Neighbourly regular strength of a graph

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Abstract

A graph is said to be a neighbourly irregular graph (or simply an NI graph) if no two adjacent vertices have the same degree. In this paper we introduce the neighbourly regular strength of a graph. Let G be a simple graph of order n. Let NI(G) denote the set of all NI graphs in which G is an induced subgraph. The neighbourly regular strength of G is denoted by NRS(G) and is defined as the minimum k for which there is an NI graph NI(G) of order n+k in NI(G). We prove that the NRS(G) is at most n-1, with possible equality only if G is complete. In addition, we determine the NRS for some well known graphs.

Key words: Regular graph, irregular graph, neighbourly irregular graph and neighbourly regular strength of a graph.

AMS Subject Classification Code: 05C75

1 Introduction

Throughout this paper we consider only finite and simple graphs. Notations and terminology that we do not define here can be found in [8]. Let G be any graph of order n. For $0 \leq i \leq n-1$, $V_i(G)$ (or simply V_i) is defined as the set of all vertices having degree i in G. That is, $V_i(G) = \{v \in V(G) \mid d(v) = i\}$. Note that $|V_i| \leq n$ for every i, $0 \leq i \leq n-1$. For any subset W of V(G) we denote any maximum independent set of <W> (where <W> denotes the subgraph of G induced by W) by I(W) and for any I(W) we write $I^c(W) = W \setminus I(W)$. For any two graphs G and H, the join G v H is the graph obtained by joining each vertex in G to each vertex in H. A spanning 1-regular subgraph of G is called a 1-factor of G and is denoted by F.

As far as regular graphs are concerned, a lot of work has been done and plenty of results have been established. But in case of irregular graphs (graphs which are not regular [3]), one may think of the density of the irregularity. Of course, we know that in any graph, all the degrees cannot be distinct, that is, any graph has at least two vertices of the same degree.

Let I_n denote the graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_{n+1-i}v_j,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor,\ i \le j \le n-i\}$, (where $\lfloor x \rfloor$ denotes the largest integer which is less than or equal to x) which has precisely two vertices with the same degree [6]. In [11], the graph I_n is referred to as pairlone graph and is denoted by PL_n. It has been proved in [11] that, for any $n \ge 2$, there exists a unique pairlone graph of order n. For more types of irregular graphs one can refer to [1, 4, 7, 9 and 12].

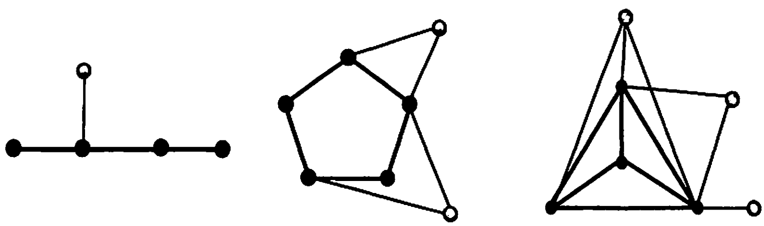
In [10], S.Gnaana Bhragsam and S.K.Ayyaswamy introduced a new concept, namely neighbourly irregular graph. A simple graph G is said to be a *neighbourly irregular graph* (or simply an NI graph) if no two adjacent vertices of G have the same degree. For example, the graph I_n is NI for every odd $n \ge 3$.

In [6], it has been stated that a simple graph G is neighbourly irregular if and only if for each i, $1 \le i \le n-1$, $V_i(G)$ is either empty or independent in G. This means that G is not an NI graph if and only if there exists at least one V_i which is neither empty nor independent in G.

In [10], it has been proved that any graph of order n is an induced subgraph of an NI graph of order at most $n(n+1)/2$. In this paper we reduced this upper bound to $2n-1$. This result motivated us to introduce a new concept called the neighbourly regular strength of a graph.

2 Neighbourly regular strength of a graph

For a simples graph G of order n, the *neighbourly regular strength* NRS(G) of G is the minimum number k for which there is an NI graph NI(G) of order n+k in NI(G), where NI(G) denotes the set of all NI graphs in which G is an induced subgraph. For example, $NRS(P_4) = 1$, $NRS(C_5) = 2$ and $NRS(W_3) = 3$. The graphs $NI(P_4)$, $NI(C_5)$ and $NI(W_3)$ are shown in Figure 1.



$NI(P_4)$ $NI(C_5)$ $NI(W_3)$

Figure 1

Note that for any graph G, NI(G) need not be unique. For example, in Figure 2 we have shown two graphs of order 8 which are in $NI(C_6)$.

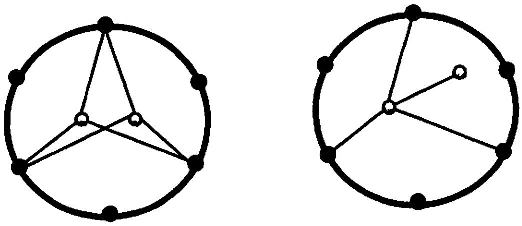


Figure 2

The following facts can be verified easily:

Fact 1 NRS(G) = 0 for any NI graph G.

Fact 2 NRS(G \ v) = 0 or 1 for any NI graph G. For example $NRS(P_3 \setminus v) = 0$ or 1 depending on the degree of v is 1 or 2.

Fact 3 Let G be the disjoint union of the graphs $G_1, G_2, \ldots, G_m$. Then NRS(G) $\leq$ $\sum_{i=1}^{m} NRS(G_i)$.

Fact 4 H is a subgraph of G does not imply that NRS(H) $\leq$ NRS(G). For example, if H is a non-NI subgraph of an NI graph G, then NRS(G) = 0 whereas NRS(H) $\geq$ 1.

Fact 5 $NRS(K_{n,m}) = \begin{cases} 0 \text{ if } n \neq m \\ 1 \text{ otherwise.} \end{cases}$

Fact 6 $NRS(P_n) = \begin{cases} 0 \text{ if } n = 1 \text{ or } 3 \\ 1 \text{ otherwise.} \end{cases}$

Fact 7 $NRS(C_n) = \begin{cases} 1 \text{ if n is even and } n \neq 6 \\ 2 \text{ otherwise.} \end{cases}$

Fact 8 $NRS(W_n) = \begin{cases} 1 \text{ for any even } n \geq 4 \\ 2 \text{ for any odd } n \geq 5 \\ 3 \text{ for } n = 3. \end{cases}$

Fact 9 $NRS(I_n) = \begin{cases} 0 \text{ if n is odd} \\ 1 \text{ otherwise.} \end{cases}$

Let G be any NI graph with clique number $\omega(G) = k$. Since the k vertices in the clique must have distinct degrees in G, $\Delta(G) \geq 2k-2$. This forces that

Fact 10 Any NI graph with clique number k has at least 2k-1 vertices.

Since $\omega(K_n) = n$, by Fact 10, it is easy to observe that

Fact 11 $NRS(K_n) \geq n-1$ for any $n \geq 1$.

In this paper we prove that $NRS(G) \leq n-1$ for any graph G of order n. We also prove that $NRS(G) = n-1$ if and only if G is isomorphic to K_n. In addition, we find NRS of the graphs $P_m(K_n)$, $K_n \setminus e$, $K_n \setminus \{e_1, e_2\}$ and $K_n \setminus \{e_1, e_2, \ldots, e_k\}$ where $e_1, e_2, \ldots, e_k$ have a common end vertex. Moreover we show that $NRS(G) \leq n-3$ for any connected irregular graph G. We also establish the result that $NRS(G \vee K_1)$ is either NRS(G) or NRS(G)±1 for any graph G. Finally, for any two integers s and n, $0 \leq s \leq n-3$, we construct a connected graph G, of order n with $NRS(G) = s$.

3 Main results

First we prove a lemma which is useful for further discussion.

Lemma 1 Let G be a non-complete connected graph. If $|V_i| > i$, for some i, $1 \leq i \leq n-2$, then G has at least two non-adjacent vertices of degree i, that is, $|I(V_i)| \geq 2$.

Proof Let $|V_i| = m$. Since V_i is non-empty, $|I(V_i)| \geq 1$. It is enough to prove that $|I(V_i)| \neq 1$. Suppose not, then any vertex w in V_i is adjacent to the remaining vertices of V_i, that is, $\langle V_i \rangle \cong K_m$. Consequently $i = d(v) \geq m-1$ for any vertex v in $\langle V_i \rangle$. But since $m > i$, then $m = i+1$. Therefore, the vertices of V_i cannot be adjacent to any vertex in $G \setminus V_i$. This means that as G is connected, $G \cong K_m$. This is a contradiction to our assumption that G is not complete. Hence $|I(V_i)| \geq 2$. ■

Let G be a connected graph of order n. Now we construct a new graph from G by the following algorithm:

Algorithm

Step 1 If G is NI, STOP. Otherwise, let a be the least positive integer such that V_a is neither empty nor independent in G. Construct a new graph H from G by joining a new vertex u to all vertices in $I^c(V_a)$.

Step 2 Take G = H. Go to Step 1.

Theorem 1 The algorithm terminates at an NI graph of order at most 2n-1 in which G is an induced subgraph.

Proof If G itself is an NI graph, then there is nothing to prove. Suppose G is not an NI graph.

It is easy to verify the result when $n = 2$. Let $n \geq 3$. For $i \geq 0$, at the i^{th} iteration of the algorithm, let G_i be the resultant graph, a_i be the corresponding least positive integer such that V_{a_i} is neither empty nor independent in G_i and u_{i+1} be the newly introduced vertex. Obviously, we may assume that $a_0 \geq 2$. Consider the graph G_{i+1}. If G_{i+1} is not an NI graph, then $a_{i+1} > a_i$. Indeed, note that $V_{a_i}(G_{i+1})$ is nothing but $I(V_{a_i}(G_i))$ with the possible inclusion of u_{i+1}. Thus V_a in G_{i+1} is either empty or independent for each $a \leq a_i$.

If $\Delta(G_i) < n$ for all $i \geq 0$, then $2 \leq a_i \leq n-2$ and so the algorithm terminates after the execution of at most n-3 iterations. Hence the NI graph obtained by the above algorithm has at most 2n-3 vertices.

Suppose j is the least positive integer such that $\Delta(G_{j+1}) = n$. It is easy to observe that if G is complete, then $d(u_{i+1}) \leq a_i$ for each $i \geq 0$. Therefore, $d(u_{i+1}) > a_i$ for some i only if G is non-complete. If $d(u_{i+1}) > a_i$, then $|V_{a_i}| > a_i$ in G_i. Now by Lemma 1, $|I(V_{a_i})| \geq 2$ in G_i. Suppose G_1 is not an NI graph. If $d(u_1) > a_0$, then as discussed earlier $|I(V_{a_0}(G_0))| \geq 2$. On the other hand, if $d(u_1) \leq a_0$, then $|I(V_{a_0}(G_0))| \geq 1$. Therefore, $|V(G_1) \setminus \bigcup_{j=1}^{a_0} V_j\ (G_1)| \leq n-1$. In general, for each $i \geq 1$, $|V(G_i) \setminus \bigcup_{j=1}^{a_{i-1}} V_j\ (G_i)| \leq n-i$. Consequently, $|V_{a_i}(G_i)| \leq n-i$ for each $i \geq 0$. Hence $d(u_i) \leq n-i$ for each $i \geq 1$. Therefore, $\Delta(G_j) = n-1$. This implies that $a_j = n-1$ and so, $|V_{n-1}(G_j)| \leq n-j$. Clearly $V_n(G_{j+1})$ is a proper subset of $V_{n-1}(G_j)$. If $V_n(G_{j+1})$ is not independent, then by the next iteration, $V_{n+1}(G_{j+2})$ is a proper subset of $V_n(G_{j+1})$. This sequence of proper subsets terminates after, say, t iterations with $V_{n+t-1}(G_{j+t})$ being independent. That is, G_{j+t} is the required NI graph with n+j+t vertices. Clearly $V_{a_{j+t}} = V_{a_{j+t-1}+1}$ is independent in G_{j+t}. As discussed earlier, $|V_{a_{j+t}}| \leq n-(j+t)$. But $|V_{a_{j+t}}| \geq 1$. Therefore, $j+t \leq n-1$. Hence $n+j+t \leq 2n-1$. ■

We illustrate the proof of Theorem 1 in Figure 3. Consider the graph G shown in Figure 3. Here $a_0 = 2$. In G, $V_2 = \{v_1, v_2, v_6, v_7, v_9, v_{10}\}$ and $I^c(V_2) = \{v_1, v_7, v_9\}$. In G_1, $a_1 = 3$, $V_3 = \{v_1, v_7, v_9, u_1\}$ and $I^c(V_3) = \{u_1\}$. Again in G_2, $a_2 = 4$, $V_4 = \{u_1, v_3, v_4\}$ and $I^c(V_4) = \{v_4\}$. G_3 is the resultant NI graph.

Theorem 1 means that

Corollary 1.1 NRS(G) $\leq$ n-1, for any graph G of order n. ■

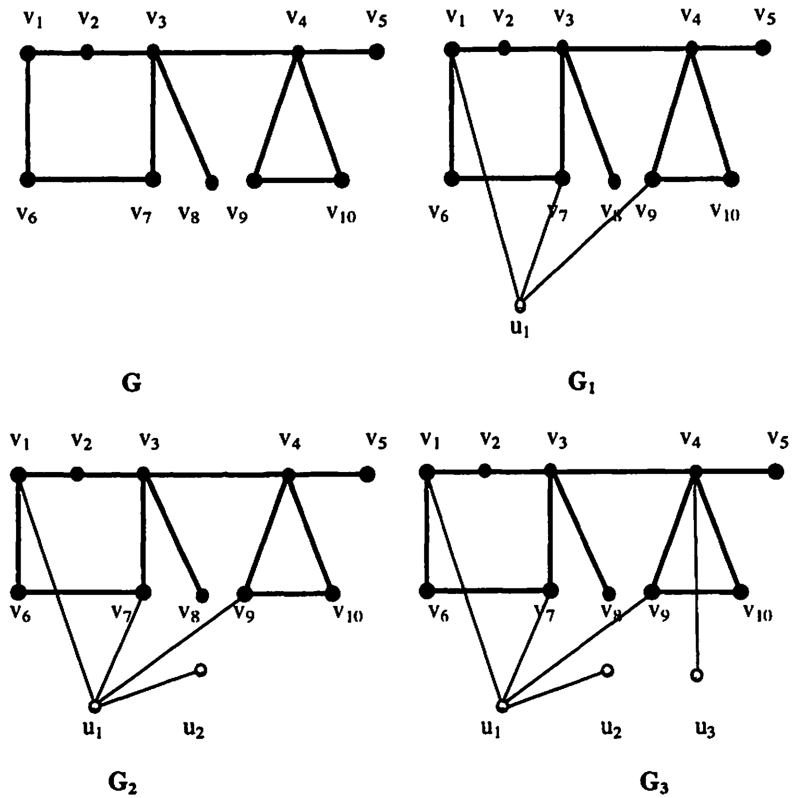


Figure 3

Notation We denote $V(G_i) \setminus \bigcup_{j=1}^{a_{i-1}} V_j\,(G_i)$ by $R[V_{a_{i-1}}(G_i)]$.

From the proof of Theorem 1, we obtain the following results which are used for further discussion. Let G_s be an NI graph obtained as in the algorithm corresponding to a connected graph G.

Result 1 If $i \geq 1$ and $|R[V_{a_{i-1}}(G_i)]| \leq (n-i)-c$ for some integer $c > 0$, then $s \leq (n-1)-c$.

For, while proving Theorem 1, we have $|R[V_{a_{i-1}}(G_i)]| \leq n-i$ for each $i \geq 1$. Now if $|R[V_{a_{i-1}}(G_i)]| \leq (n-i)-c$ for some $i \geq 1$, then $|R[V_{a_{k-1}}(G_k)]| \leq (n-k)-c$ for all $k \geq i$. Consequently $|R[V_{a_{s-1}}(G_s)]| \leq (n-s)-c$. Hence $|V_{a_s}(G_s)| \leq (n-s)-c$, where $a_s = a_{s-1}+1$. But $|V_{a_s}(G_s)| \geq 1$. Therefore, $s \leq n-1-c$.

Result 2 If $d(u_i) = a_{i-1}$ for some i, $1 \leq i \leq s$, then $|R[V_{a_{i-1}}(G_i)]| \leq n-i-1$.

For, if $d(u_i) = a_{i-1}$ for some i, $1 \leq i \leq s$, then $|V_{a_{i-1}}(G_{i-1})| > a_{i-1}$. Thus by Lemma 1, $|I(V_{a_{i-1}}(G_{i-1}))| \geq 2$ and hence $|R[V_{a_{i-1}}(G_i)]| \leq n-i-1$.

Result 3 If $d(u_i) > a_{i-1}$, $V_{a_{i-1}+1}$ is not independent in G_i and $d(u_{i+1}) \leq a_i$ for some i, $1 \leq i \leq s-1$, then $|R[V_{a_i}(G_{i+1})]| \leq n-(i+1)-1$.

For, $d(u_i) > a_{i-1}$ means that $|I^c(V_{a_{i-1}}(G_{i-1}))| > a_{i-1}$. Since the vertices in $I^c(V_{a_{i-1}}(G_{i-1}))$ have degree a_{i-1} in G_{i-1}, by Lemma 1, $|I(I^c(V_{a_{i-1}}(G_{i-1})))| \geq 2$. Note that $a_i = a_{i-1}+1$ and $I^c(V_{a_{i-1}}(G_{i-1})) \subseteq V_{a_i}(G_i)$. Thus $|I(V_{a_i}(G_i))| \geq 2$ and so $|R[V_{a_i}(G_{i+1})]| \leq n-(i+1)-1$.

Next we prove that the upper bound for NRS attained in Corollary 1.1 is sharp only when the graph G is complete.

Theorem 2 $NRS(G) = n-1$ if and only if $G \cong K_n$, for any graph G of order n.

Proof Let G be a non-complete graph of order n. If G is disconnected, then by Theorem 1 together with Fact 3, $NRS(G) \leq n-2$. Therefore, assume that G is connected. Let G_s be an NI graph obtained as in the algorithm corresponding to G. We claim that $s \neq n-1$. The result is obvious if $\Delta(G_s) < n$. Hence assume that $\Delta(G_s) \geq n$.

If $V_i(G_0)$ is non-empty for some $i < a_0$, then $|R[V_{a_0}(G_1)]| \leq (n-1)-1$ and hence by Result 1, $s \leq n-2$. Otherwise $V_i(G_0)$ is empty for each $i < a_0$.

Case(i) Suppose $V_{a_{t-1}+1}$ is independent in G_t for some t, $1 \leq t \leq s-1$.

Since $I^c(V_{a_{t-1}}(G_{t-1})) \subseteq V_{a_{t-1}+1}(G_t)$, $V_{a_{t-1}+1}$ is non-empty in G_t. Since $V_{a_{t-1}+1}$ is independent in G_t, u_{t+1} which we introduce in G_t must correspond to V_{a_t} for some $a_t > a_{t-1}+1$. Thus $|R[V_{a_t}(G_{t+1})]| \leq n-(t+1)-1$ and hence by Result 1, $s \leq n-2$.

Case(ii) Suppose $V_{a_{t-1}+1}$ is not independent in G_t for any t, $1 \leq t \leq s-1$.

If $d(u_i) = a_{i-1}$ for some i, $1 \leq i \leq s$, then by Result 2 and Result 1, $s \leq n-2$.

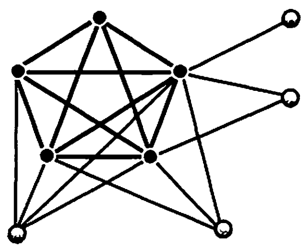
Suppose $d(u_i) > a_{i-1}$ for some i, $1 \le i \le s$. Let k be the largest positive integer such that $d(u_k) > a_{k-1}$ in G_k. If $k < s$, then by Result 3, $|R[V_{a_k}(G_{k+1})]| \le n-(k+1)-1$ and hence by Result 1, $s \le n-2$. If $k = s$, then $2 \le a_{s-1} < |R[V_{a_{s-1}}(G_s)]| \le n-s$ and hence $s < n-2$.

Suppose $d(u_i) < a_{i-1}$ for all i, $1 \le i \le s$. If $|I(V_{a_i}(G_i))| \ge 2$ for some i, $0 \le i \le s-1$, then $|R[V_{a_i}(G_{i+1})]| \le n-(i+1)-1$ and hence by Result 1, $s \le n-2$. Therefore, it is enough to prove that $|I(V_{a_i}(G_i))| \ge 2$ for some i, $0 \le i \le s-1$. Since G is non-complete, there are at least two non-adjacent vertices in G. Fix a vertex v in $V_{a_0}(G_0)$. Let $w \notin N[v]$ such that $d(w) = d$, where $d = \min\{d(u) \mid u \notin N[v]\}$. Clearly $a_0 \le d < n-1$. Thus $d = a_j$ for some j, $0 \le j \le s-1$.

If $|I(V_{a_0}(G_0))| \ge 2$, then the result follows. Otherwise $|I(V_{a_0}(G_0))| = 1$. Now choose $I(V_{a_0}(G_0))$ such that $v \notin I(V_{a_0}(G_0))$. Clearly $d(v) = a_0+1 = a_1$ in G_1 and V_{a_1} is neither empty nor independent in G_1. If $|I(V_{a_1}(G_1))| \ge 2$, then the result follows. Otherwise choose $I(V_{a_1}(G_1))$ such that $v \notin I(V_{a_1}(G_1))$. Now $d(v) = a_1+1 = a_2$ in G_2 and V_{a_2} is neither empty nor independent in G_2. Continue the process of choosing $I(V_{a_i}(G_i))$ such that $v \notin I(V_{a_i}(G_i))$ until we get $|I(V_{a_i}(G_i))| \ge 2$. Note that while continuing the process, $|I(V_{a_j}(G_j)| \ge 2$ if d(v) becomes a_j. This means that $s \le n-2$. Therefore, $NRS(G) \le n-2$ if G is non-complete.

The converse follows from Fact 11 and Corollary 1.1. ■

Note that $NI(K_n)$ obtained by the algorithm is unique and is isomorphic to I_{2n-1}. For example, $NRS(K_5) = 4$ and the corresponding $NI(K_5) \cong I_9$ is shown in Figure 4.



$NI(K_5) \cong I_9$

Figure 4

Corollary 2.1 For any graph G, $NRS(G) \ge \max\{\omega(\langle V_i \rangle)\}-1$, where maximum runs over i, $2 \le i \le n-1$.

Proof Let i be an integer such that $2 \leq i \leq n-1$ and V_i is non-empty. Suppose $\omega(< V_i >) = j$. Clearly $1 \leq j \leq n$. If $j = n$, then $i = n-1$. That is, $G \cong K_n$. Hence the result is obvious by Theorem 2. Therefore, assume that $j \leq n-1$. Clearly $< V_i >$ contains K_j. Since the vertices in K_j are adjacent to each other in G, at least one vertex in K_j must be of degree at least $i+j-1$ in NI(G). Thus NI(G) must have at least $j-1$ vertices which are not in G and hence $NRS(G) \geq j-1$. Since i is arbitrary, the result follows. ■

Clearly in $K_n \setminus e$, $< V_{n-1} > \cong K_{n-2}$. Therefore, by the above corollary, $NRS(K_n \setminus e) \geq n-3$. Also $K_n \setminus e$ is an induced subgraph of the NI graph I_{2n-3} and so $NRS(K_n \setminus e) \leq n-3$. Thus we have

Corollary 2.2 $NRS(K_n \setminus e) = n-3$ for any $n \geq 3$. ■

Lemma 2 Let G be a connected irregular graph of order n. Suppose $V_\Delta(G)$ is non-independent in G and for all $i < \Delta(G)$, $V_i(G)$ is independent if it exists. Then $NRS(G) \leq n-3$.

Proof If G is NI, then obviously the result is true. Let us assume that G is non-NI. Let G_s be an NI graph obtained as in the algorithm corresponding to G. Since G is irregular, $|V_{a_0}(G_0)| \leq n-1$. If $|V_{a_0}(G_0)| \leq n-2$, then $|R[V_{a_0}(G_1)]| \leq n-3$ and hence by Result 1, $s \leq n-3$. Next assume that $|V_{a_0}(G_0)| = n-1$. Clearly $2 \leq a_0 = \Delta \leq n-2$. Then by Lemma 1, $|I(V_{a_0}(G_0))| \geq 2$. Since $|V_i(G)| = 1$ for some $i < a_0$, if $|I(V_{a_0}(G_0))| > 2$ or $d(u_1) \leq a_0$, then $|R[V_{a_0}(G_1)]| \leq n-3$ and hence by Result 1, $NRS(G) \leq n-3$. Suppose $|I(V_{a_0}(G_0))| = 2$ and $d(u_1) > a_0$. Clearly $a_1 = a_0+1$ in G_1. In general, for each i, $1 \leq i \leq s-1$, $a_i = a_{i-1}+1$ and $V_{a_{i-1}+1}$ is not independent in G_i. Let k be the largest positive integer such that $d(u_k) > a_{k-1}$ in G_k. If $k = s$, then $2 \leq a_{s-1} < |R[V_{a_{s-1}}(G_s)]| \leq n-s$ and hence $s < n-2$. If $k < s$, then by Result 3, $|R[V_{a_k}(G_{k+1})]| \leq n-(k+1)-1$. But since $|V_i(G)| = 1$ for some $i < a_0$, $|R[V_{a_k}(G_{k+1})]| \leq n-(k+1)-2$. Hence the result follows from Result 1. ■

Theorem 3 Let G be a connected irregular graph of order n. Then $NRS(G) \leq n-3$.

Proof If G is NI, then obviously the result is true. Let us assume that G is non-NI. Let G_s be an NI graph obtained as in the algorithm corresponding to G. Since G is non-complete, obviously by Theorem 2, $s \leq n-2$. Let us assume that $s = n-2$. This implies that $\Delta(G_s) \geq n$.

First we claim that $d(u_s) = 1$. Suppose $d(u_s) \geq 2$, that is, $|I^c(V_{a_{s-1}}(G_{s-1}))| \geq 2$. Note that $|V_{a_{s-1}}(G_{s-1})| \leq |R[V_{a_{s-2}}(G_{s-1})]| \leq n-(s-1) = 3$. Since $I^c(V_{a_{s-1}}(G_{s-1})) \subseteq V_{a_{s-1}+1}(G_s)$, it is independent in G_s. Therefore, $I^c(V_{a_{s-1}}(G_{s-1}))$ is an independent set in $<V_{a_{s-1}}(G_{s-1})>$. But since $I(V_{a_{s-1}}(G_{s-1}))$ is a maximum independent set in $<V_{a_{s-1}}(G_{s-1})>$, $|I(V_{a_{s-1}}(G_{s-1}))| \geq 2$. This implies that $|V_{a_{s-1}}(G_{s-1})| \geq 4$. This is a contradiction to $|V_{a_{s-1}}(G_{s-1})| \leq 3$. Therefore, $d(u_s) = 1$.

Case(i) Suppose $V_{a_{t-1}+1}$ is independent in G_t for some t, $1 \leq t \leq s-1$.

Since $I^c(V_{a_{t-1}}(G_{t-1})) \subseteq V_{a_{t-1}+1}(G_t)$, $V_{a_{t-1}+1}$ is non-empty in G_t. Since $V_{a_{t-1}+1}$ is independent in G_t, u_{t+1} which we introduce in G_t must correspond to V_{a_t} for some $a_t > a_{t-1}+1$. If $|V_{a_{t-1}+1}(G_t)| \geq 2$, then $|R[V_{a_t}(G_{t+1})]| \leq n-(t+1)-2$ and so $s \leq n-3$. This is a contradiction to $s = n-2$. Thus $|V_{a_{t-1}+1}(G_t)| = 1$ and hence $d(u_t) = 1$. Now fuse u_t and u_s in G_s. Clearly the resultant graph is an NI graph of order 2n-3 in which G is an induced subgraph. Hence $NRS(G) \leq n-3$.

Case(ii) Suppose $V_{a_{t-1}+1}$ is not independent in G_t for any t, $1 \leq t \leq s-1$.

Let $a_0 = a$. Clearly $a_1 = a+1$. In general, for each i, $0 \leq i \leq s-1$, $a+i$ is the least positive integer such that V_{a+i} is neither empty nor independent in G_i. Note that $\Delta(G_s) = a+s$. If $a = \Delta(G)$, then by Lemma 2, $s \leq n-3$. Therefore, assume that $a \neq \Delta(G)$. That is, $a \leq n-2$.

Suppose $d(u_1) = a$. Then as discussed in Result 2, $|I(V_a(G_0))| \geq 2$. If $|I(V_a(G_0))| > 2$, then $|R[V_a(G_1)]| \leq n-3$ and so $s \leq n-3$, a contradiction to $s = n-2$. Thus $|I(V_a(G_0))| = 2$ and hence $|R[V_a(G_1)]| = n-2$. Next we claim that $d(u_i) < a+i-1$ for all i, $2 \leq i \leq s-1$. For, if $d(u_i) = a+i-1$ for some $i \geq 2$, then by Result 2, $|R[V_{a+i-1}(G_i)]| \leq n-i-1$. But $|R[V_a(G_1)]| = n-2$. Thus $|R[V_{a+i-1}(G_i)]| \leq n-i-2$. This results to a contradiction to $s = n-2$. Suppose $d(u_i) > a+i-1$ for some $i \geq 2$. Let k be the largest positive integer such that $d(u_k) > a+k-1$. Thus by Result 3, $|R[V_{a+k}(G_{k+1})]| \leq n-(k+1)-1$. But since $|R[V_a(G_1)]| = n-2$, $|R[V_{a+k}(G_{k+1})]| \leq n-(k+1)-2$, which leads to a contradiction to $s = n-2$. Therefore, $d(u_i) < a+i-1$ for all i, $2 \leq i \leq s-1$.

Now if $|I(V_{a+i}(G_i))| \geq 2$ for some i, $1 \leq i \leq s-1$, then $|R[V_{a+i}(G_{i+1})]| \leq n-(i+1)-2$, which leads to a contradiction to $s = n-2$. Thus $|I(V_{a+i}(G_i))| = 1$ for all i, $1 \leq i \leq s-1$. Let u and v be any two vertices in $R[V_a(G_1)]$. If u and v are non-adjacent, then take $i = \max\{d(u), d(v)\}$. Then we can find an independent set $I(V_{a+i}(G_i))$ such that $\{u, v\} \subseteq I(V_{a+i}(G_i))$, that is, $|I(V_{a+i}(G_i))| \geq 2$. This is a contradiction to $|I(V_{a+i}(G_i))| = 1$. Therefore, any two vertices in $R[V_a(G_1)]$ must be adjacent to each other in G_1. Let $w \in I^c(V_a(G_0))$. Since $I^c(V_a(G)) \subseteq R[V_a(G_1)]$, $d(w) = n-3$ in $<R[V_a(G_1)]>$. But w must adjacent to at least one vertex in $I(V_a(G_0))$. Therefore, $d(w) \geq n-2$ in G_0. Thus $a = n-2$ and hence $d(u_1) = |R[V_a(G_1)]|$. Consequently $|V_a(G_0)| = n$. This is a contradiction to the irregularity of G. Therefore, $d(u_1) \neq a$.

Suppose $d(u_1) > a$. Let k be the largest positive integer such that $d(u_k) > a+k-1$. Then by Result 3, $|R[V_{a+k}(G_{k+1})]| \leq n-(k+1)-1$. If $d(u_i) \leq a+i-1$ for some i, $1 < i < k$, then by Result 3, $|R[V_{a+i-1}(G_i)]| \leq n-i-1$. Consequently $|R[V_{a+k}(G_{k+1})]| \leq n-(k+1)-2$, which leads to a contradiction to $s = n-2$. Therefore, $d(u_i) > a+i-1$ for all i, $1 \leq i \leq k$. Suppose $|I(V_{a+i}(G_i))| > 2$ for some i, $i \leq k$. Then $|R[V_{a+k}(G_{k+1})]| \leq n-(k+1)-2$. This results to a contradiction to $s = n-2$. Therefore, $|I(V_{a+i}(G_i))| = 2$ for all i, $0 \leq i \leq k$.

If $d(u_i) = a+i-1$ for some i, $k < i < s$, then by Result 2, $|R[V_{a+i}(G_{i+1})]| \leq n-(i+1)-1$. But since $|R[V_{a+k}(G_{k+1})]| \leq n-(k+1)-1$, $|R[V_{a+i}(G_{i+1})]| \leq n-(i+1)-2$, which leads to a contradiction to $s = n-2$. Therefore, $d(u_i) < a+i-1$ for all i, $i > k$. If $|I(V_{a+i}(G_i))| = 2$ for some i, $k+1 \leq i \leq s-1$, then $|R[V_{a+i}(G_{i+1})]| \leq n-(i+1)-2$, which leads to a contradiction to $s = n-2$. Therefore, $|I(V_{a+i}(G_{i+1}))| = 1$ for all i, $k+1 \leq i \leq s-1$.

If $d(v) \geq a+k$ in G_k for a vertex v in $V(G_k) \setminus N[u_k]$, then choose $I(V_{a+k}(G_k))$ such that $I(V_{a+k}(G_k)) \subseteq I^c(V_{a+k-1}(G_{k-1}))$. Now take $i = \max\{d(u_k), d(v)\}$. Then we can find an independent set $I(V_{a+i}(G_i))$ such that $\{u_k, v\} \subseteq I(V_{a+i}(G_i))$, that is, $|I(V_{a+i}(G_i))| \geq 2$. This is a contradiction to $|I(V_{a+i}(G_i))| = 1$ for all i, $k+1 \leq i \leq s-1$. Therefore, in G_k, $d(v) \leq a+k-1$ for any vertex v in $V(G_k) \setminus N[u_k]$. Suppose $k = 1$. Since $a \neq \Delta(G_0)$, there exists a vertex v in $V(G_1) \setminus N[u_1]$ such that $d(v) \geq a+1$ in G_1. Therefore, $k > 1$.

Consider the vertex u_{k-1}. Since $d(u_{k-1}) > a+k-2$ in G_{k-1}, $d(u_{k-1}) = a+k-1$ in G_{k-1}. But $d(u_k) > a+k-1$ and $|I(V_{a+k-1}(G_{k-1}))| = 2$. This means that $|V_{a+k-1}(G_{k-1})| \geq a+k+2$. Also we have $N[u_{k-1}] \subseteq V_{a+k-1}(G_{k-1})$ and $|N[u_{k-1}]| = a+k$ and so $|V_{a+k-1}(G_{k-1}) \setminus N[u_{k-1}]| \geq 2$. Let u and v be any two vertices in $V_{a+k-1}(G_{k-1}) \setminus N[u_{k-1}]$. If u and v are non-adjacent in G_{k-1}, then choose $I(V_{a+k-1}(G_{k-1}))$ such that $\{u, v, u_{k-1}\} \subseteq I(V_{a+k-1}(G_{k-1}))$. This forces that $|I(V_{a+k-1}(G_{k-1}))| > 2$. This is a contradiction to $|I(V_{a+i}(G_i))| = 2$ for all $i \leq k$. Therefore, u and v must be adjacent in G_{k-1}. Thus either u or v must be non-adjacent to at least one vertex in $N(u_{k-1})$. Let w be a vertex in $N(u_{k-1})$ such that w is not adjacent to u. Since any vertex x in $N(u_{k-1})$ must be adjacent to at least one vertex in $V_{a+k-2}(G_{k-1})$, we have $d(x) \leq a+k-3$ in $\langle N(u_{k-1}) \rangle$. Clearly $|N(u_{k-1}) \setminus w| = a+k-2$. Now by Lemma 1, $|I(N(u_{k-1}) \setminus w)| \geq 2$. Choose $I(V_{a+k-1}(G_{k-1}))$, $I(V_{a+k}(G_k))$ and $I(V_{a+k+1}(G_{k+1}))$ such that they contain $I(N(u_{k-1}) \setminus w)$, $\{u, w\}$ and $\{v, u_{k-1}\}$ respectively. That is, $|I(V_{a+k+1}(G_{k+1}))| \geq 2$, which is a contradiction to $|I(V_{a+i}(G_{i+1}))| = 1$ for all $i > k$.

Therefore, $d(u_1) < a$ in G_1. Since $V_b(G)$ is non-empty for some $b > a$, we can find a G_s such that $N[u_1] \cap N(u_s) = \varphi$. Now fuse u_1 and u_s in G_s. Clearly the resulting graph is in NI(G) and have order 2n-3. Hence $NRS(G) \leq n-3$. ■

Clearly Corollary 2.2 shows the tightness of the bound attained in Theorem 3. Note that Lemma 2 and Theorem 3 do not hold, if the condition that

G is connected is dropped. For example, consider $G = K_n \cup K_1$. Clearly G is a disconnected graph of order n+1 whereas NRS(G) = n+1-2.

When we think of regular graphs, it has been proved that

Theorem A [5] $NRS(G) = \begin{cases} \chi(G) & \text{if } G \cong K_{n,n} \setminus F \text{ for } n \geq 3 \\ \chi(G) - 1 & \text{otherwise} \end{cases}$, for any connected regular graph G.

It has also been proved that

Theorem B [5] $NRS(G) \leq 2$, for any bipartite graph G.

Theorem C [5] For any tree T, NRS(T) = 0 or 1.

Next we consider the edge deleted subgraph $K_n \setminus \{e_1, e_2\}$. Clearly $K_4 \setminus \{e_1, e_2\} \cong I_4$ or C_4 dependent on whether e_1 is adjacent to e_2 or not, and so $NRS(K_4 \setminus \{e_1, e_2\}) = 1$.

Let $G = K_n \setminus \{e_1, e_2\}$, $n > 4$ and let NI(G) be in NI(G). If e_1 is adjacent to e_2, then $\omega(G) = n-1$. Hence the clique number of NI(G) is greater than or equal to n-1. Then by Fact 10, NI(G) must be of order at least 2n-3 and so $NRS(G) \geq n-3$. But since G is irregular, $NRS(G) \leq n-3$ and hence NRS(G) = n-3.

If e_1 is not adjacent to e_2, then $\omega(G) = n-2$. Since the vertices in the clique must have distinct degrees in NI(G) and all the n-2 vertices in the corresponding clique have degree at least n-2 in G, there must be a vertex have degree at least 2n-5 in NI(G). This forces that NI(G) must have at least 2n-4 vertices and hence $NRS(G) \geq n-4$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $e_1 = v_1v_2$ and $e_2 = v_3v_4$. Clearly the degree sequence of G is n-2, n-2, n-2, n-2, n-1, n-1, ... n-1. Construct G_1 from G by adding the vertices $u_1, u_2, \ldots, u_{n-4}$, the edges $u_1v_3, u_1v_4, \ldots, u_1v_n$ and u_iv_j for $2 \leq i \leq n-4$, $j \geq i+4$. Now the degrees of the vertices $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-4}$ are n-2, n-2, n-1, n-1, n, n+1, ..., 2n-5, n-2, n-5, n-6, ..., 1 respectively in G_1. Note that for $1 \leq i \leq 2n-5$, $V_i(G_1)$ is either empty or independent. Thus G_1 is in NI(G) and hence $NRS(G) \leq n-4$. Therefore, NRS(G) = n-4. Hence we can state that

Theorem 4 For any n > 4, $NRS(K_n \setminus \{e_1, e_2\}) = \begin{cases} n-3 & \text{if } e_1 \text{and } e_2 \text{ are adjacent} \\ n-4 & \text{otherwise.} \end{cases}$

■

Of course, it is much harder to find the $NRS(K_n \setminus \{e_1, e_2, ..., e_k\})$ for arbitrary edges $e_1, e_2, ..., e_k$. But if we assume that $e_1, e_2, ..., e_k$ have a common end vertex, it is not much more difficult to find the $NRS(K_n \setminus \{e_1, e_2, ..., e_k\})$, which is established in the following theorem.

Theorem 5 For any $n \geq 3$, $NRS(K_n \setminus \{e_1, e_2, ..., e_k\}) = n-3$ if $1 \leq k \leq n-2$ and the edges $e_1, e_2, ..., e_k$ have a common end vertex.

Proof Let $G = K_n \setminus \{e_1, e_2, ..., e_k\}$, where $e_1, e_2, ..., e_k$ have a common end vertex and, let NI(G) be in NI(G). Clearly $\omega(G) = n-1$. Hence the clique number of NI(G) is at least n-1. Then by Fact 10, NI(G) must be of order at least 2n-3 and so $NRS(G) \geq n-3$. But since G is irregular, $NRS(G) \leq n-3$. Thus NRS(G) = n-3. ■

Next we show that the difference between the NRS' of G and $G \vee K_1$, is at most 1.

Theorem 6 $NRS(G \vee K_1) = NRS(G)$ or $NRS(G) \pm 1$, for any graph G.

Proof Let G be a graph of order n with vertex set $\{v_1, v_2, ..., v_n\}$ and NRS(G) = k. Let NI(G) be any graph of order n+k in NI(G). Take $H = G \vee K_1$ and let v be the newly added vertex. Clearly $d_H(v_i) = d_G(v_i)+1$ for each i, $1 \leq i \leq n$. Let $NI(H) \in NI(H)$ and have order n+1+NRS(H). Clearly NI(H) is also in NI(G). Thus $NRS(G) \leq 1+NRS(H)$ and hence $NRS(G)-1 \leq NRS(H)$.

If there is an NI graph NI(G) of order n+k in NI(G) such that $V_{n+k-1}(NI(G))$ is empty, then H is an induced subgraph of the NI graph $NI(G) \vee K_1$. Hence $NRS(H) \leq NRS(G)$.

Suppose $V_{n+k-1}(NI(G))$ is non-empty for any NI graph NI(G) of order n+k in NI(G). Then construct a new graph H_1 from $NI(G) \vee K_1$ by introducing a new vertex w and joining it with v. Clearly H_1 is an NI graph in which H is an induced subgraph. Thus $NRS(H) \leq k+1$ and hence the result follows. ■

As an illustration, consider the graphs C_6, P_3 and C_4. Clearly $NRS(C_6) = 2$, $NRS(C_6 \vee K_1) = 1$, $NRS(P_3) = 0$, $NRS(P_3 \vee K_1) = 1$ and $NRS(C_4) = 1 = NRS(C_4 \vee K_1)$. Note that when G is an NI graph, $NRS(G \vee K_1)$ is 0 or 1 depending on whether the set $V_{n-1}(G)$ is empty or non empty.

For $m \geq 2$ and $n \geq 1$, let $P_m(K_n)$ be the graph obtained from G by identifying a pendent vertex of the path P_m with a vertex of K_n. Clearly $P_m(K_n)$ is a path when n = 1 or 2. Suppose n = 3 and $m \geq 2$. $NRS(P_m(K_n)) = 2$ if m = 4, otherwise $NRS(P_m(K_n)) = 1$. It is easy to verify that $NRS(P_m(K_n)) = 2$ for any $m \geq 2$ and n = 4. The remaining cases are discussed in the following theorem.

Theorem 7 $NRS(P_m(K_n)) = n-2$, for $m \geq 2$ and $n \geq 5$.

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n, $n \geq 5$ and let $u_1, u_2, \ldots, u_m$ be the vertices of P_m, $m \geq 2$. Let $G = P_m(K_n)$ be the graph obtained by fusing v_n with u_1. Clearly the order of G is n+m-1. Note that in G, $\langle V_{n-1}(G)\rangle \cong K_{n-1}$. Then by Corollary 2.1, $NRS(G) \geq n-2$.

Construct G_1 from G by introducing new vertices $w_1, w_2, \ldots w_{n-2}$ and joining w_i to v_j for all $i+1 \leq j \leq n$, $1 \leq i \leq n-2$. In G_1, the degrees of $v_1, v_2, \ldots, v_n$ are n-1, n, ..., 2n-2 respectively and the degrees of $w_1, w_2, \ldots, w_{n-2}$ are n-1, n-2, ..., 2 respectively. If $m \leq 3$, then G_1 is in NI(G) and have order (n+m-1) + (n-2). Hence $NRS(G) \leq n-2$.

Suppose m > 3. Since $d(u_i) = 2$ for each i, $2 \leq i \leq m-1$, G_1 will become an NI graph if we raise the degree of alternate vertices in the sequence $u_2, u_3, \ldots, u_{m-1}$ to at least three. Choose at least $\left\lfloor \frac{m-1}{2} \right\rfloor$ alternate vertices in the sequence $u_2, u_3, \ldots, u_{m-1}$. Let the number of vertices chosen be t. Now for each i, $1 \leq i \leq n-2$, w_i is adjacent to the vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ which are of degrees n+i-1, n+i, ..., 2n-2 respectively. Note that $d(w_i) \neq d(v_j)$ for all i and j, $1 \leq i \leq n-2$, $i+1 \leq j \leq n$ even if the degree of every vertex w_i is raised to (at most) n-1+2(i-1) by joining it with some of the vertices from $\{u_2, u_3, \ldots, u_{m-1}\}$. Hence we can raise the degrees of $w_1, w_2, \ldots, w_{n-2}$ by 0, 2, ..., 2(n-3) respectively. Clearly 2(n-4) + 2(n-3) > n for $n \geq 5$.

Suppose t = 1. Then construct G_2 from G_1 by joining the chosen vertex with w_{n-3}. If $1 < t \leq 2(n-3)$, then construct G_2 from G_1 by joining the t vertices with w_{n-2}. If $2(n-3) < t < n$, then construct G_2 from G_1 by joining 2(n-3) vertices with w_{n-2} and the remaining vertices with w_{n-3}. If $t \geq n$, then construct G_2 from G_1 by joining all the t vertices with w_1. Then in G_2, $d(w_1) = n-1 + t > 2n-2 = \Delta(G_1)$. Clearly G_2 is in NI(G) in all the cases. Hence $NRS(G) \leq n-2$. Therefore, NRS(G) = n-2. ■

Note that Theorem 3 means that for any connected irregular graph G of order n, $0 \leq NRS(G) \leq n-3$. Next we show the existence of a connected graph of order n such that NRS(G) = s for a given $s \geq 0$ and for all $n \geq s+3$.

Theorem 8 For any non-negative integer s and for all $n \geq s+3$, there exists a connected graph G of order n with NRS(G) = s.

Proof For s = 0, consider the graph $K_{n-1,1}$, $n \geq 3$. Clearly $NRS(K_{n-1,1}) = 0$.

If $s = 1$, consider the path P_n, $n \geq 4$, for which NRS is 1.When $s = 2$, for all $n \geq 5$ the cycle C_n when n is odd and the graph $K_{\frac{n}{2},\frac{n}{2}} \setminus F$ when n is even are the required graphs with NRS = 2.

Finally when $s \geq 3$, consider the graph $P_m(K_{s+2})$. Clearly its order is $s+m+1 \geq s+3$ and by Theorem 7 its NRS(G) is s. ■

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