The fifth coefficient of adjoint polynomial and a new invariant *

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Abstract

Two graphs are defined to be adjointly equivalent if their complements are chromatically equivalent. In [2, 7], Liu and Dong et al. gives the first four coefficients b_0, b_1, b_2, b_3 of adjoint polynomial and two invariants R_1, R_2, which are useful in determining the chromaticity of graphs. In this paper, we give the expression of the fifth coefficient b_4, which brings about a new invariant R_3. Using these new tools and the properties of the adjoint polynomials, we determine the chromatic equivalence class of $\overline{B_{n-9,1,5}}$.

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1 Introduction

The graphs considered in this paper are finite undirected and simple graphs. We follow the notation of Bondy and Murty [1], unless otherwise stated. For a graph G, let $\overline{G}$, $V(G)$, $E(G)$, $n(G)$, $m(G)$, $c(G)$ and $t(G)$ be the complement, vertex set, edge set, the order, the size, the component number and the number of triangles of graph G, respectively. $N_G(H)$ denote the number of subgraphs of G isomorphic to H, which H is a subgraph of G. If $W \subseteq V(G)$, we denote by

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$G \setminus W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them.

A partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of graph G if each A_i is nonempty independent set of G. We denote by $\alpha(G, r)$ the number of r-independent partitions of G. Thus the chromatic polynomial G is $P(G, \lambda) = \sum_{r \geq 1} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda - 1)\cdots(\lambda - r + 1)$ for all $r \geq 1$. The readers can turn to [4] for details on chromatic polynomial.

Two graphs G and H are said to be *chromatically equivalent*, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. By $[G]$ we denote the equivalence class determined by G under "$\sim$". It is obvious that "$\sim$" is an equivalence relation on the family of all graphs. A graph G is called chromatically unique (or simply χ-unique) if $H \cong G$ whenever $H \sim G$. See [5, 6] for many results on this field.

The adjoint polynomial of a graph is a useful tool for this study. We now proceed to define it. Let G be a graph of order n. If H is a spanning subgraph of G and each component of H is complete, then H is called an *ideal subgraph* of G. Two ideal subgraphs are considered to be different if they have different edge sets. For $k \geq 1$, let $N(G, k)$ be the number of ideal graphs H in G with $c(H) = k$. The number $N(G, k)$ is referred to as an *ideal subgraph number*. It is clear that $N(G, n) = 1$ and $N(G, k) = 0$ for $k > n$. Define

$$h(G,x) = \begin{cases} \sum_{i=1}^{n} N(G,k)x^k, & if\ n \geq 1. \\ 1, & otherwise. \end{cases}$$

The polynomial $h(G, x)$ is called the *adjoint polynomial* of G. Observe that $h(G, x) = h(H, x)$ if $G \cong H$. Hence $h(G, x)$ is a well defined graph-function. The notion of the adjoint polynomial of a graph was introduced by Liu [7]. Note that the adjoint polynomial is a special case of an F-polynomial[10]. Two graphs G and H are said to be *adjointly equivalent*, denoted by $G \sim^h H$, if $h(G, x) = h(H, x)$. Evidently, "$\sim^h$" is an equivalence relation on the family of all graphs. Let $[G]_h = \{H | H \sim^h G\}$. A graph G is said to be *adjointly unique*(or simply $h - unique$) if $G \cong H$ whenever $G \sim^h H$. Note that

$$\alpha(G,k) = N(\overline{G},k), \quad k = 1, 2, \cdots, n.$$

It follows that

Theorem 1.1 [3] (1) $G \sim^h H$ if and only if $\overline{G} \sim \overline{H}$.
(2) $[G]_h = \{H | \overline{H} \in [\overline{G}]\}$.
(3) G is χ-unique if and only if $\overline{G}$ is h–unique.

Hence the goal of determining $[G]$ for a given graph G can be realized by determining $[G]_h$. Thus if $m(G)$ is very large, it may be easier to study $[G]_h$

rather than $[G]$. The determination of $[G]$ for a given graph G has received much attention in [12, 13, 18, 19] recently.

Now we define some classes of graphs, which will be used later.

(1) C_n(resp. P_n) denotes the cycle (resp. the path) of order n, and write $\mathcal{C} = \{C_n | n \geq 3\}$, $\mathcal{P} = \{P_n | n \geq 2\}$ and $\mathcal{U} = \{U_{1,1,t,1,1} | t \geq 1\}$.

(2) $D_n (n \geq 4)$ denotes the graph obtained from C_3 and P_{n-2} by identifying a vertex of C_3 with a pendent vertex of P_{n-2}.

(3) T_{l_1,l_2,l_3} is a tree with a vertex v of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$ and $l_3 \geq l_2 \geq l_1$, write $\mathcal{T}_0 = \{T_{1,1,l_3} | l_3 \geq 1\}$ and $\mathcal{T} = \{T_{l_1,l_2,l_3} | (l_1, l_2, l_3) \neq (1,1,1)\}$.

(4) $\vartheta = \{C_n, D_n, K_1, T_{l_1,l_2,l_3} | n \geq 4\}$.

(5) $\xi = \{C_r(P_s), Q(r,s), B_{r,s,t}, F_n, U_{r,s,t,a,b}, K_4^-\}$.

(6) $\psi = \{\psi_n^1, \psi_n^2, \psi_n^3(r,s), \psi_n^4(r,s), \psi_n^5(r,s,t), \psi_5^6\}$.

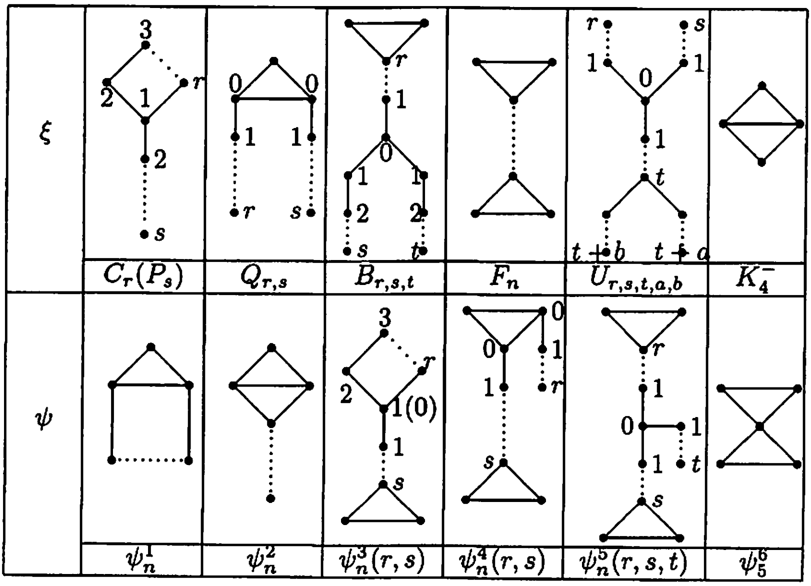

Figure 1 Families of ξ and ψ

For convenience, we simply denote $h(G,x)$ by $h(G)$ and $h_1(G,x)$ by $h_1(G)$. By $\beta(G)$ and $\gamma(G)$ we denote the smallest and the second smallest real root of $h(G)$, respectively. Let $d_G(v)$, simply denoted by $d(v)$, be the degree of vertex v. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies. By K_n we denote the complete graph with order n. Let $g(x)|f(x)$(resp. $g(x) \nmid f(x)$) denote $g(x)$ divides $f(x)$(resp. $g(x)$ does not divide $f(x)$) and $\partial(f(x))$ denote the degree of $f(x)$. By $(f(x), g(x))$ we denote the largest common factor of $f(x)$ and $g(x)$ on the real field. Let $N_G(v)$ be the neighborhood set of a vertex v.

2 The expression of the fifth coefficient

In this section, we calculate the ideal graph number $N(G, n-4)$, which is also called the fifth coefficient of adjoint polynomial.

Lemma 2.1 [8, 9] Let G be a graph with $n$vertices and m edges. Denote by $\triangle$ the set of the triangles in G and by $\triangle(i)$ the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, \cdots, d_n)$, then

(1) $b_0(G) = N(G,n) = 1$;

(2) $b_1(G) = N(G,n-1) = m$;

(3) $b_2(G) = N(G,n-2) = \binom{m+1}{2} - \frac{1}{2}\sum_{i=1}^{n} d_i^2 + N_G(K_3)$;

(4) $b_3(G) = N(G,n-3) = \frac{m}{6}(m^2+3m+4) - \frac{m+2}{2}\sum_{i=1}^{n} d_i^2 + \frac{1}{3}\sum_{i=1}^{n} d_i^3 - \sum_{ij\in E(G)} d_i d_j - \sum_{i\in\triangle} \triangle(i) d_i + (m+2)N_G(K_3) + N_G(K_4)$, where $b_i(G) = \alpha(\overline{G}, m-i)(i = 0,1,2,3)$.

Our aim here is to study the expression of the fifth coefficient of adjoint polynomial $b_4(G) = N(G, n-4)$. It is useful in determining the chromaticity of graphs.

Theorem 2.1 Let G be a graph with n vertices and m edges, and let $G' = G - \triangle$ and $\triangle$ is a triangle of G. If the degree sequence of G' is $(d_1(G'), d_2(G'), \cdots, d_n(G'))$, then

$b_4(G) = N(G,n-4) = (m(G)+6)N_G(K_4) - \sum_{x\in G} K_4(x) + \binom{t(G)}{2} - N_G(K_4^-) - N_G(\psi_5^6) + \sum_{\triangle\in G}\left(\binom{m(G')+1}{2} - \frac{1}{2}\sum_{i=1}^{n(G')} d_i^2(G')\right) + \binom{m(G)}{4} - N_G(P_5) - N_G(P_4\cup P_2) - N_G(2P_3) - \sum_{x\in V(G)}\binom{d(x)}{4} - N_G(K_{1,3}\cup K_2) - N_G(D_4) - N_G(K_3\cup P_2) - N_G(C_4)$, where $K_4(x)$ denote the number of K_4 in G which covers vertex x.

Proof. By definition, $N(G, n-4)$ is the number of ideal subgraph H in G with $c(H) = n-4$. Since $n(H) = n$, each component of H is order at most 5, we find that H is one of the following types of graphs:

(1) $4K_2 \cup (n-8)K_1$; (2) $2K_3 \cup (n-6)K_1$; (3) $K_3 \cup 2K_2 \cup (n-7)K_1$; (4) $K_4 \cup K_2 \cup (n-6)K_1$; (5) $K_5 \cup (n-5)K_1$.

Thus

$N(G,n-4) = N_G(4K_2) + N_G(2K_3) + N_G(K_3 \cup 2K_2) + N_G(K_4 \cup K_2) + N_G(K_5)$.

Observe that

(1) Let $\mathscr{K}_4$ is the set of all the subgraphs of G isomorphic to K_4. For each $K_4 \in \mathscr{K}_4$, we denote the four vertices by i, j, k, m and their degrees by d_i, d_j, d_k, d_m, respectively. Then

$$\begin{aligned} N_G(K_4 \cup K_2) &= \sum_{ijkm \in K_4} \Big(m(G) + 6 - (d_i + d_j + d_k + d_m)\Big) \\ &= (m(G)+6)N_G(K_4) - \sum_{ijkm \in K_4} (d_i + d_j + d_k + d_m) \\ &= (m(G)+6)N_G(K_4) - \sum_{x \in K_4} d(x)(K_4(x)), \end{aligned}$$

where $K_4(x)$ denote the number of K_4 in G which covers vertex x.

(2) If graph G is triangle-free or G contains one triangle, then it is easy to see that $N_G(2K_3) = 0$. If graph G contains at least two triangles, then we have

$$N_G(2K_3) = \binom{t(G)}{2} - N_G(K_4^-) - N_G(\psi_5^6),$$

where $N_G(K_4^-)$ and $N_G(\psi_5^6)$ denote the number of subgraphs isomorphic to K_4^- and ψ_5^6 in G, respectively. The definition of K_4^- and ψ_5^6 can be found in Figure 1.

(3) If graph G is triangle-free, then it is easy to $N(K_3 \cup 2K_2) = 0$. If graph G contains at least one triangle, then

$$N_G(K_3 \cup 2K_2) = \sum_{\Delta \in G} \left(\binom{m(G')+1}{2} - \frac{1}{2} \sum_{i=1}^{n(G')} d_i^2(G') \right),$$

where $G' = G - \Delta$ and Δ is a triangle of G.

The number of methods of choosing two nonadjacent edges is equal to the size of line graph of $G - \Delta$. So the number is $\frac{1}{2}\sum_{i=1}^{n(G')} d_i^2(G')$.

(4) Now consider the number of $4K_2$. Figure 2 shows all possible graphs with size 4 and no isolated vertices, where $H_1 = P_5, H_2 = P_4 \cup K_2, H_3 = 2K_3, H_4 = 4K_2, H_5 = K_{1,4}, H_6 = K_{1,3} \cup K_2, H_7 = D_4, H_8 = K_3 \cup K_2, H_9 = C_4$.

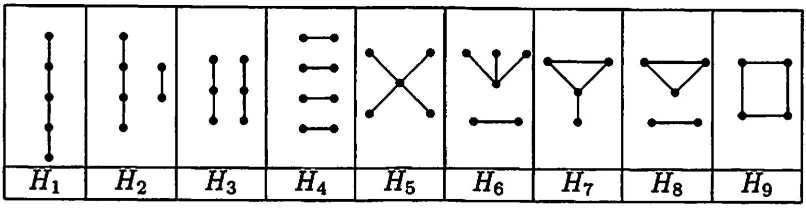

Figure 2 The combinatorial types of $4K_2$

Observe that

$$N_G(K_{1,4}) = \sum_{x\in V(G)} \binom{d(x)}{4}$$

Thus

$$\begin{aligned} N_G(4K_2) &= \binom{m(G)}{4} - N_G(P_5) - N_G(P_4 \cup K_2) - N_G(2K_3) - N_G(K_{1,4}) \\ &\quad -N_G(K_{1,3} \cup K_2) - N_G(D_4) - N_G(K_3 \cup K_2) - N_G(C_4) \\ &= \binom{m(G)}{4} - \sum_{x\in V(G)} \binom{d(x)}{4} - N_G(P_4 \cup K_2) - N_G(K_{1,3} \cup K_2) \\ &\quad -N_G(2K_3) - N_G(P_5) - N_G(D_4) - N_G(K_3 \cup K_2) - N_G(C_4) \end{aligned}$$

The result holds.

□

Theorem 2.2 (1) $b_4(P_n) = \binom{n-4}{4}$ for $n \geq 5$;
(2) $b_4(C_n) = 5n\binom{n-3}{5}$ for $n \geq 5$; $b_4(C_3) = b_4(C_4) = 0$;
(3) $b_4(D_n) = 5n\binom{n-3}{5} + n - 6$ for $n \geq 5$; $b_4(D_4) = 0$;
(4) $b_4(K_4^-) = b_4(K_4) = 0$; $b_4(C_4(P_3)) = b_4(Q_{1,2}) = 0$.

Proof. From Theorem 2.1, we can easily get the results by direct calculation. □

3 The third invariant

On the basis of the four coefficient b_0, b_1, b_2 and b_3, Liu and Dong et al.[2, 7, 9] give the following two invariants R_1 and R_2.

Definition 3.1 [2, 7, 9] Let G be a graph and $b_i(G)(0 \leq i \leq 3)$ be the first four coefficients of $h(G)$. Then

(1) The first character of a graph G is defined as

$$R_1(G) = \begin{cases} 0 & if\ q = 0, \\ b_2(G) - \binom{b_1(G)-1}{2} + 1 & if\ q > 0. \end{cases}$$

(2) The second character of a graph G is defined as

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G).$$

Now we define a new invariant $R_3(G)$, which is derived from the first fifth coefficients.

Definition 3.2 Let G be a graph. Then the third character of a graph G is defined as

$$R_3(G) = b_4(G) - \binom{b_1(G)}{4}.$$

It is obvious that $R_3(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_3(G) = R_3(H)$ if $h(G) = h(H)$.

Theorem 3.1 Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then

$$h(G) = \prod_{i=1}^{k} h(G_i) \ and \ R_3(G) = \sum_{i=1}^{k} R_3(G_i).$$

4 The chromaticity of graph $\overline{B_{n-9,1,5}}$

4.1 Preliminaries

Definition 4.1.1[8] Let G be a graph and $h_1(G,x)$ be the polynomial with a nonzero constant term such that $h(G,x) = x^{\rho(G)} h_1(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then G is called *irreducible graph.*

Theorem 4.1.1[2, 8] Let G be a graph with k components $G_1, G_2, \cdots, G_k$. Then $h(G) = \prod_{i=1}^{k} h(G_i)$ and $R_j(G) = \sum_{i=1}^{k} R_j(G_i)$ for $j = 1, 2$.

It is obvious that $R_j(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_j(G) = R_j(H)$ for $j = 1, 2$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

For an edge $e = v_1v_2$ of a graph G, the graph $G * e$ is defined as follow: the vertex set of $G * e$ is $(V(G) - \{v_1, v_2\}) \bigcup v(v \notin G)$, and the edge set of $G * e$ is $\{e' | e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv | u \in N_G(v_1) \cap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of G which are adjacent to v.

Lemma 4.1.1 [8] Let G be a graph with $e \in E(G)$. Then

$$h(G,x) = h(G - e, x) + h(G * e, x),$$

where $G - e$ denotes the graph obtained by deleting the edge e from G.

Lemma 4.1.2 [8] (1) For $n \geq 2$, $h(P_n) = \sum_{k \leq n} \binom{k}{n-k} x^k$.
(2) For $n \geq 4$, $h(D_n) = \sum_{k \leq n} (\frac{n}{k} \binom{k}{n-k} + \binom{k-2}{n-k-3}) x^k$.

(3) For $n \geq 4$, $m \geq 6$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$, $h(D_m) = x(h(D_{m-1}) + h(D_{m-2}))$.

Lemma 4.1.3 [11] Let $\{g_i(x)\}$, simply denoted by $\{g_i\}$, be a polynomial sequence with integer coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then

(1) $g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x)$.

(2) $h_1(P_n)|g_{k(n+1)+i}(x)$ if and only if $h_1(P_n)|g_i(x)$, where $0 \leq i \leq n$, $n \geq 2$ and $k \geq 1$.

Lemma 4.1.4 [7, 14] Let G be a nontrivial connected graph with n vertices. Then

(1) $R_1(G) \leq 1$, and the equality holds if and only if $G \cong P_n (n \geq 2)$ or $G \cong K_3$.

(2) $R_1(G) = 0$ if and only if $G \in \vartheta$.

(3) $R_1(G) = -1$ if and only if $G \in \xi$, especially, $q(G) = p(G) + 1$ if and only if $G \in \{F_n | n \geq 6\} \cup \{K_4^-\}$.

Lemma 4.1.5 [15] Let G be a connected graph. If $R_1(G) = 0, -1, -2$, then $n(G) - m(G) \leq |R_1(G)|$.

Lemma 4.1.6 [11] Let G be a connected graph and H a proper subgraph of G, then $\beta(G) < \beta(H)$.

Lemma 4.1.7 [11] Let G be a connected graph. Then

(1) $\beta(G) = -4$ if and only if

$$G \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2), Q(1,1), K_4^-, D_8\} \cup \mathcal{U};$$

(2) $\beta(G) > -4$ if and only if

$$G \in \{K_1, T(1,2,i)(2 \leq i \leq 4), D_i(4 \leq i \leq 7)\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_0.$$

Lemma 4.1.8 [11] Let G be a connected graph. Then $-(2+\sqrt{5}) \leq \beta(G) < -4$ if and only if G is one of the following graphs:

(1) T_{l_1,l_2,l_3} for $l_1 = 1, l_2 = 2, l_3 > 5$ or $l_1 = 1, l_2 > 2, l_3 > 3$ or $l_1 = l_2 = 2, l_3 > 2$ or $l_1 = 2, l_2 = l_3 = 3$.

(2) $U_{r,s,t,a,b}$ for $r = a = 1$, $(r,s,t) \in \{(1,1,2), (2,4,2), (2,5,3), (3,7,3), (3,8,4)\}$, or $r = a = 1$, $s \geq 1$, $t \geq t^*(s,b)$, $b \geq 1$, where $(s,b) \neq (1,1)$ and

$$t^* = \begin{cases} s+b+2, & if\ s \geq 3. \\ b+3, & if\ s = 2. \\ b, & if\ s = 1. \end{cases}$$

(3) D_n for $n \geq 9$.

(4) $C_n(P_2)$ for $n \geq 5$.

(5) F_n for $n \geq 9$.

(6) $B_{r,s,t}$ for $r = 5, s = 1$ and $t = 3$, or $r \geq 1, s = 1$ if $t = 1$, or $r \geq 4, s = 1$ if $t = 2$, or $b \geq c + 3, s = 1$ if $t \geq 3$.

(7) $G \cong C_4(P_3)$ or $G \cong Q(1, 2)$.

Corollary 4.1.1 [12] If graph G such that $R_1(G) \leq -2$, then $\beta(G) < -2 - \sqrt{5}$.

4.2 The algebraic properties of adjoint polynomials

Lemma 4.2.1 [11] For $n, m \geq 2$, $h(P_n) \mid h(P_m)$ if and only if $(n+1)|(m+1)$.

Theorem 4.2.1 (1) For $n \geq 9$, $\partial(h_1(B_{n-9,1,5})) = \begin{cases} \frac{n}{2} & if \text{ n } is\ even, \\ \frac{n-1}{2} & otherwise. \end{cases}$

(2) For $n \geq 9$, $\rho(B_{n-9,1,5}) = \begin{cases} \frac{n}{2} & if \text{ n } is\ even, \\ \frac{n+1}{2} & otherwise. \end{cases}$

(3) For $n \geq 9$, $h(B_{n-9,1,5}) = x(h(B_{n-10,1,5}) + h(B_{n-11,1,5}))$.

Proof. (1) Choosing a pendent edge $e = uv \in E(B_{n-9,1,5})$ such that $d(u) = 1$, $d(v) = 3$, and by Lemma 4.1.1, $h(B_{n-9,1,5}) = xh(D_{n-1}) + xh(P_5)h(D_{n-7})$. We have, from Lemma 4.1.2, that

$$\partial(h_1(D_{n-1})) = \lfloor n/2 \rfloor \ and\ \partial(h_1(P_5)h_1(D_{n-7})) = 2 + \lfloor (n-6)/2 \rfloor$$

If n is even, then $\partial(h_1(D_{n-1})) = \frac{n}{2} > \frac{n-2}{2} = \partial(h_1(P_5)h_1(D_{n-7}))$, which implies $\partial(h_1(B_{n-9,1,5})) = \frac{n}{2}$. If n is odd, then we arrive at $\partial(h_1(D_{n-1})) = \frac{n-1}{2} > \frac{n-3}{2} = \partial(h_1(P_5)h_1(D_{n-7}))$, which implies that $\partial(h_1(B_{n-9,1,5})) = \frac{n-1}{2}$.

(2) It obviously follows from (1).

(3) Choosing a pendent edge $e = uv \in E(B_{n-9,1,5})$ such that $d(u) = 1$, $d(v) = 3$. We have, by Lemma 4.1.2, that

$$\begin{aligned} h(B_{n-9,1,5}) &= xh(D_{n-1}) + xh(P_5)h(D_{n-7}) \\ &= x^2(h(D_{n-2}) + h(D_{n-3})) + x^2h(P_5)(h(D_{n-8}) + h(D_{n-9})) \\ &= x(xh(D_{n-2}) + xh(P_5)h(D_{n-8})) + x(xh(D_{n-3}) \\ &\quad +xh(P_5)h(D_{n-9})) \\ &= x(h(B_{n-10,1,5}) + h(B_{n-11,1,5})) \end{aligned}$$

□

Theorem 4.2.2 For $n \geq 2$, $m \geq 10$, $h(P_n) \mid h(B_{m-9,1,5})$ if and only if $n = 2$ and $m = 3k + 3$ for $k \geq 3$ or $n = 4$ and $m = 5k + 3$ for $k \geq 2$.

Proof. Let $g_0(x) = x^7 + 12x^6 + 56x^5 + 129x^4 + 155x^3 + 97x^2 + 30x + 4$, $g_1(x) = -x^7 - 11x^6 - 46x^5 - 92x^4 - 92x^3 - 46x^2 - 9x$ and $g_m(x) = x(g_{m-1}(x) + g_{m-2}(x))$. We can deduce that

$$\begin{aligned}
g_0(x) &= x^7 + 12x^6 + 56x^5 + 129x^4 + 155x^3 + 97x^2 + 30x + 4,\\
g_1(x) &= -x^7 - 11x^6 - 46x^5 - 92x^4 - 92x^3 - 46x^2 - 9x,\\
g_2(x) &= x^7 + 10x^6 + 37x^5 + 63x^4 + 51x^3 + 21x^2 + 4x,\\
g_3(x) &= -x^7 - 9x^6 - 29x^5 - 41x^4 - 25x^3 - 5x^2,\\
g_4(x) &= x^7 + 8x^6 + 22x^5 + 26x^4 + 16x^3 + 4x^2,\\
g_5(x) &= -x^7 - 7x^6 - 15x^5 - 9x^4 - x^3,\\
g_6(x) &= x^7 + 7x^6 + 17x^5 + 15x^4 + 4x^3,\\
g_7(x) &= 2x^6 + 6x^5 + 3x^4,\\
g_8(x) &= x^8 + 9x^7 + 23x^6 + 18x^5 + 4x^4,\\
g_9(x) &= x^9 + 9x^8 + 25x^7 + 24x^6 + 7x^5,\\
g_m(x) &= h(B_{m-9,1,5}),\ if\ m \geq 10.
\end{aligned} \tag{4.1}$$

Let $m = (n+1)k+i$, where $0 \leq i \leq n$. It is obvious that $h_1(P_n)|h(B_{m-9,1,5})$ if and only if $h_1(P_n)|g_m(x)$. From Lemma 4.1.3, it follows that $h_1(P_n)|g_m(x)$ if and only if $h_1(P_n)|g_i(x)$, where $0 \leq i \leq n$. We distinguish the following two cases:

Case 1 $n \geq 10$.

If $0 \leq i \leq 9$, from (4.1), it is not difficult to verify that $h_1(P_n) \nmid g_i(x)$. If $i \geq 10$, from $i \leq n$, Lemma 4.1.2 and Theorem 4.2.1, we have that $\partial(h_1(P_n)) = \lfloor \frac{n}{2} \rfloor \geq \partial(h_1(B_{i-9,1,5})) = \lfloor \frac{i}{2} \rfloor$. Suppose that $h_1(P_n)|h_1(B_{i-9,1,5})$, it must leads to $\partial(h_1(P_n)) = \partial(h_1(B_{i-9,1,5}))$ and $h_1(P_n) = h_1(B_{i-9,1,5})$, which implies $R_1(P_n) = R_1(B_{i-9,1,5})$. It is a contradiction by Lemma 4.1.4. Hence $h_1(P_n) \nmid h_1(B_{i-9,1,5})$, together with $(h_1(P_n), x^{\alpha(B_{i-9,1,5})}) = 1$, $h_1(P_n) \nmid h(B_{i-9,1,5})$.

Case 2 $2 \leq n \leq 9$.

From (1) of Lemma 4.1.2 and (4.1), we can verify that $h_1(P_n) = g_i(x)$ if and only if $n = 2$ and $i = 3$ or $n = 4$ and $i = 3$ for $0 \leq i \leq 9$. From Lemma 4.1.3, we have that $h_1(P_n)|h(B_{m-9,1,5})$ if and only if $n = 2$ and $m = 3k + 3$ or $n = 4$ and $m = 5k + 3$. From $\rho(P_2) = 1$, $\rho(P_4) = 2$ and $\rho(B_{m-9,1,5}) = \lfloor \frac{m}{2} \rfloor > 2$ for $m \geq 10$, we obtain that the result holds. □

Theorem 4.2.3 For $m \geq 10$, $h^2(P_2) \nmid h(B_{m-9,1,5})$ and $h^2(P_4) \nmid h(B_{m-9,1,5})$.

Proof. Suppose that $h^2(P_2) \mid h(B_{m-9,1,5})$, from Theorem 4.2.2, we have that $m = 3k+3$, where $k \geq 3$. Let $g_m(x) = h(B_{m-9,1,5})$ for $m \geq 10$. By (3) of Theorem 4.2.1, (1) of Lemma 4.1.2, it follows that

$$\begin{aligned} g_m(x) &= h(P_2)g_{m-2}(x) + x^2 g_{m-3}(x) \\ &= h^2(P_2)g_{m-4}(x) + 2x^2 h(P_2)g_{m-5}(x) + x^4 g_{m-6}(x) \\ &= h^2(P_2)(g_{m-4}(x) + 2x^2 g_{m-7}(x)) + 3x^4 h(P_2)g_{m-8}(x) + x^6 g_{m-9}(x) \\ &= h^2(P_2)(g_{m-4}(x) + 2x^2 g_{m-7}(x) + 3x^4 g_{m-10}(x)) \\ &\quad +4x^6 h(P_2)g_{m-11}(x) + x^8 g_{m-12}(x) \\ &= \cdots \\ &= h^2(P_2)\sum_{s=1}^{k-2} g_{m-3s-1}(x) + (k-1)x^{2k-4}h(P_2)g_{m+1-3(k-1)}(x) \\ &\quad +x^{2k-2}h(P_4)g_{m-3(k-1)}(x) \end{aligned}$$

According to the assumption and $m = 3k+3$, we arrive at, by (4.1), that

$$h^2(P_2) \mid ((k-1)x^{2k-4}h(P_2)g_7(x) + x^{2k-1}g_6(x))$$

that is

$$h(P_2) \mid ((k-1)x^{2k-4}(2x^6 + 6x^5 + 3x^4) + x^{2k-1}x^3(x^3 + 6x^2 + 11x + 4))$$

By direct calculation, we obtain that $k = -1$, which contradicts to $k \geq 3$. Using the similar methods, we also prove that $h^2(P_4) \nmid h(B_{m-9,1,5})$. □

Lemma 4.2.2[2, 8] (1) $R_2(C_n) = 0$ for $n \geq 4$; $R_2(C_3) = -2$; $R_2(K_1) = 0$.
(2) $R_2(B_{r,1,1}) = 3$ for $r \geq 1$; $R_2(B_{r,1,t}) = 4$ for $r, t > 1$.
(3) $R_2(F_6) = 5$; $R_2(F_n) = 4$ for $n \geq 7$; $R_2(K_4^-) = 3$.
(4) $R_2(D_4) = 0$; $R_2(D_n) = 1$ for $n \geq 5$; $R_2(T_{1,1,1}) = -1$.
(5) $R_2(T_{1,1,l_3}) = 1$; $R_2(T_{1,l_2,l_3}) = 1$; $R_2(T_{l_1,l_2,l_3}) = 2$ for $l_3 \geq l_2 \geq l_1 \geq 2$; $R_2(P_2) = -1$; $R_2(P_n) = -2$ for $n \geq 3$.
(6) $R_2(C_r(P_2)) = 3$ for $r \geq 4$; $R_2(C_4(P_3)) = R_2(Q_{1,2}) = 4$.

Lemma 4.2.3[16] Let graph $G \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$, then
(1) $R_2(G) = 3$ if and only if $G \in \{C_{n-1}(P_2) | n \geq 5\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1} | n \geq 7\}$.
(2) $R_2(G) = 4$ if and only if $G \in \{C_r(P_s) | r \geq 4, s \geq 3\} \cup \{Q_{1,n-4} | n \geq 6\} \cup \{B_{r,1,t}, B_{1,1,1} | r, t \geq 2\}$.
(3) $R_2(G) = 5$ if and only if $G \in \{Q_{r,s} | r, s \geq 2\} \cup \{B_{1,1,t}, B_{r,s,t} | r, s, t \geq 2\}$.

(4) $R_2(G) = 6$ if and only if $G \in \{B_{1,s,t} | s, t \geq 2\}$.

Corollary 4.2.1 Let graph $G_n \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$, then $R_2(G) \geq 3$.

Lemma 4.2.4 [12, 13, 18] (1) For $n \geq 5, m \geq 4, \beta(C_n(P_2)) < \beta(C_{n-1}(P_2)) \leq \beta(D_m)$.

(2) For $n \geq 6$, $m \geq 6$, $\beta(F_n) = \beta(B_{m-5,1,1})$ if and only if $n = 2k+1$ and $m = k+2$.

(3) For $n \geq 4$, $m \geq 6$, $\beta(F_m) < \beta(F_{m-1}) < \beta(D_n) < \beta(C_n)$ and $\beta(B_{m-5,1,1}) < \beta(B_{m-4,1,1}) < \beta(D_n)$.

(4) For $n \geq 7, m \geq 6, \beta(B_{n-6,1,2}) = \beta(F_m)$ if and only if $m = n-1$.

(5) For $6 \leq i \leq 8, n \geq i+1, m \geq 6, \beta(B_{n-i,1,i-4}) < \beta(D_m)$.

(6) For $n \geq 8$, $\beta(B_{n-7,1,3}) = \beta(Q_{1,2}) = \beta(C_4(P_3))$ if and only if $n = 13$.

(7) For $n \geq 9$, $\beta(Q(1,2)) = \beta(C_4(P_3)) = \beta(B_{n-8,1,4})$ if and only if $n = 12$.

(8) For $r, t \geq 1$, $\beta(B_{r,1,t}) < \beta(B_{r+1,1,t})$.

(9) $\beta(T_{1,3,6}) = \beta(C_5(P_2))$, $\beta(T_{1,3,11}) = \beta(B_{8,1,2})$.

(10) $\beta(B_{3,1,5}) = \beta(C_4(P_3)) = \beta(Q(1,2))$.

Theorem 4.2.4 (1) For $m \geq 10$, $n \geq 21$, $\beta(B_{1,1,5}) < \beta(B_{2,1,5}) < \beta(B_{3,1,5}) < \beta(B_{4,1,5}) < \beta(B_{5,1,5}) < \beta(B_{6,1,5}) < \beta(B_{7,1,5}) < \beta(C_m(P_2)) < \beta(B_{8,1,5}) < \beta(C_9(P_2)) = \beta(B_{9,1,5}) < \beta(B_{10,1,5}) < \beta(B_{11,1,5}) = \beta(C_8(P_2)) < \beta(C_7(P_2)) < \beta(B_{n-9,1,5}) < \beta(C_6(P_2)) < \beta(C_5(P_2)) < \beta(C_4(P_2))$.

(2) For $m \geq 10$, $n \geq 21$, $\beta(B_{1,1,5}) < \beta(B_{2,1,5}) < \beta(B_{1,1,2}) = \beta(F_6) < \beta(B_{3,1,5}) < \beta(F_7) = \beta(B_{4,1,5}) < \beta(B_{5,1,5}) < \beta(F_8) < \beta(B_{6,1,5}) < \beta(B_{7,1,5}) < \beta(B_{8,1,5}) < \beta(B_{9,1,5}) < \beta(B_{10,1,5}) < \beta(F_9) = \beta(B_{11,1,5}) < \beta(B_{n-9,1,5}) < \beta(F_m)$.

(3) For $n \geq 10$, $\beta(Q(1,2)) = \beta(C_4(P_3)) = \beta(B_{n-9,1,5})$ if and only if $n = 20$.

(4) For $n \geq 10, m \geq 4, \beta(B_{n-9,1,5}) < \beta(D_m)$.

(5) For $n \geq m$, $t \geq 6$, $\beta(B_{m-t-4,1,t}) < \beta(B_{n-9,1,5})$.

(6) For $n \geq 10$, $m \geq 6$, $\beta(B_{n-9,1,5}) = \beta(B_{m-5,1,1})$ if and only if $m = 6, n = 20$.

(7) For $n \geq 10, m \geq 7, \beta(B_{n-9,1,5}) = \beta(B_{m-6,1,2})$ if and only if $m = 9$, $n = 20$ or $m = 10, n = 18$.

(8) For $n \geq 10, m \geq 8, \beta(B_{n-9,1,5}) = \beta(B_{m-7,1,3})$ if and only if $m = 13$, $n = 20$.

(9) For $n \geq 10, m \geq 9, \beta(B_{n-9,1,5}) = \beta(B_{m-8,1,4})$ if and only if $m = 16$, $n = 20$.

Proof. (1) For $n \geq 21$, it is obvious that $T_{1,3,6}$ is a proper subgraph of $B_{n-9,1,5}$. From Lemma 4.1.6 and (9) of Lemma 4.2.4, we have $\beta(B_{n-9,1,5}) < \beta(T_{1,3,6}) = \beta(C_5(P_2))$. From (1) and (8) of Lemma 4.2.4, the result holds.

(2) Using software Mathematica and by calculation, we have $\beta(B_{1,1,5}) = -4.50469 < \beta(B_{2,1,5}) = -4.40387 < \beta(B_{2,1,4}) = \beta(B_{1,1,2}) = \beta(F_6) = -4.39026 < \beta(B_{3,1,5}) = -4.34292 < \beta(B_{4,1,5}) = \beta(F_7) = -4.30278 < \beta(B_{5,1,5}) = -4.27497 < \beta(B_{6,1,5}) = -4.25517 < \beta(F_8) = \beta(B_{3,1,2}) = -4.24978 < \beta(B_{7,1,5}) = -4.24089 < \beta(B_{8,1,5}) = -4.23057 < \beta(B_{9,1,5}) = -4.22318 < \beta(B_{10,1,5}) = -4.21795 < \beta(F_9) = \beta(B_{11,1,5}) < \beta(F_{m-1}) = \beta(B_{m-6,1,2})$. For $n \geq 21$, it follows, from Lemma 4.1.6 and (9) of Lemma 4.2.4, that $\beta(B_{n-9,1,5}) < \beta(T_{1,3,11}) = \beta(B_{8,1,2})$. From (3) and (8) of Lemma 4.2.4, the result holds.

(3) From (10) of Lemma 4.2.4, the result evidently holds.

(4) By (3) of Lemma 4.2.4 and (2) of the theorem, it is easy to get the result.

(5) Since $n \geq m$ and $t \geq 6$, from (8) of Lemma 4.2.4 and Lemma 4.1.6, we have $\beta(B_{m-t-4,1,t}) \leq \beta(B_{n-t-4,1,t}) < \beta(B_{n-9,1,t}) < \beta(B_{n-9,1,5})$.

(6) Applying (2) of the theorem and (2) of Lemma 4.2.4, the result holds.

(7) From (4) of Lemma 4.2.4 and (2) of the theorem, we can get the result.

(8) By (6) of Lemma 4.2.4 and (3) of the theorem, the result evidently holds.

(9) Using (7) of Lemma 4.2.4 and (3) of the theorem, we can get the result.

□

Lemma 4.2.5 [12, 17] (1) For $t \geq 10$ and $1 \leq t_1 \leq 8$, we have

$$\gamma(U_{1,2,t,5,1}) < \gamma(U_{1,2,9,5,1}) = -4 < \gamma(U_{1,2,t_1,5,1})$$

(2) For $r, t \geq 1$, $h(U_{1,2,r,1,t}) = h(K_1 \cup B_{r,1,t})$.

Lemma 4.2.6 For $n \geq 10$ and $1 \leq n_1 \leq 8$, we have

$$\gamma(B_{n-9,1,5}) < \gamma(B_{9,1,5}) = -4 < \gamma(B_{n_1,1,5})$$

Proof. From Lemma 4.2.5, the result obviously holds. □

4.3 The chromaticity of graph $\overline{B_{n-9,1,5}}$

Theorem 4.3.1 Let G be a graph such that $G \sim^h B_{n-9,1,5}$, where $n \geq 10$. Then G contains at most two components whose first characters are 1, furthermore, one of both is P_2 and the other is P_4 or one of both is P_2 and the other is C_3.

Proof. Let G_1 be one of the components of G such that $R_1(G_1) = 1$. From Theorem 4.2.2, that $h_1(G_1)|h(B_{n-9,1,5})$ if and only if $G_1 \cong P_2$ and $n = 3k+3$, or $G_1 \cong P_4$ and $n = 5k+3$. We distinguish the following cases:

According to (1) of Lemma 4.1.3, we obtain the following equality:

$$h(B_{15k+9,1,5}) = h(P_{15})h(B_{15(k-1)+9,1,5}) + xh(P_{14})h(B_{15(k-1)+8,1,5}) \quad (4.2)$$

Noting that $\{n|n = 3k+3, k \geq 3\} \cap \{n|n = 5k+3, k \geq 2\} = \{n|n = 15k+18, k \geq 0\}$, we have

$$h(P_2)h(P_4) \mid h(B_{15(k-1)+9,1,5}) \tag{4.3}$$

By Lemma 4.2.1, we get $h(P_2) \mid h(P_{14})$ and $h(P_4) \mid h(P_{14})$, together with $(h(P_2), h(P_4)) = 1$, which leads to

$$h(P_2)h(P_4) \mid h(P_{14}) \tag{4.4}$$

From (4.2) to (4.4), $h(P_2)h(P_4) \mid h(B_{15k+9,1,5})$. Noting $h(P_4) = h(K_1 \cup C_3)$, we also have $h(P_2)h(C_3) \mid h(B_{15k+9,1,5})$, together with Theorem 4.2.3, so the theorem holds. □

Lemma 4.3.1 Let G be a graph such that $G \sim^h B_{n-9,1,5}$, where $n \geq 10$. If $n \neq 18$, then G does not contain K_4^- as one of its components.

Proof. According to Theorem 4.2.2, we arrive at $h_1(P_2) \mid h_1(B_{9,1,5})$, that is, $(x+1) \mid h_1(B_{9,1,5})$. From Lemma 4.2.6, we obtain that $\gamma(B_{9,1,5}) = -4$ and $(x+4) \mid h_1(B_{9,1,5})$ if and only if $n = 18$. Noting that $(x+1, x+4) = 1$ and $h_1(K_4^-) = (x+1)(x+4)$, we obtain that $h_1(K_4^-) \mid h_1(B_{n-9,1,5})$ if and only if $n = 18$. From this together with $\alpha(K_4^-) = 2$ and $\alpha(B_{9,1,5}) = 9$, we know that the lemma holds. □

Theorem 4.3.2 Let G be a graph such that $G \sim^h B_{n-9,1,5}$, where $n \geq 9$.

(1) If $n = 18$, then $[G]_h = \{B_{9,1,5}, P_4 \cup K_4^- \cup C_9(P_2), C_9(P_2) \cup D_8, K_4^- \cup \psi_9^3(6,1)\}$;

(2) If $n \neq 18$, then $[G]_h = \{B_{n-9,1,5}\}$.

Proof. **(1)** When $n = 18$, let graph G satisfy $h(G) = h(B_{9,1,5})$. From Lemma 4.1.4, $n(G) = m(G)$ and $R_1(G) = -1$. We distinguish the following cases:

Case 1 G is a connected graph.

By $R_2(G) = R_2(B_{9,1,5}) = 4$ and (2) of Lemma 4.2.3, we have $G \in \mathscr{G} = \{C_r(P_s) \mid r+s = 19\} \cup \{Q_{1,22}\} \cup \{B_{r,1,t} \mid r+t = 15\}$. Combining this with Theorem 2.1 and Definition 3.2, we get $R_3(G) = R_3(B_{9,1,5}) = -1827$. By direct calculation, we have $R_3(C_r(P_s)), R_3(Q_{1,22}), R_3(B_{r,1,t}) \neq -1827$, where $r+s = 19$, $r+t = 15$. Thus $\mathscr{G} \not\subseteq [G]_h$.

Case 2 G is not a connected graph.

By calculation, we have $h(G) = h(B_{9,1,5}) = x^9(x+1)(x+4)(x^2+3x+1)(x^5+10x^4+34x^3+45x^2+19x+1)$. Since $x+4$ is not an adjoint polynomial of a graph, we have $h(G) = h(B_{9,1,5}) = x^4 f_1(x) f_2(x) f_3(x)$, where $f_1(x) = (x+1)(x+4)$, $f_2(x) = x^2+3x+1$ and $f_3(x) = x^5+10x^4+34x^3+45x^2+19x+1$. By calculation, $R_1(f_1(x)) = -1$. Noting that $b_1(f_1(x)) = 5$, we obtain $f_1(x) = h_1(K_4^-)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph.

Subcase 2.1 K_4^- is a component of G.

Since G is not connected, the expression of G must be $G = K_4^- \cup G_1$. Noting that $h(P_4) = x^4+3x^3+x^2$ and $h(C_3) = x^3+3x^2+x$, we consider the following cases:

Subcase 2.1.1 P_4 is a component of G.

In this subcase, $G = K_4^- \cup P_4 \cup G_1$, where $h(G_1) = x^{10} + 10x^9 + 34x^8 + 45x^7+19x^6+x^5$. Noting that $R_1(G_1) = -1$ and $n(G_1) = m(G_1) = 10$, we have from Lemma 4.1.4 that $G_1 \in \xi$. From Lemma 4.2.3, $R_2(G) = R_2(B_{9,1,5}) = 4$. From this together with $R_2(G) = R_2(K_4^-) + R_2(P_4) + R_2(G_1)$, we have $R_2(G_1) = 3$, which results in $G_1 \in \{C_9(P_2), B_{5,1,1}\}$ by (1) of Lemma 4.2.3. By calculating the third invariant of these graphs, $K_4^- \cup P_4 \cup C_9(P_2) \in [G]_h$.

Subcase 2.1.2 C_3 is a component of G.

The expression of G is $G = K_4^- \cup C_3 \cup G_1$, where $h(G_1) = x^{11} + 10x^{10} + 34x^9 + 45x^8 + 19x^7 + x^6$. From $R_1(G_1) = -1$ and $n(G_1) = m(G_1) + 1 = 11$, it follows from Lemma 4.1.4 that $G \in \mathscr{G} = \{K_4^- \cup C_3 \cup T_{r,s,t,a,b}\}$, where $r + s + t + a + b = 11$. By calculating the third invariant of these graphs, $\mathscr{G} \not\subset [G]_h$.

Subcase 2.1.3 Neither C_3 nor P_4 is a component of G.

The expression of G must be $G = K_4^- \cup G_1$, where G_1 is connected. Noting that $R_1(G_1) = -2$ and $n(G_1) = m(G_1) + 2 = 11$, we have from Lemma 4.1.4 that $G_1 \in \psi$. Then $R_3(G) = R_3(B_{9,1,5}) = -1827 = R_3(K_4^-) + R_3(G_1)$ by Theorem 3.1, which implies $R_3(G_1) = -1827$ by Theorem 2.2 and Definition 3.2. From $n(G) = n(B_{9,1,5}) = 18$, we consider that $G_1 \in \{\psi_9^1, \psi_9^2, \psi_9^3(6,1), \psi_9^3(5,2), \psi_9^3(4,3), \psi_9^4(3,1), \psi_9^4(2,2), \psi_9^5(1,1,2), \psi_9^5(2,1,1)\} \cup \{\psi_9^4(1,3)\}$ (Figure 1). By calculating the third invariant of these graphs, $K_4^- \cup \psi_9^3(6,1) \in [G]_h$.

Subcase 2.2 K_4^- is not a component of G.

Let $G = G_1 \cup G_2$ and $h_1(G_1) = x^4(x+1)(x+4)(x^2+3x+1)$ and $h_1(G_2) = x^{10}+10x^9+34x^8+45x^7+19x^6+x^5$. Noting that $R_1(G_1) = 0$ and $n(G_1) = m(G_1)$, we have $G_1 \cong C_8$ or $G_1 \cong D_8$. It is easy to see that $R_1(G_2) = -1$ and $n(G_2) = m(G_2)$. Combining this with Lemma 4.2.2 and Theorem 4.1.1, $R_2(G) = R_2(B_{9,1,5}) = 4 = R_2(G_1) + R_2(G_2)$. If $G_1 \cong C_8$, then $R_2(G_2) = 4$, which leads to $G_2 \cong B_{1,2,2}$. So $R_3(G) = R_3(C_8) + R_3(B_{1,2,2}) \neq R_3(B_{9,1,5})$. Thus $C_8 \cup B_{1,2,2} \notin [G]_h$. If $G_1 \cong D_8$, then $R_2(G) = 3$, which results in $G_2 \cong C_9(P_2)$ or $G_2 \cong B_{5,1,1}$. By calculating the third invariant of these graphs, $C_9(P_2) \cup D_8 \in [G]_h$.

(2) When $n \geq 10$ and $n \neq 18$, let $G = \bigcup_{i=1}^{t} G_i$. From Theorem 4.1.1, we have

$$h(G) = \prod_{i=1}^{t} h(G_i) = h(B_{n-9,1,5}), \tag{4.5}$$

which results in $\beta(G) = \beta(B_{n-9,1,5}) \in [-2-\sqrt{5}, -4)$ by Lemma 4.1.8. Let s_i denote the number of components G_i such that $R(G_i) = -i$, where $i \geq -1$. From Theorems 4.1.1 and 4.3.1, it follows that $0 \leq s_{-1} \leq 2$,

$$R_1(G) = \sum_{i=1}^{t} R_1(G_i) = -1 \text{ and } n(G) = m(G), \tag{4.6}$$

which results in

$$s_{-1} = s_1 + 2s_2 + 3s_3 - 1. \tag{4.7}$$

Let $\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\cup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\cup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\cup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} | l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} | l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} | l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by T for short, $A = \{i | i \geq 4\}$ and $B = \{j | j \geq 5\}$.

We distinguish the following cases by $0 \leq s_{-1} \leq 2$:

Case 1 $s_{-1} = 0$.

It follows, from (4.7), that $s_3 = s_2 = 0, s_1 = 1$. From (4.6), we set

$$G = G_1 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.8}$$

where $R_1(G_1) = -1$.

From Lemmas 4.2.2, 4.2.3 and Theorem 4.1.1, we arrive at

$$R_2(G) = R_2(B_{n-9,1,5}) = 4 = R_2(G_1) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.9}$$

.

From (4.6) and (4.8), it follows that $0 \leq n(G_1) - m(G_1) \leq 1$, which brings about the following subcases:

Subcase 1.1 $n(G_1) = m(G_1) + 1$.

Applying Lemmas 4.1.4 and 4.3.1, we have $G_1 \cong F_m$. Recalling that $n(G) = m(G)$, we obtain that

$$a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1 \tag{4.10}$$

Using (3) of Lemma 4.2.2, we arrive at $|B| = a = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. From this together with (4.10), it follows that $b = 1$ and $G = F_m \cup (\cup_{i\in A} C_i) \cup f D_4 \cup T_{1,1,1}$. From (3) of Lemma 4.2.2, Lemmas 4.1.7 and 4.1.8, $\beta(B_{n-9,1,5}) = \beta(G) = \beta(F_m)$. From (2) of Theorem 4.2.4, we have $\beta(F_m) = \beta(B_{n-9,1,5})$ if and only if $m = 7, n = 13$ or $m = 9, n = 20$. It is impossible for $m = 7, n = 13$. If $n(G) = n(B_{n-9,1,5}) = 20$, then $G \cong F_9 \cup C_7 \cup T_{1,1,1}$. From Theorem 3.1, we get $R_3(G) = R_3(F_7) + R_3(C_7) + R_3(T_{1,1,1}) \neq R_3(B_{9,1,5})$. So $\mathcal{G}_1 \not\subseteq [G]_h$.

Subcase 1.2 $n(G_1) = m(G_1)$.

Recalling that $n(G) = m(G)$, we arrive at, from (4.8), $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, which leads to

$$G = G_1 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4. \tag{4.11}$$

From (3) of Lemma 4.1.4 and Lemma 4.1.8, it follows that

$$G_1 \in \{B_{m-t-4,1,t}, C_r(P_2), Q(1,2), C_4(P_3)\}, \quad (4.12)$$

where $m-t-4, t$ and r satisfy the conditions of Lemma 4.1.8.

We distinguish the following subcases by (4.12):

Subcase 1.2.1 $G_1 \cong C_r(P_2)$.

From Lemma 4.1.7, (3) of Lemma 4.2.4, it follows that $\beta(G) = \beta(C_r(P_2))$. Since $\beta(G) = \beta(B_{n-9,1,5})$, we have, from (1) of Theorem 4.2.4, that $\beta(G) = \beta(C_r(P_2))$ if and only if $n = 20, r = 8$ or $n = 18, r = 9$. The latter has been discussed in Case 1. We now only consider the former case. If $n = 20, r = 8$, then $G \in \mathscr{G}_2 = \{C_8(P_2) \cup C_{11}, C_8(P_2) \cup D_{11}, C_8(P_2) \cup C_7 \cup D_4, C_8(P_2) \cup D_7 \cup D_4\} \cup \{C_8(P_2) \cup C_i \cup C_j, C_8(P_2) \cup C_i \cup D_j, C_8(P_2) \cup D_i \cup D_j\}$, where $i + j = 11$, $4 \leq i, j \leq 7$. By calculating the third invariant of these graphs, we obtain that $\mathscr{G}_2 \not\subseteq [G]_h$.

Subcase 1.2.2 $G_1 \cong Q(1,2)$ or $G_1 \cong C_4(P_3)$.

From (3) of Theorem 4.2.4 and (3) of Lemma 4.2.4, we know that $\beta(G) = \beta(G_1) = \beta(B_{n-9,1,5})$ if and only if $n(G) = 20$, which brings about $G \in \mathscr{G}_3 = \{G_1 \cup C_{14}, G_1 \cup C_i \cup C_j, G_1 \cup C_i \cup D_j, G_1 \cup D_i \cup D_j, G_1 \cup C_r \cup C_s \cup C_t, G_1 \cup C_r \cup C_s \cup D_t, G_1 \cup C_r \cup D_s \cup D_t, G_1 \cup D_r \cup D_s \cup D_t,\}$, where $i+j = 14, 4 \leq i, j \leq 10$, $r + s + t = 14, 4 \leq i, j \leq 7$. By calculation, $\mathscr{G}_3 \not\subseteq [G]_h$.

Subcase 1.2.3 $G_1 \cong B_{m-t-4,1,t}$.

Subcase 1.2.3.1 $1 \leq t \leq 5$.

We only prove the case of $t = 1$, other cases can be similarly discussed by Lemma 4.2.4 and Theorem 4.2.4. From (3) of Lemma 4.2.4, $\beta(G) = \beta(B_{m-5,1,1})$. According to (6) of Theorem 4.2.4, $\beta(B_{m-5,1,1}) = \beta(B_{n-9,1,5})$ if and only if $m = 6, n = 20$. Note that $h(B_{1,1,1}) = h(C_4(P_3)) = h(Q(1,2))$, we can not find adjoint equivalence class by the same method as Subcase 1.2.2.

Subcase 1.2.3.2 $t \geq 6$.

From (4) and (5) of Theorem 4.2.4 and (3) of Lemma 4.2.4, we arrive at $\beta(G) = \beta(B_{m-t-4,1,t}) < \beta(B_{n-9,1,5})$, which contradicts to $\beta(G) = \beta(B_{n-9,1,5})$.

From the above arguments, we have $t = 5$. From Lemma 4.1.7 and (4) of Theorem 4.2.4, $\beta(G) = \beta(G_1) = \beta(B_{m-9,1,5})$, together this with $\beta(G) = \beta(B_{n-9,1,5})$ and (8) of Lemma 4.2.4, we arrive at $m = n$. So $G \cong B_{n-9,1,5}$.

Case 2 $s_{-1} = 1$.

It follows, from (4.7), that $s_1 + 2s_2 = 2$, which implies the following subcases:

Subcase 2.1 $s_2 = 1, s_1 = 0$.

Without loss of generality, let G_1 be the component such that $R_1(G_1) = -2$. From Corollary 4.1.1, we know $\beta(G_1) < -2 - \sqrt{5}$, which contradicts $\beta(B_{n-9,1,5}) \in [-2 - \sqrt{5}, -4)$.

Subcase 2.2 $s_2 = 0, s_1 = 2$.

Without loss of generality, let

$$G = G_1 \cup G_2 \cup G_3 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.13}$$

where $G_1 \in \{P_2, P_4, C_3\}$, $R_1(G_2) = R_1(G_3) = -1$.

From Lemmas 4.2.2, 4.2.3 and Theorem 4.1.1, we arrive at

$$R_2(G) = R_2(B_{n-9,1,5}) = R_2(G_1) + \sum_{i=2}^{3} R_2(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.14}$$

Subcase 2.2.1 $G_1 \cong P_2$ or $G_1 \cong P_4$.

Recalling that $n(G) = m(G)$, we obtain that $1 \leq \sum_{i=2}^{3}(n(G_i) - m(G_i)) \leq 2$. Thus we have the following subcases to consider:

Subcase 2.2.1.1 $\sum_{i=2}^{3}(n(G_i) - m(G_i)) = 1$.

From Lemmas 4.1.4, 4.3.1 and (4.13), it follows that $G_2 \cong F_m$, $G_3 \in \xi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. If $G_1 \cong P_2$, then $4 = -1 + \sum_{i=2}^{3} R_2(G_i) + |B|$. Combining this with (3) of Lemma 4.2.2, $R_2(G_3) = 5 - R_2(F_m) - |B| \leq 1 - |B| \leq 1$, which contradicts to $G_3 \in \xi$ by Corollary 4.2.1. If $G_1 \cong P_4$, then $4 = -2 + \sum_{i=2}^{3} R_2(G_i) + |B|$. From this together this with (3) of Lemma 4.2.2, $R_2(G_3) = 6 - R_2(F_m) - |B| \leq 2 - |B| \leq 2$, which also contradicts to $G_3 \in \xi$.

Subcase 2.2.1.2 $\sum_{i=2}^{3}(n(G_i) - m(G_i)) = 2$.

It is obvious that $G_i \cong F_m (i = 2, 3)$ and $a + b + |\mathcal{T}_1| + |\mathcal{T}_2| + |\mathcal{T}_3| = 1$ by Lemmas 4.1.4, 4.3.1 and (4.13). From these together with (4.14), we have $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $4 = R_2(G_1) + 2R_2(F_m) + |B| + a + |\mathcal{T}_1|$. Thus $R_2(G_1) = -4 - |B| - a - |\mathcal{T}_1| \leq -4$, which contradicts to $-2 \leq R_2(G_1) \leq -1$ by (5) of Lemma 4.2.2.

Subcase 2.2.2 $G_1 \cong C_3$.

Recalling that $n(G) = m(G)$, it follows that $0 \leq \sum_{i=2}^{3}(n(G_i) - m(G_i)) \leq 2$. If $\sum_{i=2}^{3}(n(G_i) - m(G_i)) = 0$, then we have, from Lemma 4.1.4, 4.3.1 and (4.6), that $G_i \in \xi (i = 2, 3)$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Then $4 = -2 + \sum_{i=2}^{3} R_2(G_i) + |B|$, which contradicts to $G_i \in \xi (i = 2, 3)$ by Corollary 4.2.1. Other two subcases can be similarly discussed.

Case 3 $s_{-1} = 2$.

It follows, from (4.7), that $s_1 + 2s_2 + 3s_3 = 3$. We have the following subcases to consider:

Subcase 3.1 $s_3 = 1, s_2 = s_1 = 0$.

Without loss of generality, let the component G_1 such that $R_1(G_1) = -3$. From Corollary 4.1.1, we have $\beta(G) < -2 - \sqrt{5}$, which contradicts to $\beta(G) \in [-2 - \sqrt{5}, -4)$.

Subcase 3.2 $s_2 = 1, s_1 = 1, s_3 = 0$.

With the same method as that of Subcase 3.1, we get a contradiction.

Subcase 3.3 $s_1 = 3, s_2 = s_3 = 0$.

Without loss of generality, let

$$G = P_2 \cup G_1 \cup (\bigcup_{i=2}^{4} G_i) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup b T_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \quad (4.15)$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_i) = -1 (i = 2,3,4)$.

From Lemmas 4.2.2, 4.2.3 and Theorem 4.1.1,

$$R_2(G) = R_2(B_{n-9,1,5}) = -1 + R_2(G_1) + \sum_{i=2}^{4} R_2(G_i) + |B| + a + |T_1| + 2|T_2| + 3|T_3| \quad (4.16)$$

Subcase 3.3.1 $G_1 \cong P_4$.

Recalling that $n(G) = m(G)$, we get $2 \le \sum_{i=2}^{4}(n(G_i) - m(G_i)) \le 3$. If $\sum_{i=2}^{4}(n(G_i) - m(G_i)) = 2$, then $G_i \cong F_m (i = 2,3)$, $G_4 \in \xi$ and $a = b = |T_1| = |T_2| = |T_3| = 0$. Combining these with (4.16), we have $R_2(G_4) = 7 - 2R_2(F_m) - |B| \le -1 - |B| \le -1$ by (3) of Lemma 4.2.2, which contradicts to $G_4 \in \xi$ by Corollary 4.2.1. We can get a similar contradiction for $\sum_{i=2}^{4}(n(G_i) - m(G_i)) = 3$.

Subcase 3.3.2 $G_1 \cong C_3$.

From (4.6), it follows that $1 \le \sum_{i=2}^{4}(n(G_i) - m(G_i)) \le 3$. We only prove the case of $\sum_{i=2}^{4}(n(G_i) - m(G_i)) = 1$, other two cases can be discussed similarly. Applying Lemmas 4.1.4, 4.3.1 and (4.15), it follows that $G_2 \cong F_m$, $G_i \in \xi (i = 3,4)$ and $a = b = |T_1| = |T_2| = |T_3| = 0$. From these together with (4.16), $4 = -3 + \sum_{i=2}^{4} R_2(G_i) + |B|$, which results in $R_2(G_3) + R_2(G_4) = 3 - |B|$ by (3) of Lemma 4.2.2. It contradicts to $G_i \in \xi (i = 3,4)$ by Corollary 4.2.1.

This completes the proof of the theorem. □

Corollary 4.3.1 If $n \ge 10$, graph $B_{n-9,1,5}$ is adjoint uniqueness if and only if $n \ne 18$.

Corollary 4.3.2 If $n \ge 10$, the chromatic equivalence class of $\overline{B_{n-9,1,5}}$ only contains the complements of graphs that are in Theorem 4.3.2.

Corollary 4.3.3 If $n \ge 10$, graph $\overline{B_{n-9,1,5}}$ is chromatic uniqueness if and only if $n \ne 18$.

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