $x \in V$), the subtournament $T[V - X]$ (resp. $T[V - \{x\}]$) is denoted by $T - X$ (resp. $T - x$). For tournaments $T$ and $T'$, if $T'$ is isomorphic to a subtournament of $T$, we say that $T'$ embeds into $T$. The *dual* of $T$ is the tournament $T^\star = (V, \{(x, y) : (y, x) \in A\})$. The tournament $T$ is then said to be *self-dual* if $T \simeq T^\star$. Given a class $\mathcal{C}$ of tournaments, $\mathcal{C}^\star$ denotes the class $\{T^\star : T \in \mathcal{C}\}$.

The indecomposability plays an important role in this paper. Given a tournament $T = (V, A)$, a subset $I$ of $V$ is an *interval* [7, 9, 11] (or a *clan* [6]) of $T$ provided that for every $x \in V \setminus I$, $x \sim I$. This definition generalizes the notion of interval of a total order. Given a tournament $T = (V, A)$, $\emptyset$, $V$ and $\{x\}$, where $x \in V$, are clearly intervals of $T$, called *trivial* intervals. A tournament is then said to be *indecomposable* [1, 9] (or *primitive* [6]) if all of its intervals are trivial. It is said to be *decomposable* otherwise. For example, the 3-cycle $P_3 = (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$ is indecomposable whereas a total order of cardinality $\geq 3$ is decomposable. The tournaments $T$ and $T^\star$ have the same intervals. Thus, $T$ is indecomposable if and only if $T^\star$ is. A vertex $x$ of an indecomposable tournament $T$ is said to be *critical* if the subtournament $T - x$ is decomposable.

The *indecomposability graph* [2, 3] of an indecomposable tournaments $T$ is the graph, denoted by $\mathbb{I}(T)$, whose vertices are those of $T$ and the edges are the pairs $\{x, y\}$ of distinct vertices such that $T - \{x, y\}$ is indecomposable. This graph was introduced by Ille [8].

Sayar [10] improved [9, Theorem 1] in the case of tournaments as follows.

**Proposition 1.1** ( Sayar [10]) *Given an indecomposable tournament* $T = (V, A)$*, consider a subset* $X$ *of* $V$ *such that* $\mid X \mid \geq 3$ *and* $T[X]$ *is indecomposable. If* $\mid V \setminus X \mid \geq 4$*, then* $\binom{V \setminus X}{2} \cap E(\mathbb{I}(T)) \neq \emptyset$.

Since, for each vertex $x$ of an indecomposable tournament $T = (V, A)$, with $\mid V \mid \geq 3$, there exists $X \subseteq V$ such that $T[X] \simeq P_3$ and $x \in X$, the following corollary is an immediate consequence of Proposition 1.1.

**Corollary 1.2** *For an indecomposable tournament* $T$*, with* $v(T) \geq 7$*,* $\mathbb{I}(T)$ *is not 1-covered.*

The next problem follows from Corollary 1.2.

**Problem 1.3** ( **Ille** [4]) *Characterize the indecomposable tournaments* $T$ *such that* $\mathbb{I}(T)$ *is 2-covered.*

This problem is a natural question in the study of the (−2)-recognition ([4]). An important tool in this work is the notion of minimal tournaments defined as follows. Given two distinct vertices $x$ and $y$ of an indecomposable tournament $T$, we say that $T$ is *minimal* for $\{x, y\}$, or $\{x, y\}$*-minimal*, whenever for each proper subset $X$ of $V(T)$, if $x, y \in X$ and $\mid X \mid \geq 3$, then $T[X]$ is decomposable. The minimal tournaments for two vertices where characterized by Cournier and Ille [5]. In order to recall this characterization, we introduce the tournaments $P_k$ and $Q_k$.

- For $k \geq 3$, the tournament $P_k = (\mathbb{N}_k, A_k)$ is defined as follows. For $x \neq y \in \mathbb{N}_k$, $(x, y) \in A_k$ if

$$\begin{cases} y = x + 1 \\ \text{or} \\ y \leq x - 2. \end{cases}$$

- For $k \geq 5$, the tournament $Q_k$ is defined on $\mathbb{N}_k$ as follows.

$$Q[\mathbb{N}_{k-2}] = P_{k-2},\ \mathbb{N}_{k-2} \to k,\ \mathbb{N}_{k-3} \to k-1 \text{ and } k \to k-1 \to k-2.$$

For $k \geq 5$, the tournaments $P_k$ (see Figure 1) and $Q_k$ (see Figure 2) are indecomposable and $\{1, k\}$-minimal. Conversely,

**Theorem 1.4** ( **Cournier and Ille** [5]) *Given a tournament* $T$*, with* $v(T) \geq 6$*, consider two distinct vertices* $a$ *and* $b$ *of* $T$*. The tournament* $T$ *is* $\{a, b\}$*-minimal if and only if there is an isomorphism* $f$ *from* $T$ *onto* $P_{v(T)}$*,* $Q_{v(T)}$ *or* $(Q_{v(T)})^*$ *such that* $f(\{a, b\}) = \{1, v(T)\}$.

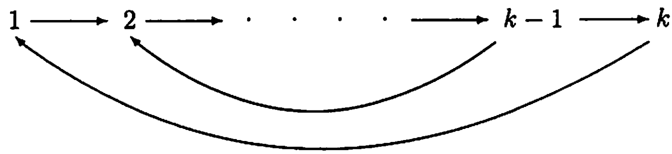


Figure 1: The tournament $P_k$.

## 2 Preliminaries

We recall some properties of indecomposable tournaments. Given a tournament $T = (V, A)$, consider a subset $X$ of $V$ such that $\mid X \mid \geq 3$ and $T[X]$ is indecomposable. We use the following subsets of $V \setminus X$.

- Ext(X) is the set of $v \in V \setminus X$ such that $T[X \cup \{v\}]$ is indecomposable;
- $\langle X \rangle$ is the set of $v \in V \setminus X$ such that $X$ is an interval of $T[X \cup \{v\}]$, that is,

$$\langle X \rangle = N_T^-(X) \cup N_T^+(X)$$

- for each $u \in X$, $X(u)$ is the set of $v \in V \setminus X$ such that $\{u, v\}$ is an interval of $T[X \cup \{v\}]$.

The family constituted by Ext(X), $\langle X \rangle$ and $X(u)$, where $u \in X$, is denoted by $p_X$.

**Lemma 2.1** ( Ehrenfeucht and Rozenberg [6]) *Given a tournament $T = (V, A)$, consider a subset $X$ of $V$ such that $\mid X \mid \geq 3$ and $T[X]$ is indecomposable. The family $p_X$ realizes a partition of $V \setminus X$. Moreover, the following hold.*

- *Let $u \in X$, $v \in X(u)$ and $w \in V \setminus (X \cup X(u))$. If $T[X \cup \{v, w\}]$ is decomposable, then $\{u, v\}$ is an interval of $T[X \cup \{v, w\}]$.*
- *Let $v \in \langle X \rangle$ and $w \in V \setminus (X \cup \langle X \rangle)$. If $T[X \cup \{v, w\}]$ is decomposable, then $X \cup \{w\}$ is an interval of $T[X \cup \{v, w\}]$.*
- *Let $v \neq w \in$ Ext(X). If $T[X \cup \{v, w\}]$ is decomposable, then $\{v, w\}$ is an interval of $T[X \cup \{v, w\}]$.*

As a consequence of the above lemma, we obtain the following.

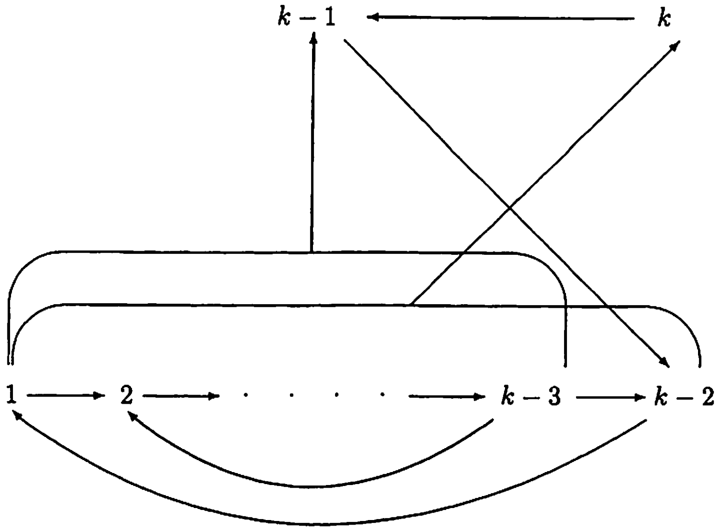


Figure 2: The tournament $Q_k$.

**Corollary 2.2** ( Ehrenfeucht and Rozenberg [6]) *Let $T = (V, A)$ be an indecomposable tournament. If $X$ is a subset of $V$ such that $| X | \geq 3$, $| V \backslash X | \geq 2$ and $T[X]$ is indecomposable, then there are two distinct vertices $x$ and $y$ of $V \setminus X$ such that $T[X \cup \{x, y\}]$ is indecomposable.*

Given Corollary 2.2, we introduce the following graph. Given a tournament $T = (V, A)$, consider a subset $X$ of $V$ such that $| X | \geq 3$ and $T[X]$ is indecomposable. The graph $G_X$ is defined on $V \setminus X$ by
given $x \neq y \in V \setminus X$, $\{x, y\} \in E(G_X)$ if $T[X \cup \{x, y\}]$ is indecomposable.

# 3 The tournaments whose indecomposability graph is 2-covered

To begin, notice the following. If $T$ is an $\{a, b\}$-minimal tournament, with $a \neq b$ and $v(T) \geq 5$, then $\mathbb{I}(T)$ is $\{a, b\}$-covered. Indeed, if $\mathbb{I}(T)$ is not $\{a, b\}$-covered, then there exists $\{c, d\} \in E(\mathbb{I}(T))$ such that $\{c, d\} \cap \{a, b\} = \emptyset$. Thus, $T - \{c, d\}$ is indecomposable, which contradicts the $\{a, b\}$-minimality of $T$. Also, we have to notice the following:

- For $k \geq 7$, $\mathbb{I}(P_k) = (\{1, \ldots, k\}, \{\{1, 2\}, \{1, k\}, \{k - 1, k\}\})$.
- For $k \geq 7$, $\mathbb{I}(Q_k) = (\{1, \ldots, k\}, \{\{1, 2\}, \{1, k\}, \{2, k\}, \{k - 1, k\}\})$.

We use the following proposition.

**Proposition 3.1** *Let $T$ be an indecomposable tournament with $v(T) \geq 9$. Given $a \neq b \in V(T)$, if $\mathbb{I}(T)$ is $\{a,b\}$-covered, then $T$ contains an $\{a,b\}$-minimal subtournament of cardinality $v(T)$, $v(T)-1$ or $v(T)-3$.*

*Proof*. Given $a \neq b \in V$, assume that $\mathbb{I}(T)$ is $\{a,b\}$-covered. Consider a minimal subset $X$ of $V$ under inclusion among the subsets $Y$ of $V$ satisfying $|Y| \geq 3$, $\{a,b\} \subseteq Y$ and $T[Y]$ is indecomposable. By minimality of $X$, $T[X]$ is $\{a,b\}$-minimal. It remains to verify that $|V \setminus X| = 0, 1$ or $3$. As $\mathbb{I}(T)$ is $\{a,b\}$-covered, $|V \setminus X| \neq 2$. Moreover, since $\mathbb{I}(T)$ is $\{a,b\}$-covered, it follows from Proposition 1.1 that $|V \setminus X| \leq 3$. □

The above proposition leads us to describe the tournaments $T$ such that $\mathbb{I}(T)$ is $\{a,b\}$-covered, from the $\{a,b\}$-minimal tournaments embedding into $T$. We introduce the following tournaments classes.

- $\mathcal{P}$ is the set of $P_n$ where $n \geq 9$.
- $\mathcal{Q}$ is the set of $Q_n$ where $n \geq 9$.
- $\mathcal{P}_{-1}$ is the set of indecomposable tournaments $T$ defined on $\mathbb{N}_n$ for some $n \geq 9$, such that $\mathbb{I}(T)$ is $\{1, n-1\}$-covered and $T - n = P_{n-1}$.
- $\mathcal{Q}_{-1}$ is the set of indecomposable tournaments $T$ defined on $\mathbb{N}_n$ for some $n \geq 11$, such that $\mathbb{I}(T)$ is $\{1, n-1\}$-covered and $T - n = Q_{n-1}$.
- $\mathcal{P}_{-3}$ is the set of indecomposable tournaments $T$ defined on $\mathbb{N}_n$ for some $n \geq 12$, such that $\mathbb{I}(T)$ is $\{1, n-3\}$-covered and $T - \{n, n-1, n-2\} = P_{n-3}$.
- $\mathcal{Q}_{-3}$ is the set of indecomposable tournaments $T$ defined on $\mathbb{N}_n$ for some $n \geq 12$, such that $\mathbb{I}(T)$ is $\{1, n-3\}$-covered and $T - \{n, n-1, n-2\} = Q_{n-3}$.

Clearly, $\mathcal{P}$ is a subset of $\mathcal{P}_{-1}$. As $Q_n - 1 \simeq Q_{n-1}$, we have $\mathcal{Q} \subseteq \mathcal{Q}_{-1}$ up to isomorphism. Similarly, since $P_n$ is self-dual, $(\mathcal{P}_{-1})^\star \subseteq \mathcal{P}_{-1}$ and $(\mathcal{P}_{-3})^\star \subseteq \mathcal{P}_{-3}$ up to isomorphism.

Our description is done by the following result.

**Theorem 3.2** *Given a tournament $T$, with $v(T) \geq 12$, $T$ is indecomposable and $\mathbb{I}(T)$ is 2-covered if and only if $T \simeq T'$ where $T' \in \mathcal{P}_{-1} \cup \mathcal{P}_{-3} \cup \mathcal{Q}_{-1} \cup \mathcal{Q}_{-3} \cup \mathcal{Q}^\star_{-1} \cup \mathcal{Q}^\star_{-3}$ with $v(T') \geq 12$.*

Hence, the remainder of the paper is devoted to describe each of the classes $\mathcal{P}_{-1}$, $\mathcal{P}_{-3}$, $\mathcal{Q}_{-1}$ and $\mathcal{Q}_{-3}$.

## 3.1 The class $\mathcal{P}_{-1}$

The next proposition describes the class $\mathcal{P}_{-1}$,

**Proposition 3.3** *Given a tournament $T$ defined on $\mathbb{N}_n$ where $n \geq 9$,*

$$T \in \mathcal{P}_{-1} \quad \textit{if and only if} \quad T - n = P_{n-1} \textit{ and}$$

$$N_T^+(n) = \begin{cases} \mathbb{N}_{k-1} \textit{ where } k \in \{4, \ldots, n-3\} \cup \{2, n-1\} \\ \textit{or} \\ \mathbb{N}_{k-1} \cup \{k+1\} \textit{ where } k \in \{2, \ldots, n-3\}. \end{cases}$$

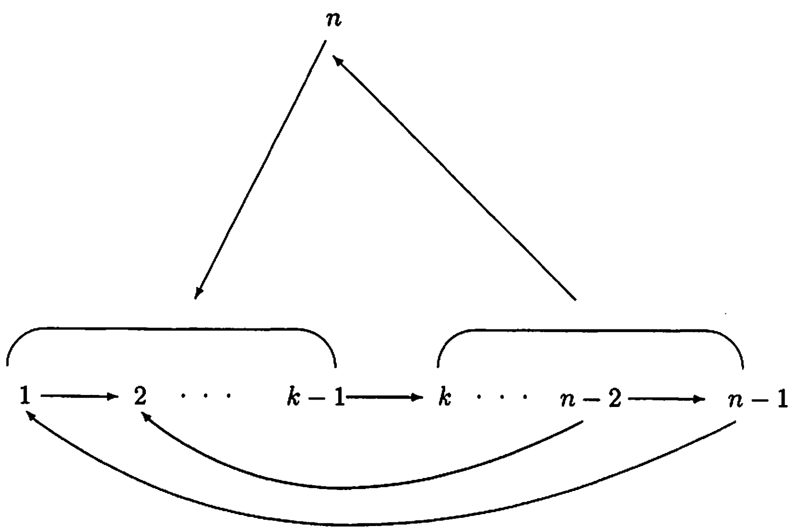


Figure 3: $N_T^+(n) = \mathbb{N}_{k-1}$.

*Proof.* Consider a tournament $T$ defined on $\mathbb{N}_n$, where $n \geq 9$, such that $T - n = P_{n-1}$. We use the following permutation of $\mathbb{N}_n$

$$\begin{array}{rccl} \varphi: & \mathbb{N}_n & \longrightarrow & \mathbb{N}_n \\ & i & \longmapsto & n-i \quad \text{for } 1 \leq i \leq n-1 \\ & n & \longmapsto & n. \end{array}$$

We denote by $\varphi(T^\star)$ the unique tournament such that $\varphi$ is an isomorphism from $T^\star$ onto $\varphi(T^\star)$. Observe that $\varphi(T^\star) - n = P_{n-1}$.

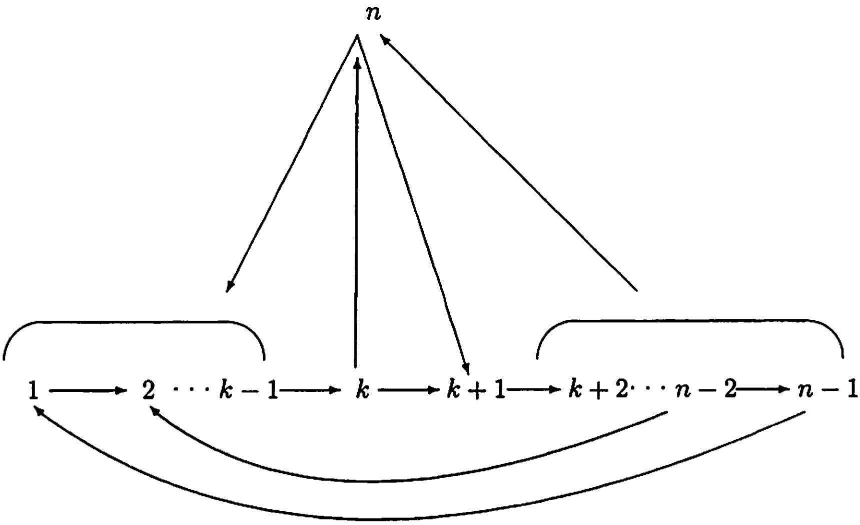


Figure 4: $N_T^+(n) = \mathbb{N}_{k-1} \cup \{k+1\}$.

First, assume that $N_T^+(n) = \mathbb{N}_{k-1}$ where $k \in \{2\} \cup \{4, \ldots, n-3\} \cup \{n-1\}$. To begin, we show that $T$ is indecomposable. If $k \in \{2, n-1\}$, then $T \simeq P_n$ and hence $T$ is indecomposable. Assume that $k \in \{4, \ldots, n-3\}$. Since $T[\mathbb{N}_{n-1}] = P_{n-1}$ is indecomposable, we use the partition $p_{(\mathbb{N}_{n-1})}$ as follows. We have $n \notin \langle \mathbb{N}_{n-1} \rangle$ because $n-1 \longrightarrow n \longrightarrow 1$. As $n \longrightarrow 3 \longrightarrow 1$, $n \notin \mathbb{N}_{n-1}(1)$. Furthermore $n \notin \mathbb{N}_{n-1}(i)$ for $2 \leq i \leq k$ because $n \longrightarrow i-1 \longrightarrow i$. Thus

$$n \notin \bigcup_{i=1}^{i=k} \mathbb{N}_{n-1}(i). \tag{1}$$

Observe that $N^+_{\varphi(T^*)}(n) = \mathbb{N}_{l-1}$ where $l = n-k+1 \in \{4, \ldots, n-3\}$. By applying (1) to $\varphi(T^*)$, we get

$$n \notin \bigcup_{i=1}^{i=l} \mathbb{N}_{n-1}(i) \text{ in } \varphi(T^*), \text{ that is, } n \notin \bigcup_{i=k-1}^{i=n-1} \mathbb{N}_{n-1}(i) \quad \text{in T.}$$

Therefore $n \notin \langle \mathbb{N}_{n-1} \rangle$ and $n \notin \mathbb{N}_{n-1}(i)$ for each $i \in \mathbb{N}_{n-1}$. Since $p_{(\mathbb{N}_{n-1})}$ is a partition by Lemma 2.1, $n \in \text{Ext}(\mathbb{N}_{n-1})$ or equivalently, $T[\mathbb{N}_{n-1} \cup \{n\}] = T$ is indecomposable. Now we prove that $\mathbb{I}(T)$ is $\{1, n-1\}$-covered. Given $i < j \in \mathbb{N}_n \setminus \{1, n-1\}$, we have to verify that $T - \{i, j\}$

is decomposable. If $i \geq k+1$, then $\{1, \ldots, i-1\} \cup \{n\}$ is a non trivial interval of $T - \{i, j\}$. If $i \leq k$, then $T - \{i, j\}$ is decomposed into $\{i+1, \ldots, n\} \setminus \{j\} \longrightarrow \{1, \ldots, i-1\}$. Consequently $\{1, \ldots, i-1\}$ is a non trivial interval of $T - \{i, j\}$ when $i \geq 3$, and $\{i+1, \ldots, n\} \setminus \{j\}$ is a non trivial interval of $T - \{i, j\}$ when $i \leq n-3$.

Second, assume that $N_T^+(n) = \mathbb{N}_{k-1} \cup \{k+1\}$ where $k \in \{2, \ldots, n-3\}$. Suppose for a contradiction that $T$ admits a non trivial interval $I$. Denote by $T_{\{k,k+1\}^\star}$ the tournament obtained from $T$ by uniquely reversing the arc between $k$ and $k+1$. The permutation of $\mathbb{N}_n$ defined by

$$\begin{array}{rcll} i & \longmapsto & i & \text{for } 1 \leq i \leq k, \\ n & \longmapsto & k+1, & \\ i & \longmapsto & i+1 & \text{for } k+1 \leq i \leq n-1, \end{array}$$

is an isomorphism from $T_{\{k,k+1\}^\star}$ onto $P_n$. Thus $T_{\{k,k+1\}^\star}$ is indecomposable. It follows that $\mid I \cap \{k, k+1\} \mid = 1$. Moreover, as $T[\mathbb{N}_{n-1}] = P_{n-1}$, either $I = \mathbb{N}_{n-1}$ or $\mid I \cap \mathbb{N}_{n-1} \mid = 1$ and $n \in I$. Therefore $I = \{k, n\}$ or $\{k+1, n\}$. But $\{k, n\}$ is not an interval of $T$ because $n \longrightarrow k-1 \longrightarrow k$, and $\{k+1, n\}$ also since $k+1 \longrightarrow k+2 \longrightarrow n$. Consequently $T$ is indecomposable. To prove that $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, we proceed as previously.

Conversely, consider $T \in \mathcal{P}_{-1}$. For a contradiction, suppose that $1 \longrightarrow n$. As $\mathbb{N}_{n-1}$ is not an interval of $T$, there is $i \in \{2, \ldots, n-1\}$ such that $n \longrightarrow i$. Set

$$m = \max(\{i \in \{2, \ldots, n-1\} : n \longrightarrow i\}).$$

Clearly $2 \leq m \leq n-1$ and $\{m+1, \ldots, n-1\}$ (when $m \leq n-2$ ) $\longrightarrow n \longrightarrow m$. If $m = 2$ or $m = 3$, then $\{m-1, n\}$ would be a non trivial interval of $T$. Moreover, if $m = n-1$, then $T[\{1, n-1, n\}] \simeq P_3$ is indecomposable. It would follow from Proposition 1.1 that $\mathbb{I}(T)$ is not $\{1, n-1\}$-covered. Thus

$$4 \leq m \leq n-2.$$

Set $X = \{4, \ldots, n-1\}$. Since $T[X] \simeq P_{n-4}$, $T[X]$ is indecomposable. As $X \longrightarrow 1$, $1 \in \langle X \rangle$. We have $m, m+1 \in X$ because $m \leq n-2$. Since $m+1 \longrightarrow n \longrightarrow m$, $m \notin \langle X \rangle$. As $X \longrightarrow 1 \longrightarrow n$, it follows from Lemma 2.1 that $T[X \cup \{1, n\}] = T - \{2, 3\}$ would be indecomposable which contradicts the fact that $\mathbb{I}(T)$ is $\{1, n-1\}$-covered. Consequently

$$n \longrightarrow 1.$$

Since $\mathbb{N}_{n-1}$ is not an interval of $T$, there is $i \in \{2, \ldots, n-1\}$ such that $i \longrightarrow n$. Set

$$\mu = \min(\{i \in \{2, \ldots, n-1\} : i \longrightarrow n\}).$$

Clearly $2 \leq \mu \leq n-1$ and $\mu \longrightarrow n \longrightarrow \{1, \ldots, \mu-1\}$. Furthermore $\{n-1, n\}$ would be a non trivial interval of $T$ if $\mu = n-2$. Therefore

$$\mu \in \{2, \ldots, n-3\} \cup \{n-1\}. \tag{2}$$

Observe that $\varphi(T^*) \in \mathcal{P}_{-1}$. By applying (2) to $\varphi(T^*)$, we obtain $\nu \in \{2, \ldots, n-3\} \cup \{n-1\}$ such that $\nu \longrightarrow n \longrightarrow \{1, \ldots, \nu-1\}$ in $\varphi(T^*)$. We get $\{n-\nu+1, \ldots, n-1\} \longrightarrow n \longrightarrow n-\nu$ (in $T$). If $\mu = n-1$, then $N_T^+(n) = \mathbb{N}_{k-1}$ with $k = n-1$. Similarly, if $\nu = n-1$, then $N_T^+(n) = \mathbb{N}_{k-1}$ with $k = 2$. Assume that

$$\mu, \nu \in \{2, \ldots, n-3\}. \tag{3}$$

As $\mu \longrightarrow n \longrightarrow \{1, \ldots, \mu-1\}$ and $\{n-\nu+1, \ldots, n-1\} \longrightarrow n \longrightarrow n-\nu$, we have

$$\begin{cases} \mu \leq n-\nu+1 \\ \text{and} \\ \mu \neq n-\nu. \end{cases} \tag{4}$$

Assume that $\mu = n-\nu+1$. We get $N_T^+(n) = \mathbb{N}_{\mu-1}$. Furthermore, $\mu \geq 4$ because $\nu \leq n-3$. Hence

$$\begin{cases} N_T^+(n) = \mathbb{N}_{\mu-1} \\ \text{and} \\ 4 \leq \mu \leq n-3. \end{cases}$$

So assume that $\mu \neq n-\nu+1$. It follows from (3) and (4) that

$$5 \leq \mu+\nu+1 \leq n. \tag{5}$$

For a contradiction, suppose that $n > \mu+\nu+1$. Set $X = \{1, \ldots, \mu\} \cup \{n\} \cup \{n-\nu, \ldots, n-1\}$. The function

$$\begin{array}{cccl} X & \longrightarrow & \{1, \ldots, \mu+\nu+1\} & \\ i & \longmapsto & i & \text{for } 1 \leq i \leq \mu, \\ n & \longmapsto & \mu+1, & \\ i & \longmapsto & i-(n-\mu-\nu-2) & \text{for } n-\nu \leq i \leq n-1, \end{array}$$

is an isomorphism from $T[X]$ onto $P_{\mu+\nu+1}$. Thus $T[X]$ is indecomposable with

$$\mid \mathbb{N}_n \setminus X \mid = n-(\mu+\nu+1). \tag{6}$$

Since $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, $n-(\mu+\nu+1) \neq 2$. Moreover, it follows from Proposition 1.1 that $n-(\mu+\nu+1) \leq 3$. Thus $n-(\mu+\nu+1) = 1$ or 3. If $n-(\mu+\nu+1) = 1$, then $\{\mu+1, n\}$ would be a non trivial interval of $T$.

Suppose that $n - (\mu + \nu + 1) = 3$. We have $\mathbb{N}_n \setminus X = \{\mu+1, \mu+2, \mu+3\}$. As $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, we should have $T[X \cup \{i\}]$ is decomposable for $i \in \mathbb{N}_n \setminus X$. Using Lemma 2.1, we obtain the following contradiction

- if $T[X \cup \{\mu+1\}]$ is decomposable, then $\mu + 1 \in X(1)$ and $\mu = 3$;
- if $T[X \cup \{\mu+3\}]$ is decomposable, then $\mu+3 \in X(n-1)$ and $\mu = n-7$;
- if $T[X \cup \{\mu+2\}]$ is decomposable, then

$$\begin{cases} \mu + 2 \in X(1) \text{ and } \mu = 2 \\ \text{or} \\ \mu + 2 \in X(n-1) \text{ and } \mu = n-6. \end{cases}$$

It follows that $n = \mu + \nu + 1$. We obtain $N_T^+(n) = \mathbb{N}_{\mu-1} \cup \{\mu+1\}$. Moreover $2 \leq \mu \leq n-3$ by (3).

□

The next remark describes the indecomposability graph of the tournaments of $\mathcal{P}_{-1}$.

**Remark 3.4** *Consider $T \in \mathcal{P}_{-1}$. Applying Proposition 3.3, we have to distinguish the following two cases according to $N_T^+(n)$.*

1. *If $N_T^+(n) = \mathbb{N}_{k-1}$ where $k \in \{4, \ldots, n-3\} \cup \{2, n-1\}$, then*

$$E(\mathbb{I}(T)) = \begin{cases} \{\{1,n\},\{n,n-1\},\{n-1,n-2\}\} & \text{if } k = 2, \\ \{\{1,2\},\{1,n\},\{n,n-1\},\{n-1,n-2\}\} & \text{if } k = 4, \\ \{\{1,n\},\{1,n-1\},\{n-1,n\}\} & \text{if } k = 5 \text{ and } n = 9, \\ \{\{1,n\},\{1,n-1\},\{n-1,n\},\{n-1,n-2\}\} & \text{if } k = 5 \text{ and } n \geq 10, \\ \{\{1,2\},\{1,n\},\{1,n-1\},\{n-1,n\},\{n-1,n-2\}\} & \text{if } k \in \{6,\ldots,n-5\}, \\ \{\{1,2\},\{1,n\},\{1,n-1\},\{n-1,n\}\} & \text{if } k = n-4 \text{ and } n \geq 10, \\ \{\{1,2\},\{1,n\},\{n-1,n\},\{n-1,n-2\}\} & \text{if } k = n-3, \\ \{\{1,2\},\{1,n\},\{n-1,n\}\} & \text{if } k = n-1. \end{cases}$$

*2. If* $N_T^+(n) = \mathbb{N}_{k-1} \cup \{k+1\}$ *where* $k \in \{2, \ldots, n-3\}$, *then*

$$E(\mathbb{I}(T)) = \begin{cases} \{\{1,2\},\{1,n\},\{n,n-1\},\{n-1,n-2\}\} & \text{if } k=2, \\ \{\{1,n\},\{n,n-1\},\{1,n-1\},\{n-1,n-2\}\} & \text{if } k=3, \\ \{\{1,2\},\{1,n\},\{1,n-1\},\{n-1,n\},\{n-1,n-2\}\} & \\ \qquad\qquad \text{if } k \in \{4,\ldots,n-5\}, & \\ \{\{1,2\},\{1,n\},\{1,n-1\},\{n-1,n\}\} & \text{if } k=n-4, \\ \{\{1,2\},\{1,n\},\{n-1,n\},\{n-1,n-2\}\} & \text{if } k=n-3. \end{cases}$$

□

## 3.2 The class $\mathcal{Q}_{-1}$

The next proposition describes the class $\mathcal{Q}_{-1}$,

**Proposition 3.5** *Given a tournament* $T$ *defined on* $\mathbb{N}_n$, *where* $n \geq 11$, *such that* $T - n = Q_{n-1}$,

$T \in \mathcal{Q}_{-1}$ *if and only if*

$$N_T^+(n) = \begin{cases} \mathbb{N}_{n-4} \cup \{n-1\} \\ or \\ \mathbb{N}_{n-3} \cup \{n-1\} \\ or \\ \mathbb{N}_{k-1} \cup \{k+1, n-2, n-1\} \text{ where } k \in \{2,\ldots,n-4\} \\ or \\ \mathbb{N}_{k-1} \cup \{n-1, n-2\} \text{ where } k \in \{2\} \cup \{4,\ldots,n-3\}. \end{cases} \quad (7)$$

*Proof.* Let $T$ be a tournament defined on $\mathbb{N}_n$ such that $T - n = Q_{n-1}$ where $n \geq 11$. To begin assume that $T$ satisfies (7). To verify that $T$ is indecomposable and $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, we proceed as at the beginning of the proof of Proposition 3.3.

Conversely, assume that $T$ is indecomposable and $\mathbb{I}(T)$ is $\{1, n-1\}$-covered. Set $X = \mathbb{N}_{n-3}$. We have $T[X] = P_{n-3}$ is indecomposable. Clearly $n-1 \in \langle X \rangle$. Similarly set $Y = \mathbb{N}_{n-4}$. We have $T[Y] = P_{n-4}$ is indecomposable and $n-1 \in \langle Y \rangle$. Also set $Z = \{4, \ldots, n-1\}$. We have $T[Z] \simeq Q_{n-4}$ is indecomposable. Observe that $1 \in Z(n-1)$.

Let $u \in Y$. For a contradiction, suppose that $n \in X(u)$. We have $n \in Y(u)$ as well. Since $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, $T - \{n-3, n-2\} =$

$T[Y \cup \{n-1, n\}]$ is decomposable . By Lemma 2.1, $\{u, n\}$ is an interval of $T[Y \cup \{n-1, n\}]$. In particular $n \longrightarrow n-1$. Now we prove that $n \longrightarrow n-2$, which implies that $\{u, n\}$ would be a non trivial interval of $T$. We distinguish the following two cases.

- Assume that $u \neq 1$. Set $Y' = (Y \setminus \{u\}) \cup \{n\}$. As $\{u, n\}$ is an interval of $T[Y \cup \{n\}]$, $T[Y] \simeq T[Y']$ and hence $T[Y']$ is indecomposable. We have $n-1 \in \langle Y' \rangle$ because $n \longrightarrow n-1$. Since $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, $T - \{u, n-3\} = T[Y' \cup \{n-2, n-1\}]$ is decomposable. As $Y' \cup \{n-2\}$ is not a interval of $T[Y' \cup \{n-2, n-1\}]$, it follows from Lemma 2.1 that $n-2 \in \langle Y' \rangle$. In particular $n \longrightarrow n-2$.
- Assume that $u = 1$. For a contradiction, suppose that $n-2 \longrightarrow n$. Set $Z' = \mathbb{N}_{n-5}$. The tournament $T[Z'] = P_{n-5}$ is indecomposable. Moreover $n \in Z'(1)$, $n-2 \in \langle Z' \rangle$ and $1 \longrightarrow n-2 \longrightarrow n$. It follows from Lemma 2.1 that $T[Z' \cup \{n-2, n\}]$ is indecomposable. Set $Z" = Z' \cup \{n-2, n\}$. We have $n-1 \notin \langle Z" \rangle$ because $n \longrightarrow n-1 \longrightarrow n-2$. Furthermore, since $T[\{n-1, n-2, n\}]$ is indecomposable, $n-1 \notin Z"(n-2) \cup Z"(n)$. By Lemma 2.1 $n-1 \notin Z'(v)$ for $v \in Z'$ because $n-1 \in \langle Z' \rangle$. Thus $n-1 \notin Z"(v)$ for $v \in Z'$. It follows from Lemma 2.1 that $n-1 \in \text{Ext}(Z")$. Thus $T[Z" \cup \{n-1\}] = T - \{n-4, n-3\}$ is indecomposable which contradicts the fact that $\mathbb{I}(T)$ is $\{1, n-1\}$-covered.

Consequently

$$n \notin \bigcup_{u=1}^{u=n-4} X(u).$$

Since $p_X$ is a partition of $\{n-2, n-1, n\}$ by Lemma 2.1, we obtain

$$n \in X(n-3) \cup \langle X \rangle \cup \text{Ext(X)}.$$

First, assume that $n \in X(n-3)$. As $n \in X(n-3)$, $n \longrightarrow 4$. Thus $n \longrightarrow 4 \longrightarrow n-1$ and hence $n \notin Z(n-1)$. Furthermore $T[Z \cup \{1, n\}] = T - \{2, 3\}$ is decomposable because $\mathbb{I}(T)$ is $\{1, n-1\}$-covered. Since $1 \in Z(n-1)$, $\{1, n-1\}$ is an interval of $T[Z \cup \{1, n\}]$. In particular $n \longrightarrow n-1$, so that $\{n-3, n\}$ is an interval of $T-(n-2)$. Therefore $\{n-3, n\}$ is not an interval of $T[\{n-3, n-2, n\}]$ and $n \longrightarrow n-2$. For $k = n-4$, we obtain

$$N_T^+(n) = \begin{cases} \mathbb{N}_{k-1} \cup \{k+1, n-2, n-1\} & \text{if } \ n \longrightarrow n-3 \\ \mathbb{N}_{k-1} \cup \{n-2, n-1\} & \text{if } \ n \longrightarrow n-3. \end{cases}$$

Second, assume that $n \in \langle X \rangle$. Suppose for a contradiction that $n \in Z(n-1)$. We have $\{4, \ldots, n-3\} \longrightarrow n \longrightarrow n-2$. As $n \in \langle X \rangle$, we obtain $\{1, \ldots, n-3\} \longrightarrow n \longrightarrow n-2$ and $\{n-1, n\}$ would be a non trivial interval of $T$. Thus

$$n \notin Z(n-1).$$

Since $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, $T - \{2, 3\} = T[Z \cup \{1, n\}]$ is decomposable. As $1 \in Z(n-1)$ and $n \notin Z(n-1)$, $\{1, n-1\}$ is an interval of $T[Y \cup \{1, n\}]$. We obtain either $\{1, n-1\} \longrightarrow n$ or $n \longrightarrow \{1, n-1\}$. Suppose for a contradiction that $\{1, n-1\} \longrightarrow n$. Since $n \in \langle X \rangle$, $\mathbb{N}_{n-3} \longrightarrow n$. If $n-2 \longrightarrow n$, then $\mathbb{N}_{n-1}$ would be a non trivial interval of $T$, and if $n \longrightarrow n-2$, then $\{n-1, n\}$ would be a non trivial interval of $T$. Therefore

$$n \longrightarrow \{1, n-1\}.$$

As $n \in \langle X \rangle$, $n \longrightarrow \mathbb{N}_{n-3}$. Since $\mathbb{N}_{n-1}$ is not an interval of $T$, $n-2 \longrightarrow n$ and we obtain

$$N_T^+(n) = \mathbb{N}_{n-3} \cup \{n-1\}.$$

Third, assume that $n \in \mathrm{Ext(X)}$. For a contradiction, suppose that $n-1 \longrightarrow n$. As $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, $T - \{n-3, n-2\} = T[Y \cup \{n-1, n\}]$ is decomposable. Since $Y \longrightarrow n-1 \longrightarrow n$, it follows from Lemma 2.1 that $n \in \langle Y \rangle$. Furthermore, as $n \in \mathrm{Ext(X)}$, $T[X \cup \{n\}] = T[Y \cup \{n-3, n\}]$ is indecomposable. Thus, either $Y \longrightarrow n \longrightarrow n-3$ or $n-3 \longrightarrow n \longrightarrow Y$. If $Y \longrightarrow n \longrightarrow n-3$, then $\{n-2, n\}$ would be a non trivial interval of $T$. Suppose that $n-3 \longrightarrow n \longrightarrow Y$. Since $n \longrightarrow 4 \longrightarrow n-1$, $n \notin Z(n-1)$. As $1 \in Z(n-1)$ and $n-1 \longrightarrow n \longrightarrow 1$, it would follow from Lemma 2.1 that $T[Z \cup \{1, n\}] = T - \{2, 3\}$ is indecomposable and $\mathbb{I}(T)$ would not be $\{1, n-1\}$-covered. Consequently

$$n \longrightarrow n-1.$$

Lastly, consider $X' = X \cup \{n\}$. We have $T[X']$ is indecomposable because $n \in \mathrm{Ext(X)}$. We verify that $\mathbb{I}(T[X'])$ is $\{1, n-3\}$-covered. Otherwise, there exist $x \neq y \in X' \setminus \{1, n-3\}$ such that $T[X'] - \{x, y\}$ is indecomposable. Set $Y' = X' \setminus \{x, y\}$. We have $Y' \longrightarrow n-1$ because $n \longrightarrow n-1$. Therefore $n-1 \in \langle Y' \rangle$. Moreover $n-2 \notin \langle Y' \rangle$ because $1 \longrightarrow n-2 \longrightarrow n-3$. Since $Y' \longrightarrow n-1 \longrightarrow n-2$, $T[Y' \cup \{n-2, n-1\}]$ is indecomposable by Lemma 2.1. As $x, y \in X' \setminus \{1, n-3\} \subseteq \mathbb{N}_n \setminus \{1, n-1\}$, $\mathbb{I}(T)$ would not be $\{1, n-1\}$-covered. It follows that

$$\mathbb{I}(T[X']) \quad \text{is} \quad \{1, n-3\} - \textit{covered}.$$

Consider the bijection

$$\begin{array}{rccc} \varphi: & X' & \longrightarrow & \mathbb{N}_{n-2} \\ & x \in X & \longmapsto & x \\ & n & \longmapsto & n-2 \end{array}$$

and denote by $T'$ the unique tournament defined on $\mathbb{N}_{n-2}$ such that $\varphi$ is an isomorphism from $T[X']$ onto $T'$. We obtain that $T' \in \mathcal{P}_{-1}$. By Proposition 3.3,

$$N^+_{T'}(n-2) = \begin{cases} \mathbb{N}_{k-1} & \text{where } k \in \{2\} \cup \{4, \dots, n-5\} \cup \{n-3\} \\ \text{or} & \\ \mathbb{N}_{k-1} \cup \{k+1\} & \text{where } k \in \{2, \dots, n-5\}. \end{cases}$$

Thus

$$N^+_{T[X']}(n) = \begin{cases} \mathbb{N}_{k-1} & \text{where } k \in \{2\} \cup \{4, \dots, n-5\} \cup \{n-3\} \\ \text{or} & \\ \mathbb{N}_{k-1} \cup \{k+1\} & \text{where } k \in \{2, \dots, n-5\}. \end{cases} \tag{8}$$

Since $n \longrightarrow n-1$, we obtain

$$N^+_{T[X']}(n) = \begin{cases} \mathbb{N}_{n-4} \cup \{n-1\} \\ \text{or} \\ \mathbb{N}_{k-1} \cup \{n-2, n-1\}. \end{cases} \quad \text{when} \quad k = n-3.$$

Assume that $k \neq n-3$. We show that $n \longrightarrow n-2$. Set $Y' = \mathbb{N}_{n-5}$. We have $T[Y'] = P_{n-5}$ is indecomposable and $n-2, n-1 \in \langle Y' \rangle$. Moreover, by (8), $n \notin \langle Y' \rangle$ because $k \neq n-3$. As $\mathbb{I}(T)$ is $\{1, n-1\}$-covered, $T - \{n-4, n-3\} = T[Y' \cup \{n-2, n-1, n\}]$ admits a non trivial interval $I$. Since $T[Y']$ is indecomposable, $I \cap Y'$ is a trivial interval of $T[Y']$. Therefore $I \cap Y' = \emptyset$, $\{u\}$, where $u \in Y'$, or $Y'$.

- Assume that $I \cap Y' = \emptyset$. As $n-2, n-1 \in \langle Y' \rangle$ and $n \notin \langle Y' \rangle$, $I = \{n-2, n-1\}$. Since $n \longrightarrow n-1$, $n \longrightarrow n-2$.
- Assume that $I \cap Y' = \{u'\}$, where $u' \in Y'$. For every $x \in I \setminus \{u'\}$, we have $x \in Y'(u')$. As $n-2, n-1 \in \langle Y' \rangle$, $n-2, n-1 \notin Y'(u')$ by Lemma 2.1. Hence $I = \{u', n\}$ and $n \longrightarrow n-2$ because $u' \longrightarrow n-2$.

- Assume that $I \cap Y' = Y'$. For every $x \in \{n-2, n-1, n\} \setminus I$, we have $x \in \langle Y' \rangle$. Thus $n \in I$. Since $n \longrightarrow n-1 \longrightarrow n-2$, $I \neq Y' \cup \{n-2, n\}$. It follows that $n-2 \notin I$. As $Y' \longrightarrow n-2$, we obtain $n \longrightarrow n-2$.

  It follows that

$$n \longrightarrow n-2.$$

Consequently $n \longrightarrow \{n-2, n-1\}$ and it follows from (8) that

$$N^+_{T[X']}(n) = \begin{cases} \mathbb{N}_{k-1} \cup \{k+1, n-2, n-1\} \text{ where } k \in \{2, \ldots, n-5\} \\ \text{or} \\ \mathbb{N}_{k-1} \cup \{n-2, n-1\} \text{ where } k \in \{2\} \cup \{4, \ldots, n-5\}. \end{cases}$$

□

The next remark describes the indecomposability graph of the tournaments of $\mathcal{Q}_{-1}$.

**Remark 3.6** *Consider $T \in \mathcal{Q}_{-1}$. Applying Proposition 3.5, we have to distinguish the following cases according to $N^+_T(n)$.*

1. *If $N^+_T(n) = \mathbb{N}_{n-3} \cup \{n-1\}$, then $E(\mathbb{I}(T)) = \{\{1,2\}, \{2, n-1\}, \{n, 1\}, \{n, n-1\}, \{1, n-1\}\}$.*
2. *If $N^+_T(n) = \mathbb{N}_{n-4} \cup \{n-1\}$, then $E(\mathbb{I}(T)) = \{\{1,2\}, \{2, n-1\}, \{n, 1\}, \{n, n-1\}, \{1, n-1\}, \{n-1, n-2\}\}$.*
3. *If $N^+_T(n) = \mathbb{N}_{k-1} \cup \{n-1, n-2\}$ where $k \in \{4, \ldots, n-3\} \cup \{2\}$, then*

$$E(\mathbb{I}(T)) = \begin{cases} \{\{1,n\}, \{n, n-1\}, \{1, n-1\}, \{n-1, n-2\}\} \text{ if } k = 2 \text{ or } k = 5, \\ \{\{1,2\}, \{1,n\}, \{n, n-1\}, \{n-1, 2\}, \{n-1, n-2\}\} \text{ if } k = 4, \\ \{\{1,2\}, \{1,n\}, \{1, n-1\}, \{n-1, n\}, \{n-1, n-2\}, \{2, n-1\}\} \\ \qquad \text{if } k \in \{6, \ldots, n-5\} \cup \{n-3\}, \\ \{\{1,2\}, \{1,n\}, \{1, n-1\}, \{n-1, n\}, \{n-1, 2\}\} \text{ if } k = n-4. \end{cases}$$

4. *If $N^+_T(n) = \mathbb{N}_{k-1} \cup \{k+1, n-1, n-2\}$ where $k \in \{2, \ldots, n-4\}$, then*

$$E(\mathbb{I}(T)) = \begin{cases} \{\{1,2\},\{1,n\},\{n,n-1\},\{n-1,n-2\},\{2,n-1\}\} \ \textit{if} \ k=2, \\ \{\{1,n\},\{n,n-1\},\{1,n-1\},\{n-1,n-2\}\} \ \textit{if} \ k=3, \\ \{\{1,2\},\{1,n\},\{1,n-1\},\{n-1,n\},\{n-1,n-2\},\{n-1,2\}\} \\ \qquad \textit{if} \ k \in \{4,\ldots,n-5\}, \\ \{\{1,2\},\{1,n\},\{1,n-1\},\{n-1,n\},\{n-1,2\}\} \ \textit{if} \ k=n-4. \end{cases}$$

□

## 3.3 The class $\mathcal{P}_{-3}$

**Proposition 3.7** *Up to isomorphism, the elements of $\mathcal{P}_{-3}$ are the tournaments $T$ defined on $\mathbb{N}_n$, where $n \geq 12$, such that $T[\mathbb{N}_{n-3}] = P_{n-3}$, $n-2 \in N_T^-(\mathbb{N}_{n-3})$ and satisfying one and only one of the following assertions.*

1. $n-1 \in N_T^-(\mathbb{N}_{n-3})$, $n \in \mathbb{N}_{n-3}(n-3)$ *and*

$$E(G_{\mathbb{N}_{n-3}}) = \{\{n-2,n\}\} \quad \textit{with} \ \ n-2 \longrightarrow n-1.$$

2. $n-1 \in \mathbb{N}_{n-3}(n-4)$, $n \in \mathbb{N}_{n-3}(n-3)$ *and*

$$\begin{cases} E(G_{\mathbb{N}_{n-3}}) = \{\{n-2,n-1\},\{n-2,n\}\} \\ \textit{or} \\ \{n-1,n\} \in E(G_{\mathbb{N}_{n-3}}), \mid E(G_{\mathbb{N}_{n-3}}) \mid \geq 2 \ \textit{with} \ \ n \longrightarrow n-3. \end{cases}$$

3. $n-1, n \in \mathbb{N}_{n-3}(u)$, *where* $u = n-4$ *or* $n-3$, *and*

$$E(G_{\mathbb{N}_{n-3}}) = \begin{cases} \{\{n-2,n-1\}\} \quad \textit{with} \ \ n-1 \not\longrightarrow \{u,n\} \\ \textit{or} \\ \{\{n-2,n-1\},\{n-2,n\}\} \quad \textit{with} \ \ u \not\longrightarrow \{n-1,n\}. \end{cases}$$

The proof is analogous to that of Proposition 3.3. The next lemma is helpful.

**Lemma 3.8** *Let $T$ be an indecomposable tournament defined on $\mathbb{N}_{n-1}$ where $n \geq 12$, verifying : $T[\mathbb{N}_{n-3}] = P_{n-3}$ and for each vertex $i$ of $\mathbb{N}_{n-1} - \{1, n-3\}$, $i$ is critical. Then one and only one of the following assertions holds, where $\{\alpha, \beta\} = \{n-2, n-1\}$.*

1. $N_T^-(\mathbb{N}_{n-3}) = \{\alpha\}$, $\mathbb{N}_{n-3}(n-3) \cup \mathbb{N}_{n-3}(n-4) = \{\beta\}$.
2. $N_T^+(\mathbb{N}_{n-3}) = \{\alpha\}$, $\mathbb{N}_{n-3}(1) \cup \mathbb{N}_{n-3}(2) = \{\beta\}$.
3. $\mathbb{N}_{n-3}(n-4) = \{\alpha\}$, $\mathbb{N}_{n-3}(n-3) = \{\beta\}$ *with* $\beta \longrightarrow n-3$.
4. $\mathbb{N}_{n-3}(1) = \{\alpha\}$, $\mathbb{N}_{n-3}(2) = \{\beta\}$ *with* $1 \longrightarrow \alpha$.

## 3.4 The class $\mathcal{Q}_{-3}$

The proof of the last proposition is similar to that of Proposition 3.5. For convenience, we use the following notation. Given a tournament $T = (V, A)$, consider a subset $X$ of $V$ such that $\mid X \mid \geq 3$ and $T[X]$ is indecomposable. For $u \in X$, $X(u)$ is divided into $X^-(u)$ and $X^+(u)$ as follows

- $X^-(u)$ is the set of the elements $x$ of $X(u)$ such that $x \longrightarrow u$;
- $X^+(u)$ is the set of the elements $x$ of $X(u)$ such that $u \longrightarrow x$.

**Proposition 3.9** *Up to isomorphism, the elements of $\mathcal{Q}_{-3}$ are the tournaments $T$ defined on $\mathbb{N}_n$, where $n \geq 12$, such that $T[X] = Q_{n-3}$ and satisfying one and only one of the following assertions, where $X = \mathbb{N}_{n-3}$.*

1. $n-2 \in N_T^+(X)$, $n-1 \in X^-(n-4)$, *and*

$$\begin{cases} n \in X^+(n-4) \text{ and } E(G_X) = \{\{n-2, n-1\}, \{n-2, n\}\} \\ \text{or} \\ n \in X(n-4),\ n \longrightarrow n-1 \text{ and } E(G_X) = \{\{n-2, n-1\}\}. \end{cases}$$

2. $n-2 \in X^+(n-3)$, $n-1 \in N_T^-(X)$, $n \in N_T^+(X)$ *and* $E(G_X) = \{\{n-2, n-1\}, \{n-2, n\}\}$.
3. $n-2 \in X^+(n-3)$, $n-1 \in \langle X \rangle$, *and*

$$\begin{cases} n \in \langle X \rangle,\ n \not\sim \{n-2, n-1\} \text{ and } E(G_X) = \{\{n-2, n-1\}\} \\ \text{or} \\ n \in X^+(n-3),\ n-2 \longrightarrow n \text{ and } E(G_X) = \{\{n-2, n-1\}\}. \end{cases}$$

4. $n-2 \in X^+(n-3)$, $n-1 \in X(n-4)$, *and*

$$\begin{cases} n \in \langle X \rangle \;\; \text{and} \;\; E(G_X) = \{\{n-2, n-1\}, \{n-2, n\}\} \\ \text{or} \\ n \in N_T^+(X) \;\; \text{and} \;\; E(G_X) = \{\{n-2, n\}, \{n-1, n\}\} \\ \text{or} \\ n \in N_T^+(X), \; n-1 \longrightarrow n-4 \;\; \text{and} \\ \qquad\qquad E(G_X) = \{\{n-2, n-1\}, \{n-2, n\}, \{n-1, n\}\}. \end{cases}$$

5. $n-2, n-1 \in X^-(n-3)$, $n-1 \longrightarrow n-2$, $n \in X(1)$ *and* $E(G_X) = \{\{n-2, n\}\}$.

6. $n-2 \in X^-(n-3)$, $n-1 \in X^+(1)$, *and*

$$\begin{cases} n \in X^-(1) \;\; \text{and} \;\; E(G_X) = \{\{n-2, n-1\}, \{n-2, n\}\} \\ \text{or} \\ n \in X(1), \; n-1 \longrightarrow n \;\; \text{and} \;\; E(G_X) = \{\{n-2, n-1\}\} \\ \text{or} \\ n \in X(2) \;\; \text{and} \;\; E(G_X) \supseteq \{\{n-2, n-1\}\}. \end{cases}$$

7. $n-2 \in X^-(n-3)$, $n-1 \in X^+(2)$, *and*

$$\begin{cases} n \in X^-(2), \; n \longrightarrow n-1 \;\; \text{and} \;\; E(G_X) = \{\{n-2, n-1\}, \{n-2, n\}\} \\ \text{or} \\ n \in X(2), \; n-1 \longrightarrow n \;\; \text{and} \;\; E(G_X) = \{\{n-2, n-1\}\}. \end{cases}$$

8. $n-2 \in X^-(n-3)$, $n-1 \in X^-(1)$, $n \in X(2)$ *and* $E(G_X) = \{\{n-2, n-1\}, \{n-2, n\}\}$.


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# REFERENCES


[1] H. Belkhechine and I. Boudabbous, Indecomposable tournaments and their indecomposable subtournaments on 5 and 7 vertices, to appear in *Ars Combinatoria*.

[2] H. Belkhechine, I. Boudabbous and M. B. Elayech, Les graphes (-1)-critiques, to appear in *Ars Combinatoria.*

[3] I. Boudabbous and P. Ille, Critical and infinite directed graphs, *Discrete Math.* **307** (2007), pp. 2415-2428.

[4] A. Boussaïri, A. Chaïchaâ and P. Ille, Indecomposability graph and indecomposability recognition, submitted.

[5] A. Cournier and P. Ille, Minimal indecomposable graphs, *Discrete Math* **183** (1998), pp. 61-80.

[6] A. Ehrenfeucht and G. Rozenberg, Primitivity is hereditary for 2-structures, *Theoret. Comput. Sci.* **70** (1990), pp. 343-358.

[7] R. Fraïssé, L'intervalle en théorie des relations, ses généralisations, filtre intervallaire et clôture d'une relation, *in: Orders, Description and Roles, M. Pouzet et D. Richard éd. North-Holland.* (1984), pp. 313-342.

[8] P. Ille, Recognition problem in reconstruction for decomposable relations, *in: B. Sands, N. Sauer, R. Woodrow (Eds.), Finite and Infinite Combinatorics in Sets and Logic, Kluwer Academic Publishers.* (1993), pp. 189-198.

[9] P. Ille, Indecomposable graphs, *Discrete Math.* **173** (1997), pp. 71-78.

[10] M.Y. Sayar, Partially critical tournaments and partially critical supports, *Contrib. Discrete Math.* **6** (2011), pp. 52-76.

[11] J.H. Schmerl and W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, *Discrete Math.* **113** (1993), pp. 191-205.