Shape and pattern containment of separable permutations

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Abstract

A word has a *shape* determined by its image under the Robinson-Schensted-Knuth correspondence. We show that when a word w contains a separable (i.e., 3142- and 2413-avoiding) permutation σ as a pattern, the shape of w contains the shape of σ. As an application, we exhibit lower bounds for the lengths of supersequences of sets containing separable permutations.

The Robinson-Schensted-Knuth (RSK) correspondence associates to a word w a pair of *Young tableaux*, each of equal partition shape λ. We say that w has *shape* $\text{sh}(w) = \lambda$ and that the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ *contains* the partition $\mu = (\mu_1, \mu_2, \ldots)$ if $\mu_i \leq \lambda_i$ for all $i \geq 1$. It is natural to expect that if σ is a subsequence of w, then $\text{sh}(\sigma) \subseteq \text{sh}(w)$. However, this is not necessarily the case: If $\sigma = 2413$ and $w = 24213$, then

$$(P(\sigma), Q(\sigma)) = \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right) \text{ and } (P(w), Q(w)) = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right).$$

We see that $\text{sh}(w) = (3,1,1) \not\supseteq (2,2) = \text{sh}(\sigma)$. The main theorem of this paper is that the inclusion does hold when σ is a *separable* permutation. Furthermore, σ need only be contained as a pattern rather than as an actual subsequence.

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Theorem 1. If a word w contains a separable permutation σ as a pattern, then $\text{sh}(w) \supseteq \text{sh}(\sigma)$.

Our discovery of Theorem 1 was motivated by an application involving lower bounds for *shortest containing supersequences.* Such supersequences arise in bioinformatics [13, 14] through the design of DNA microarrays, in planning [5] and in data compression [15]. This application to supersequences is described in Section 3. Section 1 introduces the notation required for the proof of Theorem 1 appearing in Section 2. Section 2.1 discusses the relationship between Greene's Theorem, separable permutations, and the contents of this paper.

Remark 2. It is *not* true that if σ is not separable, then there exists a word w containing it for which $\text{sh}(w) \not\supseteq \text{sh}(\sigma)$: Let $\sigma = 24513$ with $\text{sh}(\sigma) = (3,2)$. The permutation σ contains the pattern 2413 and hence is not separable. If w contains σ but not its shape then $\text{sh}(w)$ is a hook. In other words, w is the shuffle of one increasing and one decreasing subsequence. The restriction to 24513 should, therefore, also split into two sequences — one decreasing and one increasing. This is impossible.

1 Background and setup

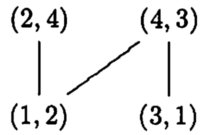
Let $[n]^*$ denote the set of finite-length words on $[n] := \{1, 2, \ldots, n\}$ and let $[n]^a$ denote the subset of length-a words. The set of permutations of length n is denoted by S_n (here a subset of $[n]^n$). Permutations will be denoted by Greek letters and written in one-line notation. For example, the permutation $\tau \in S_3$ defined by $\tau(1) = 3$, $\tau(2) = 1$ and $\tau(3) = 2$ is written 312. When referring to a subsequence of a permutation τ we make no distinction between the actual subsequence and the corresponding subset of elements; the subsequence can be reconstructed by the positions in τ. The length of a word u is denoted $|u|$.

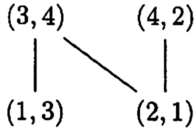
Given a word $w \in [n]^a$ and a permutation $\pi \in S_m$, $m \leq a$, we say that *w contains the pattern π* if there exist indices $1 \leq i_1 < i_2 < \cdots < i_m \leq a$ such that, for all $1 \leq j, k \leq m$, $w(i_j) < w(i_k)$ if and only if $\pi(j) < \pi(k)$ and $w(i_j) > w(i_k)$ if and only if $\pi(j) > \pi(k)$. If w does not contain the pattern π, then we say *w avoids π*.

It is important to note that Theorem 1 considers the relationship between a *word* w and a *permutation* σ. The word w is assumed to contain σ as a pattern. In turn, the permutation σ is assumed to be separable. One characterization of the class of *separable* permutations (see [4]) is as those permutations that simultaneously avoid the patterns 3142 and 2413.

Given a permutation $\pi \in S_n$, let P_π denote its *inversion poset.* P_π has elements $(i, \pi(i))$ for $1 \leq i \leq n$ under the partial order $\prec$, in which

$(a,b) \prec (c,d)$ if and only if $a < c$ and $b < d$. Increasing subsequences in π correspond to chains in P_π. A longest increasing subsequence of π corresponds to a maximal chain in P_π. In the pictorial representations of posets in this paper, we indicate $(a,b) \prec (c,d)$ (as opposed to $(c,d) \prec (a,b)$) by placing (a,b) lower on the page than (c,d).

Example 3. The inversion poset of 2413 is 

and that of 3142 is 

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Example 3 above immediately gives the following fact.

Fact 4. A permutation π is separable if and only if its inversion poset P_π has no (induced) subposet isomorphic to 

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We write our partitions with parts in decreasing order and make no distinction between the positive and zero parts. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n (denoted $\lambda \vdash n$), the associated *Ferrers diagram* consists of λ_i left-justified cells in the i-th row from the top. A *semistandard Young tableau of shape* λ is a filling of the cells in this diagram with positive integers such that the rows weakly increase from left to right and the columns strictly increase from top to bottom. The set of such tableaux with entries from $[n]$ is denoted by $\mathrm{SSYT}_n(\lambda)$. A tableau $T \in \mathrm{SSYT}_n(\lambda)$ with $\lambda \vdash n$, is *standard* if each number from 1 to n appears in its filling. The set of all such tableaux is denoted by $\mathrm{SYT}(\lambda)$. Given a semistandard tableau T, the *reading word* of T, $\mathrm{rw}(T)$, is the word obtained by reading off the rows from left to right starting with the bottom row. For $\lambda \vdash n$, define the *superstandard tableau* $T \in \mathrm{SYT}(\lambda)$ by filling in the rows from top to bottom. That is, by placing $1, 2, \ldots, \lambda_1$ in the first row, $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ in the second row, etc.

The RSK correspondence yields a bijection between the set of length-a words $[n]^a$ and $\cup_{\lambda \vdash a} \mathrm{SSYT}_n(\lambda) \times \mathrm{SYT}(\lambda)$ [9]. We give a brief description of how to compute the pair $(P(w), Q(w))$ to which a word $w \in [n]^a$ corresponds. Write $w = w'x$ with $w' \in [n]^{a-1}$. By induction, we know that w' maps to some pair $(P(w'), Q(w'))$. We *row insert* x in the first row of $P(w')$ as follows: If $x = x_1$ is greater than or equal to all elements in this row, place x_1 at the end of the row. Otherwise, find the leftmost entry,

x_2, in the row that is strictly greater than x_1. Place x_1 in this position and "bump" x_2 to be inserted into the next row. This process generates a finite sequence $x_1, \ldots, x_k$ of bumped elements and ends by adding x_k at the end of the k-th row, creating a new semistandard tableaux $P(w)$. Set $Q(w)$ to have an a in the new box (end of row k) created by the bumping process. The *shape* of w, $\mathrm{sh}(w)$, is the shape of $P(w)$ (or, equivalently, of $Q(w)$). Throughout this paper σ will denote a separable permutation with $\mathrm{sh}(\sigma) = \mu = (\mu_1, \mu_2, \ldots)$.

Example 5. The permutation $\pi = 7135264$ contains the pattern 4231 but avoids 3412. Under the RSK correspondence, $w = 2214312$ maps to $(P(w), Q(w)) = \left(\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 4 & & \end{array}, \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & & \end{array} \right)$ with $\mathrm{rw}(P(w)) = 4223112$. Finally, the superstandard tableau of shape $(3,3,2)$ is $\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{array}$.

2 Proof of Theorem 1

Many properties of a word w translate to natural properties of the associated tableaux. For example, the length of the longest weakly increasing subsequence of w equals the length of the first row of $P(w)$. In fact, the minor generalization of Greene's Theorem [7] to words with repetitions (see [16, Theorem 4.8.10]) gives a much more precise correspondence.

Theorem 6 (Greene's Theorem). Let w be a word of shape λ. For any $d \geq 0$ the sum $\lambda_1 + \cdots + \lambda_d$ equals the maximum number of elements in a disjoint union of d weakly increasing subsequences of w.

In order to prove Theorem 1, we will combine the insight afforded by Greene's Theorem with the ability to exchange collections of disjoint increasing subsequences with other collections for which the number of intersections has, in a certain sense, been reduced. (We will take the adjective "weakly" to be understood.) Lemma 7, which is the only place separability explicitly appears in our proof, allows us to perform these exchanges.

Lemma 7. Let u, α, and β be increasing subsequences of a separable permutation σ. Assume further that α and β are disjoint. Then there exist two disjoint increasing subsequences γ and δ, such that $\gamma \cup \delta = \alpha \cup \beta$ and $\gamma \cap u = \emptyset$.

Proof. Write $u = u_0 \cup u_1$ with $u_0 \cap (\alpha \cup \beta) = \emptyset$ and $u_1 \subset \alpha \cup \beta$. Since $\gamma \subset \alpha \cup \beta$, the requirement that γ and u_1 be disjoint ensures that γ and u

are disjoint as well. Hence, without loss of generality, we may restrict our attention in the proof to the case in which $u \subset \alpha \cup \beta$.

We prove by contradiction. Consider the inversion poset P_σ of the separable permutation σ. Increasing subsequences are in correspondence with chains and we will regard them as such. Assume there is no chain $\delta \subset (\alpha \cup \beta)$ such that $u \subset \delta$ and $(\alpha \cup \beta) \setminus \delta$ is also a chain. Let $\omega \subset (\alpha \cup \beta)$ be a maximal chain such that $u \subset \omega$. Then there exist two incomparable points $x, y \in (\alpha \cup \beta) \setminus \omega$. (We will write $x \| y$ to indicate the incomparability of these two elements.) Then x and y belong to the two different chains, e.g. $x \in \alpha$, $y \in \beta$. By maximality, $x \cup \omega$ and $y \cup \omega$ are not chains. Hence there exist $a, b \in \omega$ for which $x \| a$, $y \| b$, so we must have $a \in \beta$ and $b \in \alpha$. Without loss of generality, assume $a \succ b$. Then we must have $x \succ b$ and $y \prec a$. We have $\begin{array}{ccc} x & & a \\ | & \diagup & | \\ b & & y \end{array}$ with $x \| a$, $x \| y$, and $y \| b$. This is a subposet of P_σ isomorphic to $\begin{array}{ccc} * & & * \\ | & \diagup & | \\ * & & * \end{array}$, contradicting Fact 4. □

Lemma 7 can also be proved constructively.

Algorithm UNRAVEL

INPUT: $(\sigma, u, \alpha, \beta)$. A separable permutation σ along with increasing subsequences α, β and u. The sequences α and β are disjoint.

OUTPUT: (γ, δ). Disjoint increasing subsequences γ and δ of σ satisfying $\gamma \cup \delta = \alpha \cup \beta$ and $\gamma \cap u = \emptyset$.

Step 0: Initialize variables.

Set $z = \alpha \cup \beta$ and let ℓ and n denotes the lengths of $u \cap z = u_1 \cdots u_\ell$ and $z = z_1 \cdots z_n$, respectively. (As was the case for Lemma 7, the elements of u not in $\alpha \cup \beta$ are irrelevant to the construction of γ and δ.) There exist indices $i_1 < i_1 < \cdots < i_\ell$ such that $u_j = z_{i_j}$ for each $1 \le j \le \ell$. Augment the two sequences by prepending a $u_0 = z_0 < \min\{z_i\}_{1 \le i \le n}$ and appending a $u_{\ell+1} = z_{n+1} > \max\{z_i\}_{1 \le i \le n}$.

Step 1: Determine δ.

For each $1 \le j \le \ell$, let δ^j be the sequence of left-to-right maxima from $z_{i_j} \cdots z_{i_{j+1}-1}$ whose values are greater than or equal to u_j and less than u_{j+1}. Define δ^0 analogously except with values *greater than* u_0 and less than u_1. Define $\delta = \delta^0 \cdots \delta^\ell$.

Step 2: OUTPUT $\gamma = (\alpha \cup \beta) \setminus \delta$ and δ.

Lemma 8. Let σ be a separable permutation. The sequences $\gamma = (\alpha \cup \beta) \setminus \delta$ and δ returned by $\mathsf{UNRAVEL}(\sigma, u, \alpha, \beta)$ are increasing sequences that satisfy $\gamma \cup \delta = \alpha \cup \beta$ and $\gamma \cap u = \emptyset$.

Proof. That $\gamma \cup \delta = \alpha \cup \beta$ and that δ is increasing follow directly by construction. (Note that δ does not include u_0 or $u_{\ell+1}$.) Furthermore, each $u_j = z_{i_j}$ is a left-to-right maximum of the sequence $z_{i_j} \cdots z_{i_{j+1}}$ and hence is part of δ. So $\gamma \cap u = \emptyset$.

It remains to show that γ is increasing. Suppose not. Then there exist indices a and j such that $\gamma_j = z_a > \gamma_{j+1}$. Let m be the unique value such that $i_m < a < i_{m+1}$. Note that since z is a shuffle of two increasing disjoint words, z also avoids the pattern 321.

We now split into cases in order to obtain a contradiction by arguing that z must contain one of the three patterns 321, 3142 or 2413.

1. Suppose $\gamma_j > u_{m+1}$. This implies $m < \ell$ (and hence that u_{m+1} is an element of z). We argue according to the region in which the point γ_{j+1} lies (see Figure 1).

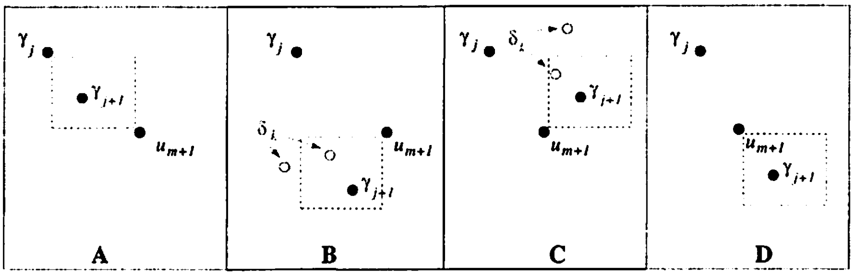


Figure 1: The cases in the proof of Lemma 8 for which $\gamma_j > u_{m+1}$. Points are labeled by their y-values.

A) Then $\gamma_j \gamma_{j+1} u_{m+1}$ forms a 321 pattern.

B) Since γ_{j+1} is not a left-to-right maximum, there must be some element δ_k lying to the northwest of γ_{j+1} yet below u_{m+1}. If δ_k lies to the left of γ_j, then $\delta_k \gamma_j \gamma_{j+1} u_{m+1}$ forms a 2413 pattern. Otherwise, $\gamma_j \delta_k \gamma_{j+1}$ forms a 321 pattern.

C) Since γ_{j+1} is not a left-to-right maximum, there must be some element δ_k lying to the northwest of γ_{j+1} yet to the right of u_{m+1}. If δ_k lies above γ_j, then $\gamma_j u_{m+1} \delta_k \gamma_{j+1}$ forms a 3142 pattern. Otherwise, $\gamma_j \delta_k \gamma_{j+1}$ forms a 321 pattern.

D) Then $\gamma_j u_{m+1} \gamma_{j+1}$ forms a 321 pattern.

2. Suppose $\gamma_j < u_{m+1}$. Since γ_j is not a left-to-right maximum, there must be some element δ_k (possibly u_m) lying to the northwest of γ_j. Hence $\delta_k \gamma_j \gamma_{j+1}$ forms a 321 pattern.

□

Example 9. Figure 2 illustrates the sequences γ and δ that arise from Algorithm UNRAVEL. The two original sequences shuffled together are connected by dotted lines. The elements of u are illustrated by open circles. The boxes indicate the regions in which the elements of δ (other than those of u itself) are required to lie. Finally, the sequence δ is connected by the thick, dashed line.

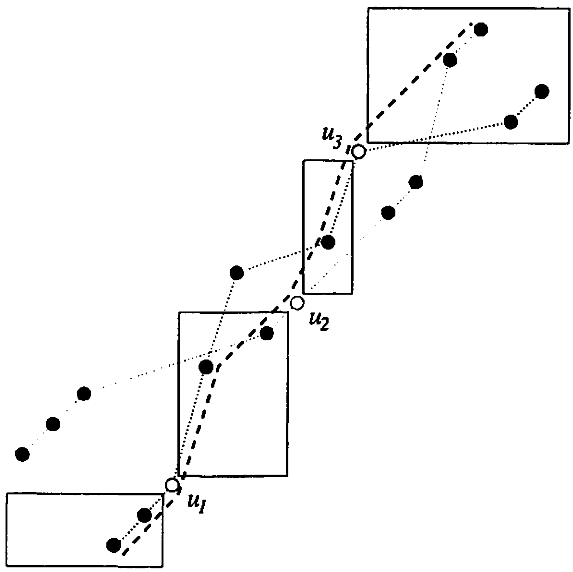


Figure 2: Example application of Algorithm UNRAVEL.

Proposition 10. Let $k \geq 0$ and $u^1, \ldots, u^k$ be disjoint (possibly empty) increasing subsequences of the separable permutation σ. Then there exists an increasing subsequence u^{k+1}, disjoint from each u^i, $1 \leq i \leq k$ such that $|u^{k+1}| \geq \mu_{k+1}$.

Proof. Let $V = (v^1, \ldots, v^{k+1})$ be a sequence of $k+1$ disjoint, increasing subsequences of σ of maximum total length. Let $m = m(V) \leq k+1$ be the largest value such that if $p \leq m$ then $u^i \cap v^j = \emptyset$ for all i, j with $1 \leq i < p \leq j \leq k+1$. Such a value of m exists since the requirement is vacuous for $p = 1$. Suppose $m = k+1$; in particular, v^{k+1} is disjoint from each u^i. Since the elements of V are of maximum total length, $|v^1| + \cdots + |v^{k+1}| = \mu_1 + \cdots + \mu_{k+1}$. If $|v^{k+1}|$ were less than μ_{k+1}, then $v^1, \ldots, v^k$ would have total length greater than $\mu_1 + \cdots + \mu_k$. This is impossible. Hence $|v^{k+1}| \geq \mu_{k+1}$ as desired.

Therefore, it suffices to show that given any such V and associated value $m = m(V) < k+1$, we can transform V into a new sequence V' (still of maximal total length) with $m(V') \geq m+1$. We do this by repeated applications of Algorithm UNRAVEL. (These applications will corral all of the elements of $v^m \cup \cdots \cup v^{k+1}$ that are also in u^m into a single sequence δ_{k-m+1}.) Set

$$\begin{aligned}
(\gamma_1, \delta_1) &= \mathsf{UNRAVEL}(\sigma, u^m, v^m, v^{m+1}), \\
(\gamma_2, \delta_2) &= \mathsf{UNRAVEL}(\sigma, u^m, \delta_1, v^{m+2}), \\
\vdots \quad &= \quad \vdots \\
(\gamma_{k-m+1}, \delta_{k-m+1}) &= \mathsf{UNRAVEL}(\sigma, u^m, \delta_{k-m}, v^{k+1}).
\end{aligned}$$

We will set $V' = (v^1, \ldots, v^{m-1}, \delta_{k-m+1}, \gamma_1, \ldots, \gamma_{k-m+1})$. It follows from the definition of Algorithm UNRAVEL and the inputs given to it that each γ_i is an increasing subsequence of σ that is disjoint from u^m. Consider the intermediate $(k+1)$-tuples:

$$\begin{aligned}
V = V_0 &= (v^1, \ldots, v^{m-1}, v^m, v^{m+1}, \ldots, v^{k+1}) \\
V_1 &= (v^1, \ldots, v^{m-1}, \delta_1, \gamma_1, v^{m+2}, v^{m+3}, \ldots, v^{k+1}) \\
V_2 &= (v^1, \ldots, v^{m-1}, \delta_2, \gamma_1, \gamma_2, v^{m+3}, \ldots, v^{k+1}) \\
\vdots \quad &= \quad \vdots \\
V' = V_{k-m+1} &= (v^1, \ldots, v^{m-1}, \delta_{k-m+1}, \gamma_1, \gamma_2, \ldots, \gamma_{k-m+1}).
\end{aligned}$$

We still need to show that V' satisfies the following properties:

1. $(u^1 \cup \cdots \cup u^{m-1}) \cap (\delta_{k-m+1} \cup \gamma_1 \cup \cdots \cup \gamma_{k-m+1}) = \emptyset$ (i.e., $m(V') \geq m$),

2. $\gamma_i \cap \gamma_j = \emptyset$ for $1 \leq i < j \leq k-m+1$, and

3. $\delta_{k-m+1} \cap \gamma_i = \emptyset$ for $1 \leq i \leq k-m+1$.

By the definition of V and $m(V)$, we know that

$$\left(u^1 \cup \cdots \cup u^{m-1}\right) \cap \left(v^m \cup \cdots \cup v^{k+1}\right) = \emptyset.$$

The definition of Algorithm UNRAVEL implies that $v^m \cup \cdots \cup v^{k+1} = \delta_{k-m+1} \cup \gamma_1 \cup \cdots \cup \gamma_{k-m+1}$. This yields Property 1.

$V_0 = V$ consists of disjoint sequences by hypothesis. Let $1 \le \ell \le k - m + 1$. We will prove by induction that V_ℓ consists of disjoint sequences as well. (Write δ_0 for v^m.) V_ℓ differs from $V_{\ell-1}$ only in that $\delta_{\ell-1}$ and $v^{m+\ell}$ have been replaced by the two disjoint sequences δ_ℓ and γ_ℓ. Since Algorithm UNRAVEL ensures that $\delta_\ell \cup \gamma_\ell = \delta_{\ell-1} \cup v^{m+\ell}$ and $\delta_\ell \cap \gamma_\ell = \emptyset$, it follows immediately that V_ℓ consists of disjoint sequences as well. This shows that Properties 2 and 3 also hold.

Properties 1 through 3 ensure that V' is a sequence of $k+1$ disjoint increasing subsequences of σ of maximum total length with $m(V') \ge m(V)$. By construction, each γ_i is disjoint from u^m. Hence, $m(V') \ge m(V) + 1$ as required. This concludes the proof of the proposition.

□

Example 11. Consider $\sigma = 10652438ba97$ (where we use a for 10 and b for 11). The shape of σ is $(5,3,2,2)$. Suppose we have $u^1 = 0248b$, $u^2 = 167$ and $u^3 = 5a$ and wish to find a disjoint increasing subsequence u^4 of length 2. We could, of course, simply use the remaining two elements, 3 and 9. However, in order to illustrate the proofs of Proposition 10 and Theorem 13, we show how to generate this sequence from an arbitrarily chosen 4-tuple of disjoint increasing subsequences of maximum total length: $V = \{68b, 049, 237, 15a\}$.

Set $k = 3$. Consider the argument of Proposition 10. $m = m(V) = 1$. Let $u = u^1$, $\alpha = 68b$ and $\beta = 049$. Algorithm UNRAVEL yields $\gamma = 69$ and $\delta = 048b$. Applying the algorithm again with $\alpha = 048b$ and $\beta = 237$ yields $\gamma = 37$ and $\delta = 0248b$. Since $15a$ is already disjoint from u^1, a third application of Algorithm UNRAVEL trivially sets $\delta = \alpha = 0248b$ and $\gamma = \beta = 15a$. This produces the new 4-tuple $V' = \{0248b, 69, 37, 15a\}$ with $m(V') = 2$.

Now set $u = u^2$. Once again, an application of the algorithm with $\alpha = 69$ and $\beta = 15a$ yields $\gamma = 59$ and $\delta = 16a$, while a following application to $\alpha = 16a$ and $\beta = 37$ yields $\gamma = 3a$ and $\delta = 167$. This produces the new 4-tuple $V'' = \{0248b, 167, 59, 3a\}$ with $m(V'') = 3$.

A final application of Algorithm UNRAVEL with $u = u^3 = 5a$, $\alpha = 59$ and $\beta = 3a$ yields the sought for $u^4 = \gamma = 39$.

Proof of Theorem 1. Let $\mathrm{sh}(w) = \lambda = (\lambda_1, \lambda_2, \ldots)$. Let σ' be any subsequence of w in the same relative order as the elements of σ; i.e., w contains σ at the positions of σ'. By Greene's Theorem applied to w, for any $k \ge 1$ there exist k disjoint increasing subsequences $w^1, \ldots, w^k$ with $|w^1| + \cdots + |w^k| = \lambda_1 + \cdots + \lambda_k$. The intersection $\sigma' \cap w^i$ induces a subsequence of σ we denote by u^i. These u^i are then k disjoint increasing subsequences of σ. By Proposition 10, there is an increasing subsequence u of σ, disjoint from the u^is, with length at least μ_{k+1}. The mapping $\sigma \mapsto \sigma'$

induces a corresponding map of u to a subsequence u' of w. It follows then that u' is disjoint from each w^i as well. Then $w^1, \dots, w^k, u'$ are $k+1$ disjoint increasing subsequences in w. By Greene's Theorem,

$$|w^1| + \cdots + |w^k| + |u'| \leq \lambda_1 + \cdots + \lambda_k + \lambda_{k+1}.$$

Hence $|u'| \leq \lambda_{k+1}$. We also know by construction that $\mu_{k+1} \leq |u| = |u'|$. Combining these equalities and running over all k yields $\mu \subseteq \lambda$ as desired. □

2.1 Relationship to Greene's Theorem

Greene's Theorem only tells us about the maximum *sum* of lengths of disjoint increasing sequences. It is *not* generally true that one can find d disjoint increasing subsequences $u^1, u^2, \dots, u^d$ of σ with u^i of length μ_i for each i. In other words, the shape of a permutation does not tell us the lengths of the subsequences in a set of d disjoint increasing subsequences of maximum total length; it just tells us the maximum total length.

Example 12. Consider the permutation $\sigma = 236145$ of shape $(4, 2)$. The only increasing subsequence of length four is 2345. However, the remaining two entries appear in decreasing order. Greene's Theorem tells us that we should be able to find two disjoint increasing subsequences of total length 6. Indeed, 236 and 145 work.

Nonetheless, such a collection of subsequences $\{u^i\}$ does exist when σ is a separable permutation.

Proposition 13. Let σ be a separable permutation of shape μ. For any $d \geq 1$, there exist d disjoint, increasing subsequences $u^1, \dots, u^d$ such that the length of each u^i is given by μ_i.

Theorem 1 and Proposition 13 are superficially similar. We have already shown how Theorem 1 follows from Proposition 10 (and Greene's Theorem). Proposition 13 follows even more immediately.

Proof of Proposition 13. We can construct such a sequence via d applications of Proposition 10. In particular, given the $u^1, \dots, u^i$ for some $0 \leq i < d$, produce u^{i+1} by applying the proposition with $k = i$. □

Proposition 13 has a very simple proof relying on the recursive definition of a separable permutation as one that can be built up by direct and skew sums [4]. (The proof follows directly from Proposition 4 of [2].) However, we have been unable to follow a correspondingly direct proof of Theorem 1.

3 Supersequences

Let $B \subseteq S_n$ be a set of permutations. A word w is a *supersequence of* B if, for all $\sigma \in B$, σ is a subsequence of w. Note that for w to be a supersequence of $\{\sigma\}$, the actual entries of σ must occur (in the same order) in w; this is in contrast to pattern containment in which we need only find elements of w in the same relative order.

Example 14. The word $w = 2214312$ is a supersequence of 132 but not of 321. In fact, w is a supersequence of the set $B = \{132, 312, 213\}$.

Let $\mathrm{scs}_n(B)$ denote the minimum length of a supersequence of the set B. An upper bound of $\mathrm{scs}_n(S_n) \leq n^2 - 2n + 4$ has been proven by a number of different researchers in various contexts and generalities. See in particular [1, 6, 10, 11, 12, 17]. Recently, an upper bound of $n^2 - 2n + 3$ was proven constructively for $n \geq 10$ by Zălinescu [18]. Kleitman and Kwiatkowski [8] have shown that $\mathrm{scs}_n(S_n) \geq n^2 - Cn^{7/4+\varepsilon}$ where $\varepsilon > 0$ and C depends on ε.

For the remainder of the paper, we will think of the shape of a permutation as a Ferrers diagram. As such, taking the "union" of several shapes will amount to overlaying their Ferrers diagrams (This is in contrast to the usual notion of union for integer partitions in which all parts get interleaved.) More precisely, take shapes $\mu^1 = \mathrm{sh}(\sigma_1), \ldots, \mu^k = \mathrm{sh}(\sigma_k)$ corresponding to k permutations $\sigma_1, \ldots, \sigma_k$. We define the *union* $\bigcup_i \mathrm{sh}(\sigma_i)$ to be $\nu = (\nu_1, \nu_2, \ldots)$ where $\nu_i = \max\{\mu_i^1, \mu_i^2, \ldots, \mu_i^k\}$.

It turns out that for certain sets B, we can construct a lower bound for $\mathrm{scs}_n(B)$ by considering the union of $\mathrm{sh}(\sigma)$ as σ runs over the elements of B. To do this, we need the following fact.

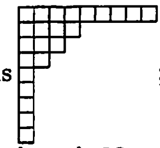
Lemma 15. If μ is a partition then there exists a separable permutation θ with $\mathrm{sh}(\theta) = \mu$.

Proof. Let T be the superstandard tableau of shape μ. For $i < j$, the entries in row j are greater than, and precede, the entries in row i. It follows that $\mathrm{rw}(T)$ avoids the pattern 213 and hence that $\mathrm{rw}(T)$ is a 2413,3142-avoiding permutation (i.e., is separable). By construction $\mathrm{sh}(\mathrm{rw}(T)) = \mu$. Hence we can set $\theta = \mathrm{rw}(T)$. □

Fix $k > 0$ and $B = \{\sigma_1, \ldots, \sigma_k\}$ with each σ_i separable. It follows then from Proposition 13 that for any supersequence w of B, $\mathrm{sh}(w) \supseteq \bigcup_i \mathrm{sh}(\sigma_i)$. Hence, if we choose the σ_i so that the Ferrers diagrams of shapes $\mathrm{sh}(\sigma_i)$ overlap as little as possible, we force any supersequence w to be relatively long.

Example 16. Let $n = 9$ and $k = 5$. Choose the permutations $B = \{\sigma_1, \ldots, \sigma_5\}$ as

$$\begin{aligned} \sigma_1 &= 123456789, & \mathrm{sh}(\sigma_1) &= (9), \\ \sigma_2 &= 678912345, & \mathrm{sh}(\sigma_2) &= (5,4), \\ \sigma_3 &= 789456123, & \mathrm{sh}(\sigma_3) &= (3,3,3), \\ \sigma_4 &= 978563412, & \mathrm{sh}(\sigma_4) &= (2,2,2,2,1), \\ \sigma_5 &= 987654321, & \mathrm{sh}(\sigma_5) &= (1,1,1,1,1,1,1,1,1). \end{aligned}$$

The union of the corresponding Ferrers diagrams is  ; we see that $|\cup_{i=1}^{5} \mathrm{sh}(\sigma_i)| = 23$. A computer search provides the length-23 supersequence

$$69787596543123456789123,$$

thereby showing that this bound is optimal.

Let $\nu(n)$ be the Ferrers diagram obtained by taking the union of all Ferrers diagrams of size n.

Proposition 17. Let $\tau(i)$ denote the number of divisors of i. Then $|\nu(n)| = \sum_{i=1}^{n} \tau(i)$ and the number of corners (i.e., distinct row lengths of $\nu(n)$) is given by $\lfloor\sqrt{4n+1}\rfloor - 1$.

Proof. For each divisor d of n, the shape $(\overbrace{n/d, \ldots, n/d}^{d \text{ times}})$ will be contained in $\nu(n)$. Furthermore, the cells $(d, n/d)$ are the only corners that are not part of $\nu(n-1)$. The result $|\nu(n)| = \sum_{i=1}^{n} \tau(i)$ then follows by induction. (In fact, the nested sequence of Ferrers diagrams $\nu(1) \subset \nu(2) \subset \cdots \subset \nu(n)$ can be thought of as a semistandard Young tableau of shape $\nu(n)$ in which the label i occurs $\tau(i)$ times.)

We now prove that the number of corners of $|\nu(n)|$ is $\lfloor\sqrt{4n+1}\rfloor - 1$. Let k be the largest integer for which a $k \times k$ square is contained in the diagram of $\nu(n)$, that is, k is the number of cells on the main diagonal in $\nu(n)$. We have that $k^2 \leq n$. The cell (k, k) is a corner of $\nu(n)$ if and only if $k(k+1) > n$, i.e. $(\underbrace{k, \ldots, k}_{k+1})$ is not contained in any diagram of size n. We claim that the rows $1, \ldots, k$ of $\nu(n)$ will each contain a corner of $\nu(n)$. For $1 \leq i < k$, row i ends at $(i, \lfloor n/i \rfloor)$ while the row below ends at $(i+1, \lfloor n/(i+1) \rfloor)$. Using the fact that $k \leq \sqrt{n}$, a short algebraic computation shows that $\lfloor n/(i+1) \rfloor < \lfloor n/i \rfloor$. Hence we have a corner in the i-th row for $1 \leq i < k$. We also have a corner in the k-th row by choice of k. The same argument holds for the first k columns, so the total number of corners is $2k-1$ if (k,k) is a corner and $2k$ otherwise.

Now we need to show that our above formulas for the number of corners (i.e., $2k-1$ or $2k$) can be expressed in terms of n as $\lfloor\sqrt{4n+1}\rfloor - 1$. For reference, note that $(2k)^2 = 4k^2$, $(2k+1)^2 = 4k^2+4k+1$ and $(2k+2)^2 = 4k^2+8k+4$. Suppose (k,k) is a corner. As mentioned above, this implies that $k(k+1) > n$. So $n = k^2 + a$ for some $0 \leq a < k$. Then $\lfloor\sqrt{4n+1}\rfloor - 1 = \lfloor\sqrt{4(k^2+a)+1}\rfloor - 1 = 2k-1$, as desired. Similarly, if (k,k) is not a corner, then $n \geq k(k+1)$ and, by choice of k, $(k+1)^2 > n$. So $n = k^2 + a$ for some $k \leq a \leq 2k$. In this case, $\lfloor\sqrt{4n+1}\rfloor - 1 = \lfloor\sqrt{4(k^2+a)+1}\rfloor - 1 = (2k+1) - 1 = 2k$ as desired.

□

It is a standard fact that $\sum_{i=1}^{n} \tau(i) = n(\log n + 2\gamma - 1) + O(\sqrt{n})$ where $\gamma \approx 0.57721\cdots$ is the Euler-Mascheroni constant (see, e.g., [3, Theorem 3.3]). Hence, for any n we can find $\lfloor\sqrt{4n+1}\rfloor - 1$ permutations whose supersequence is of length at least $n(\ln n + 2\gamma + \cdots)$. Compare this with $n!$ permutations having a supersequence of length $O(n^2)$.

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References

[1] Leonard Adleman, *Short permutation strings*, Discrete Math. **10** (1974), 197–200.

[2] Michael Albert, *Young classes of permutations*, arXiv:1008.4615v2 [math.CO].

[3] Tom M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976, Undergraduate Texts in Mathematics.

[4] Prosenjit Bose, Jonathan F. Buss, and Anna Lubiw, *Pattern matching for permutations*, Inform. Process. Lett. **65** (1998), no. 5, 277–283.

[5] D. Foulser, M. Li and Q. Yang, *Theory and algorithms for plan merging*, Artificial Intelligence, **57** (1992) no. 2-3, 143–181.

[6] G. Galbiati and F. P. Preparata, *On permutation-embedding sequences*, SIAM J. Appl. Math. **30** (1976), no. 3, 421–423.

[7] Curtis Greene, *An extension of Schensted's theorem*, Advances in Math. **14** (1974), 254–265.

[8] D. J. Kleitman and D. J. Kwiatkowski, *A lower bound on the length of a sequence containing all permutations as subsequences*, J. Combinatorial Theory Ser. A **21** (1976), no. 2, 129–136.

[9] Donald E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. **34** (1970), 709–727.

[10] P. J. Koutas and T. C. Hu, *Shortest string containing all permutations*, Discrete Math. **11** (1975), 125–132.

[11] S. P. Mohanty, *Shortest string containing all permutations*, Discrete Math. **31** (1980), no. 1, 91–95.

[12] M. C. Newey, *Notes on a problem involving permutations as subsequences*, Technical Report **CS-TR-73-340** (1973).

[13] K. Ning and H. W. Leong, *Towards a better solution to the shortest common supersequence problem: the deposition and reduction algorithm*, BMC Bioinformatics **7** (2006), Suppl 4:S12.

[14] S. Rahmann, *The shortest common supersequence problem in a microarray production setting*, Bioinformatics (2003), Suppl 2:ii, 156–61.

[15] M. Rodeh, V. R. Pratt and S. Even, *Linear algorithm for data compression via string matching*, J. ACM **28** (1981), no. 1, 16–24.

[16] Bruce E. Sagan, *The symmetric group*, Spinger-Verlag, New York, 2000, Graduate Texts in Mathematics. Second edition.

[17] Carla Savage, *Short strings containing all k-element permutations*, Discrete Math. **42** (1982), no. 2-3, 281–285.

[18] Eugen Zălinescu, *Shorter strings containing all k-element permutations*, Inform. Process. Lett. **111** (2011), no. 12, 605–608.