A note on acyclic total coloring of plane graphs

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Abstract

An acyclic total coloring is a proper total coloring of a graph G such that there are at least 4 colors on vertices and edges incident with a cycle of G. The acyclic total chromatic number of G, $\chi''_a(G)$, is the least number of colors in an acyclic total coloring of G. In this paper, we prove that for every plane graph G with maximum degree Δ and girth $g(G)$, $\chi''_a(G) = \Delta + 1$ if (1) $\Delta \geq 9$ and $g(G) \geq 4$; (2) $\Delta \geq 6$ and $g(G) \geq 5$; (3) $\Delta \geq 4$ and $g(G) \geq 6$; (4) $\Delta \geq 3$ and $g(G) \geq 14$.

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1 Introduction

Graphs considered in this note are finite, simple and undirected. Unless stated otherwise, we follow the notations and terminology in [3].

For a plane graph G, we denote its vertex set, edge set, face set, minimum degree and maximum degree by $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$, respectively. For a vertex v, $d_G(v)$ and $N_G(v)$ denote its degree and the set of its neighbors in G, respectively. We use $g(G)$ to denote the *girth* of G, i.e. the length of the shortest cycle of G.

We use $b(f)$ to denote the boundary walk of a face f and write $f = [v_1v_2v_3 \cdots v_n]$ if $v_1, v_2, v_3, \cdots, v_n$ are the vertices of $b(f)$ in a cyclic order. The *degree*, $d(f)$, of a face f is the number of edges in its boundary walk $b(f)$, where

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cut edges are counted twice. A k (k^- or k^+)-*vertex* is a vertex of degree (at most or at least) k. A k (k^- or k^+)-*face* is defined similarly.

A proper vertex k-*coloring* of G is a mapping ϕ from $V(G)$ to a set of size k such that $\phi(x) \neq \phi(y)$ for any adjacent vertices x and y. A graph is vertex k-*colorable* if it has a proper vertex k-coloring.

A proper vertex coloring of a graph G is called *acyclic* if there is no 2-colored cycle in G. The acyclic vertex chromatic number $\chi_a(G)$ is the smallest integer k such that G has an acyclic vertex k-coloring. Grünbaum [10] proved that every planar graph has an acyclic 9-coloring and conjectured that all planar graphs have acyclic 5-colorings. Borodin [4] finally proved the conjecture.

A mapping $C : E(G) \to \{1, 2, \ldots, k\}$ is called an *acyclic edge* k-*coloring* of a graph G provided that any two adjacent edges receive different colors and there are no bichromatic cycles in G under the coloring C. In other words, for every pair of distinct colors i and j, the subgraph of G induced by all the edges which have either color i or j is acyclic. The smallest number k of colors such that G has an acyclic edge k-coloring is called the *acyclic chromatic index* of G and denoted by $\chi'_\alpha(G)$. It is certain that $\chi'_\alpha(G) \geq \Delta(G)$. The concept of acyclic edge coloring was first introduced by Fiamcik in [8]. In [1], Alon et al. proved that $\chi'_\alpha(G) \leq 64\Delta(G)$. In 2001, Alon et al. [1] conjectured that $\Delta(G) \leq \chi'_\alpha(G) \leq \Delta(G) + 2$ for any graph G and proved that the conjecture holds for almost all regular graphs. The conjecture is still open, even for planar graphs. For more information, interested readers are referred to [7], [9], [11], [14].

A proper *total* k-*coloring* of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest integer k such that G has a total k-coloring. Behzad [2] and Vizing [19] independently conjectured that $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$. For a plane graph G, the conjecture is proved to be true except $\Delta(G) = 6$. Interested readers may see [5], [6], [12], [15], [18] for more information.

An *acyclic total* k-*coloring* is a proper total k-coloring of a graph G such that there are at least 4 colors on vertices and edges incident with a cycle of G. The *acyclic total chromatic number* of G, $\chi''_a(G)$, is the smallest integer k such that G has an acyclic total k-coloring. The acyclic total coloring was introduced by Sun and Wu [16]. They conjectured that $\Delta(G) + 1 \leq \chi''_a(G) \leq \Delta(G) + 2$. In the same paper, they proved that the acyclic total chromatic number of a planar graph G is at most $\Delta(G) + 2$ if $\Delta(G) \geq 12$, or $\Delta(G) \geq 6$ and $g(G) \geq 4$, or $\Delta(G) \geq 5$ and $g(G) \geq 5$, or $g(G) \geq 6$. Furthermore, they proved that $\chi''_a(G) = \Delta(G) + 1$ if G is a series-parallel graph with $\Delta(G) \geq 3$. In [17], it is proved that the acyclic total chromatic number of a planar graph G of maximum degree at least k and without l cycles is at most $\Delta(G) + 2$ if $(k, l) \in \{(6,3),(7,4),(6,5),(7,6)\}$.

It is obvious that $\chi''_a(G) \geq \Delta(G) + 1$. In this paper, we mainly investigate the sufficient conditions for plane graphs to attain the lower bound. In fact, we prove that for every plane graph G with maximum degree Δ and girth $g(G)$, $\chi''_a(G) = \Delta + 1$ if (1) $\Delta \geq 9$ and $g(G) \geq 4$; (2) $\Delta \geq 6$ and $g(G) \geq 5$; (3) $\Delta \geq 4$ and $g(G) \geq 6$; (4) $\Delta \geq 3$ and $g(G) \geq 14$.

2 Structure of $(\Delta + 1)$-minimal graphs

A graph G is called *k-minimal* if G is not acyclic total k-colorable, but any proper subgraph of G does.

Note that if a graph G has maximum degree at most 2 and contains cycles, then G does not admit any acyclic total coloring using exactly $\Delta + 1$ colors. Hence, in this section, we always assume that G is a $(\Delta + 1)$-minimal graph with $\Delta \geq 3$.

For convenience, we introduce the following terminology. Let ϕ be an acyclic total $(\Delta + 1)$-coloring of G. For each $x \in V(G)$, we use $F(x)$ to denote the set of the colors assigned to the edges incident with x. A *k-thread* is a graph which we obtain from a path on k vertices, $k \geq 2$, by adding at least two 1-vertices to each of the end vertices of the path. The 3^+-vertices of the k-thread we call the *end vertices* of the k-thread. Let $S = \{1, 2, \cdots, \Delta + 1\}$ be the color set.

Lemma 2.1 *G contains no 1-vertex.*

Proof. It follows from $(\Delta + 1)$-minimality of G.

Lemma 2.2 *G contains no 2-vertex adjacent to $(\Delta - 1)^-$-vertices if $\Delta \geq 4$.*

Proof. Suppose on the contrary that v is a 2-vertex of G adjacent to a $(\Delta-1)^-$-vertex u. We consider the worst possibility. Let w be the other neighbor of v and $d(w) = \Delta$. Since G is $(\Delta + 1)$-minimal, $G - vw$ admits an acyclic total $(\Delta + 1)$-coloring ϕ. Without loss of generality, assume that $\phi(w) = \Delta$ and $F(w) = \{1, 2, \cdots, \Delta - 1\}$. We will extend ϕ to the whole graph G. First, we erase the colors of v, uv and color vw with $\Delta+1$. Since $d(u) \leq \Delta - 1$ and $\Delta \geq 4$, we can properly color uv and v in sequence. If $\phi(uv) \neq \Delta$ or $\phi(u) \neq \Delta+1$, then we obtain an acyclic total coloring of G, since $|\{\phi(v), \phi(w), \phi(vw), \phi(uv)\}| = 4$ or $|\{\phi(v), \phi(w), \phi(vw), \phi(u)\}| = 4$. If $\phi(uv) = \Delta$ and $\phi(u) = \Delta + 1$, then we check the color of v. Note that $d(u) \leq \Delta - 1$ and $|S\backslash(F(u) \cup \{\phi(u)\})| \geq 1$. Hence, if $\phi(v) \in S\backslash(F(u) \cup \{\phi(u)\})$, then it is not difficult to verify that ϕ is an acyclic total coloring of G. Otherwise, we recolor v with some color $\alpha \in S\backslash(F(u) \cup \{\phi(u)\})$. Thus, we obtain a contradiction to $(\Delta + 1)$-minimality of G.

Lemma 2.3 *G contains no 3-thread.*

Proof. If $\Delta(G) \geq 4$, then the lemma holds by Lemma 2.2. So we assume that $\Delta(G) = 3$.

Suppose on the contrary that $xuvwy$ is a 3-thread with $d(u) = d(v) = d(w) = 2$. By the choice of G, $G - v$ admits an acyclic total coloring ϕ using at most 4 colors. For convenience, assume that $\phi(x) = 3$ and $\phi(xu) = 4$. We erase the colors of u and w. We will show that we can extend ϕ to the whole graph G, which is a contradiction to the choice of G. We have the following cases.

Case 1. $\phi(wy) \in \{3, 4\}$. If $\phi(wy) = 4$, then we can color u, uv, v and vw with 2, 1, 4 and 3, respectively. If $\phi(wy) = 3$, then we can color u, uv, v and vw with 2, 1, 3 and 4, respectively. Moreover in both cases, if $\phi(y) = 1$, then $\phi(w) = 2$, otherwise $\phi(w) = 1$.

Case 2. $\phi(wy) \in \{1, 2\}$. If $\phi(wy) = 1$, then we can color u, uv and vw with 2, 1 and 2, respectively. If $\phi(wy) = 2$, then we can color u, uv and vw with 1, 2 and 1, respectively. Moreover in both cases, if $\phi(y) = 3$, then $\phi(w) = 4$ and $\phi(v) = 3$, otherwise $\phi(w) = 3$ and $\phi(v) = 4$.

Lemma 2.4 *G does not contain a 3-vertex u such that u is the end vertex of two 2-threads.*

Proof. If $\Delta(G) \geq 4$, then the lemma holds by Lemma 2.2. So we assume that $\Delta(G) = 3$.

Suppose on the contrary that u is a 3-vertex adjacent to two 2-threads, named ux_2x_1x and uz_2z_1z, respectively. Let y_2 be the other neighbor of u. By the choice of G, $G - x_1x_2$ admits an acyclic total coloring ϕ using at most 4 colors. We erase the colors of x_1 and x_2. Without loss of generality, assume that $\phi(x) = 3$ and $\phi(xx_1) = 4$. It is not difficult to verify that ϕ can be extended from $G - x_1x_2$ to the whole graph G except $\phi(u) = 4$ and $\phi(ux_2) = 3$.

Hence, we assume that $\phi(u) = 4$ and $\phi(ux_2) = 3$. In all the cases we want to recolor some vertices to get the case $\phi(u) \neq 4$ or $\phi(ux_2) \neq 3$. By symmetry, let $\phi(uz_2) = 1$ and $\phi(uy_2) = 2$. If $\phi(z_2) \neq 3$ and $\phi(y_2) \neq 3$, then we can exchange the colors of u and ux_2. It is easy to extend ϕ to the whole graph G. Now, we assume that $\phi(z_2) = 3$. We have the following possibilities.

Case 1. $\phi(y_2) = 3$.

Case 1.1 First, we assume that $\phi(z_2z_1) = 4$. If $\phi(z_1) = 2$, then we first exchange the colors of z_2 and uz_2, and next recolor x_2u with 1. Otherwise, if $\phi(z_1) = 1$, we first recolor z_2 with 2. Next, we exchange the colors of z_2u and x_2u.

Case 1.2 If $\phi(z_2z_1) = 2$, then we exchange the colors of u and uz_2.

Case 2. $\phi(y_2) = 1$.

Case 2.1 If $\phi(z_2z_1) = 4$, then we recolor y_2 with 3 and reduce this case to Case 1.1.

Case 2.2 Now, suppose $\phi(z_2z_1) = 2$. First, assume that $\phi(z_1) = 1$. If $\phi(z_1z) = 3$, then we recolor z_1z_2 with 4 and reduce this to Case 2.1. Otherwise, if $\phi(z_1z) = 4$, then we first exchange the colors of z_2, z_1z_2, and next recolor u with 3 and ux_2 with 4.

Finally, we assume that $\phi(z_1) = 4$. If $\phi(z_1z) = 1$, then we first exchange the colors z_2, z_1z_2, and next recolor u with 3 and ux_2 with 4. Otherwise, if $\phi(z_1z) = 3$, then we recolor z_1z_2, z_2, uz_2, ux_2 with 1, 2, 3, 1, respectively.

Hence, ϕ can be extended to the whole graph G, which is impossible.

Lemma 2.5 *If $\Delta \geq 4$, then G contains no even cycle $C = v_1v_2\cdots v_{2n}$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $n \geq 2$.*

Proof. Suppose on the contrary that $C = v_1v_2\cdots v_{2n}$ is an even cycle with $d(v_{2i-1}) = 2$ for $1 \leq i \leq n$. By Lemma 2.2, $d(v_{2i}) = \Delta$, for $1 \leq i \leq n$. Assume $N(v_{2i}) = \{u_{2i,j} | j = 1, 2, \cdots, \Delta - 2\} \cup \{v_{2i-1}, v_{2i+1}\}$, where the index i is taken module $2n$. Since G is $(\Delta+1)$-minimal, $G - E(C) - \{v_{2i-1} | 1 \leq i \leq n\}$ admits an acyclic total $(\Delta+1)$-coloring ϕ.

For each $v_{2i-1}v_{2i} \in E(C)$, $L(v_{2i-1}v_{2i}) = S\backslash(F(v_{2i}) \cup \{\phi(v_{2i})\})$, where $i = 1, 2, \cdots, n$. We can properly color $E(C)$, since $|L(v_{2i-1}v_{2i})| = 2$ for $1 \leq i \leq n$, and each even cycle is 2-edge choosable. Let $T(v_{2i}) = F(v_{2i})\backslash\{\phi(v_{2i-1}v_{2i}), \phi(v_{2i}v_{2i+1})\}$.

Lastly, we color each 2-vertex v_{2i-1} of C for $i = 1, 2, \cdots, n$, (we assume that $v_0 = v_n$) according to the following rules.

(*a*) If $|F(v_{2i-1}) \cup \{\phi(v_{2i-2}), \phi(v_{2i})\}| \geq 3$, then we color v_{2i-1} with a color $\alpha \in S\backslash(F(v_{2i-1}) \cup \{\phi(v_{2i-2}), \phi(v_{2i})\})$.

(b) If $|F(v_{2i-1}) \cup \{\phi(v_{2i-2}), \phi(v_{2i})\}| = 2$ and $T(v_{2i-2}) = T(v_{2i})$, then we color v_{2i-1} with a color $\alpha \in S \backslash (T(v_{2i-2}) \cup \{\phi(v_{2i-2}), \phi(v_{2i-2}v_{2i-1})\})$.

(c) If $|F(v_{2i-1}) \cup \{\phi(v_{2i-2}), \phi(v_{2i})\}| = 2$ and $T(v_{2i-2}) \neq T(v_{2i})$, then we color v_{2i-1} with a color $\alpha \in T(v_{2i-2}) \backslash T(v_{2i})$.

We will show that after coloring each 2-vertex from v_1 to v_{2i-1} in sequence, we can extend ϕ to the whole graph G, which contradicts the choice of G.

First, we claim that the color assigned to each 2-vertex is proper. If we use (a), since $|F(v_{2i-1}) \cup \{\phi(v_{2i-2}), \phi(v_{2i})\}| \leq 4$, $|(F(v_{2i-1}) \cup \{\phi(v_{2i-2}), \phi(v_{2i})\})| \leq \Delta$, then α is available. If we use (b), since $|(T(v_{2i-2}) \cup \{\phi(v_{2i-2}), \phi(v_{2i-2}v_{2i-1})\})| \leq \Delta$ and $\alpha \notin F(v_{2i-1})$, then α is available, too. Otherwise, if we use (c), then we can color v_{2i-1} with some color $\alpha \in T(v_{2i-2}) \backslash T(v_{2i})$.

Now, we show that in the coloring of each 2-vertex v_{2i-1}, $i = 1, 2, \cdots, n$, there is no cycle for which at most 3 color are used with the exception of C.

Consider a 2-vertex v_{2k-1} of C, where $k = 1, 2, \cdots, n$. If v_{2k-1} is colored according to (a), then it is easy to check that $|F(v_{2k-1}) \cup \{\phi(v_{2k-1}), \phi(v_{2k-2}), \phi(v_{2k})\}| \in \{4, 5\}$, and no 3-colored cycle will go through the segment $v_{2k-2}v_{2k-1}v_{2k}$. If v_{2k-1} is colored according to (b), then $|\{\phi(v_{2k-1})\} \cup F(v_{2k-1})| = 3$. Note that $T(v_{2k-2}) = T(v_{2k})$ and $\phi(v_{2k-1}) \notin T(v_{2k})$, thus at least 4 colors appear on the segment $u_{2k-2,p}v_{2k-2}v_{2k-1}v_{2k}u_{2k,q}$, where $1 \leq p, q \leq \Delta - 2$. Hence, no cycle assigned at most 3 color will be established, except C itself. Otherwise, if v_{2k-1} is colored according to (c), then $|\{\phi(v_{2k-1})\} \cup F(v_{2k-1})| = 3$. Note that $T(v_{2k-2}) \neq T(v_{2k})$ and $\phi(v_{2k-1}) \notin T(v_{2k})$, thus at least 4 colors appear on the segment $v_{2k-2}v_{2k-1}v_{2k}u_{2k,q}$, where $1 \leq q \leq \Delta - 2$. Hence, no cycle assigned at most 3 color will be established passing through $u_{2k-2,p}v_{2k-2}v_{2k-1} \cdots v_{2l}u_{2l,q}$, where $1 \leq p, q \leq \Delta - 2$ and $k \leq l \leq n$, except C itself.

Finally, we show that after completing coloring all 2-vertices of C, if at most 3 colors appear on C, then we can recolor C such that at least 4 colors appear on C and this recoloring will not induce new 3-colored cycles.

Suppose at most 3 colors $\Delta - 1, \Delta, \Delta + 1$ appear on C. Above discussion shows that all 2-vertices are colored according to (b) and $\{\phi(u_{2i,j}) | 1 \leq j \leq \Delta - 2\} = \{1, 2, \cdots, \Delta - 2\}$, for $1 \leq i \leq n$. Hence, $|F(v_{2i-1})| = 2$ for $1 \leq i \leq n$. We arbitrarily choose a segment $v_{2k-2}v_{2k-1}v_{2k}v_{2k+1}$ of C. Without loss of generality, assume that $\phi(v_{2k-2}) = \Delta$, $\phi(v_{2k-2}v_{2k-1}) = \Delta - 1$, $\phi(v_{2k-1}) = \Delta + 1$, $\phi(v_{2k-1}v_{2k}) = \Delta$, $\phi(v_{2k}) = \Delta - 1$, $\phi(v_{2k}v_{2k+1}) = \Delta + 1$, $\phi(v_{2k+1}) = \Delta$. Now, we recolor v_{2k-1} and v_{2k+1} with 1, $v_{2k-1}v_{2k}$ with $\Delta + 1$ and $v_{2k}v_{2k+1}$ with Δ. It is not difficult to check that at least 4 colors appear on C and no new 3-colored cycles are induced. Hence, we have a contradiction.

Lemma 2.6 *If $\Delta \geq 6$, then G contains no 3-vertex v with $N(v) = \{x, y, z\}$ and $d(x) \leq d(y) \leq d(z)$ such that $d(x) + d(y) \leq \Delta$.*

Proof. Assume by a contradiction that v is a 3-vertex of G, $N(v) = \{x, y, z\}$ and $d(x) \leq d(y) \leq d(z)$ such that $d(x) + d(y) \leq \Delta$. We consider the worst case that $d(z) = \Delta$. By Lemma 2.2, $d(x) \geq 3$. Since G is $(\Delta + 1)$-minimal, $G - vz$ admits an acyclic total coloring ϕ using $\Delta + 1$ colors. For convenience, assume that $F(z) = \{1, 2, \cdots, \Delta - 1\}$ and $\phi(z) = \Delta$. Now, we will extend ϕ from $G - vz$ to the whole graph G to obtain a contradiction. First, we erase the color of v, vx, vy and color vz with $\Delta + 1$. Note that $d(x) + d(y) \leq \Delta$ and $\lfloor \frac{\Delta}{2} \rfloor \geq d(x) \geq 3$, $|(F(x) \cup \{\Delta, \Delta + 1, \phi(x)\}| \leq \Delta - 1$ and $|(F(y) \cup \{\Delta, \Delta + 1, \phi(y)\}| \leq \Delta - 1$. Thus, we can properly color vy and vx in sequence, with colors distinct from Δ, and next properly color v. For convenience, assume that $\phi(vx) = \alpha$, $\phi(vy) = \beta$ and $\phi(v) = \gamma$. Since $|\{\phi(vx), \phi(v), \phi(vz), \phi(z)\}| = 4$ and $|\{\phi(vy), \phi(v), \phi(vz), \phi(z)\}| = 4$, no 3-colored cycle will be induced except $\phi(vx) = \phi(y)$, $\phi(vy) = \phi(x)$ and $\gamma \in F(x) \cap F(y)$. In this situation, we choose a color $\delta \notin \{\Delta, \Delta + 1, \alpha, \beta, \gamma\} \cup F(x)$ to vx. Since $d(x) \leq \lfloor \frac{\Delta}{2} \rfloor$, $\Delta \geq 6$ and $\gamma \in F(x)$, δ always exists. It is easy to verify that $|\{\phi(x), \phi(xv), \phi(v), \phi(vz), \phi(z)\}| \geq 4$ and $|\{\phi(x), \phi(xv), \phi(v), \phi(vy), \phi(y)\}| \geq 4$, thus no 3-colored cycles will be induced. The obtained coloring of G is an acyclic total $(\Delta + 1)$-coloring, which is impossible.

Lemma 2.7 *If $\Delta \geq 9$, then G contains no 3-vertex v with $N(v) = \{x, y, z\}$ and $d(x) \leq d(y) \leq d(z)$ such that $d(x) + d(y) \leq \Delta + 2$.*

Proof. The proof of this lemma is quite similar to that of Lemma 2.6. Assume by a contradiction that v is a 3-vertex of G, $N(v) = \{x, y, z\}$ and $d(x) \leq d(y) \leq d(z)$ such that $d(x) + d(y) \leq \Delta + 2$. We consider the worst case that $d(z) = \Delta$. By Lemma 2.2, $d(x) \geq 3$. Since G is $(\Delta + 1)$-minimal, $G - vz$ admits an acyclic total coloring ϕ using $\Delta + 1$ colors. For convenience, assume that $F(z) = \{1, 2, \cdots, \Delta - 1\}$ and $\phi(z) = \Delta$. Now, we will extend ϕ from $G - vz$ to the whole graph G to obtain a contradiction. First, we erase the color of v, vx, vy and color vz with $\Delta + 1$. Note that $d(x) + d(y) \leq \Delta + 2$, $|F(x) \cup \{\Delta, \Delta + 1, \phi(x)\}| \leq \Delta - 1$ and $|F(y) \cup \{\Delta + 1, \phi(y)\}| \leq \Delta$. Thus we can properly color vy and vx in sequence, and next properly color v. For convenience, assume that $\phi(vx) = \alpha$, $\phi(vy) = \beta$ and $\phi(v) = \gamma$. Since $|\{\phi(vx), \phi(v), \phi(vz), \phi(z)\}| = 4$, no 3-colored cycle will be established passing through zvx. Moreover, if $\phi(y) \neq \Delta + 1$, then no 3-colored cycle will be established passing through zvy. If $\phi(y) = \Delta + 1$, then $|F(y) \cup \{\Delta + 1, \phi(y)\}| \leq \Delta - 1$ and $|F(x) \cup \{\Delta, \Delta + 1, \phi(x)\}| \leq \Delta - 1$, and we can recolor vy and vx with proper colors distinct from Δ in sequence and finally properly recolor v, which is quite similar to the case in Lemma 2.6. Hence, no 3-colored cycle will be induced except $\phi(vx) = \phi(y)$ and $\phi(vy) = \phi(x)$. If $\phi(vx) = \phi(y)$ and $\phi(vy) = \phi(x)$, we choose a color $\delta \notin \{\Delta, \Delta+1, \alpha, \beta, \gamma\} \cup F(x)$ to vx. Since $d(x) \leq \lfloor \frac{\Delta+2}{2} \rfloor$ and $\Delta \geq 9$, δ always exists. It is easy to verify that $|\{\phi(x), \phi(xv), \phi(v), \phi(vz), \phi(z)\}| \geq 4$ and $|\{\phi(x), \phi(xv), \phi(v), \phi(vy), \phi(y)\}| \geq 4$, and no 3-colored cycles will be induced. The obtained coloring of G is an acyclic total $(\Delta + 1)$-coloring, which is impossible.

Lemma 2.8 *If $\Delta \geq 9$, then G contains no 4-vertex v with $N(v) = \{v_1, v_2, v_3, v_4\}$ and $d(v_1) \leq d(v_2) \leq d(v_3) \leq d(v_4)$ such that $d(v_1) \leq \Delta - 4, d(v_2) \leq \Delta - 3$ and $d(v_3) \leq \Delta - 2$.*

Proof. Assume to the contrary that v is such a 4-vertex. By the choice of G, $G - \{vv_1, vv_2, vv_3\}$ admits an acyclic total coloring ϕ using $\Delta + 1$ colors. We consider the worst case that $d(v_4) = \Delta$ and assume that $F(v_4) = \{1, 2, \cdots, \Delta\}$, $\phi(vv_4) = \Delta$ and $\phi(v_4) = \Delta + 1$. To extend ϕ to the whole graph G, we first recolor v with $\alpha \in S \backslash \{\Delta, \Delta + 1, \phi(v_1), \phi(v_2), \phi(v_3)\}$. Next, we assign a color $\beta \in S \backslash (\{\Delta, \phi(v), \phi(v_3)\} \cup F(v_3))$ to vv_3 if $\phi(v_3) \neq \Delta$. Otherwise, we assign $\beta \in S \backslash (\{\Delta, \Delta + 1, \phi(v), \phi(v_3)\} \cup F(v_3))$ to vv_3. Since $d(v_3) \leq \Delta - 2$, β always exists, and it is not difficult to verify that no 3-colored cycles will be induced passing through the segment v_3vv_4.

Similarly, we color vv_2 with a color $\gamma \in S \backslash (\{\Delta, \phi(v), \phi(vv_3), \phi(v_2)\} \cup F(v_2))$ if $\phi(v_2) \notin \{\Delta, \phi(vv_3)\}$. Otherwise, if $\phi(v_2) = \Delta$, then we choose $\gamma \in S \backslash (\{\Delta, \Delta + 1, \phi(v), \phi(vv_3)\} \cup F(v_2))$ and if $\phi(v_2) = \phi(vv_3)$, then we choose $\gamma \in S \backslash (\{\Delta, \phi(v_3), \phi(v), \phi(vv_3)\} \cup F(v_2))$.

Finally, we color vv_1 with a color δ from S with the following rules.

(1) $\delta \in S \backslash (\{\Delta, \phi(v), \phi(vv_3), \phi(vv_2), \phi(v_1)\} \cup F(v_1))$ if $\phi(v_1) \notin \{\Delta, \phi(vv_3), \phi(vv_2)\}$.

(2) $\delta \in S \backslash (\{\Delta, \phi(v_i), \phi(v), \phi(vv_3), \phi(vv_2)\} \cup F(v_1))$ if $\phi(v_1) = \phi(vv_i)$, for some $i \in \{2, 3, 4\}$.

The obtained coloring of G is an acyclic total $(\Delta + 1)$-coloring, a contradiction.

Lemma 2.8 shows that if some 4-vertex v in $(\Delta + 1)$-minimal graph with $\Delta \geq 9$ is adjacent to a $(\Delta - 4)^-$-vertex, then v is adjacent to at least two $(\Delta - 2)^+$-vertices.

Lemma 2.9 *If $\Delta \geq 9$, then G contains no 5-vertex v adjacent to exactly five 3-vertices.*

Proof. Let $N(v) = \{v_1, v_2, \cdots, v_5\}$. Suppose on the contrary that $d(v_i) = 3$, for all $1 \le i \le 5$. By the choice of G, $G - vv_1$ admits an acyclic total coloring ϕ using $\Delta+1$ colors. We first recolor v with a color $\alpha \notin F(v) \cup \{\phi(v_i) | 1 \le i \le 5\}$. We will show that by a proper adjustment, the obtained coloring is also an acyclic total coloring of $G - vv_1$. If no 3-colored cycle is established, then we are done. Otherwise, suppose a 3-colored cycle is induced passing through v_5vv_4. It follows that $\phi(v_5) = \phi(vv_4)$ and $\phi(v_4) = \phi(vv_5)$. We can recolor v with $\beta \notin F(v) \cup \{\phi(v_i) | 1 \le i \le 5\} \cup F(v_5)$, since $|F(v) \cup \{\phi(v_i) | 1 \le i \le 5\} \cup F(v_5)| \le \Delta$. Therefore, no 3-colored cycle will be established unless $\phi(v_2) = \phi(vv_3)$, $\phi(v_3) = \phi(vv_2)$ and $\phi(v) \in F(v_2) \cap F(v_3)$. In this situation, we recolor v with a color $\gamma \notin F(v) \cup \{\phi(v_i) | 1 \le i \le 5\} \cup F(v_5) \cup F(v_2)$.

Now, we extend the obtained acyclic total coloring of $G - vv_1$ to the whole graph. If $\phi(v_1) \notin F(v)$, we properly color vv_1. This is possible since at most eight colors are forbidden and $\Delta \ge 9$. The obtained coloring is an acyclic total coloring of G. Otherwise, Without loss of generality, assume that $\phi(v_1) = \phi(vv_2)$. We choose a color $\eta \notin F(v) \cup F(v_1) \cup \{\phi(v_1), \phi(v_2), \phi(v)\}$ to vv_1. The obtained coloring of G is an acyclic total coloring, a contradiction.

3 Main results

Theorem 3.1 *Let G be a plane graph with maximum degree Δ and girth $g(G)$, then $\chi''_a(G) = \Delta + 1$ if one of the followings holds:*

(1) $\Delta \ge 9$ *and* $g(G) \ge 4$;
(2) $\Delta \ge 6$ *and* $g(G) \ge 5$;
(3) $\Delta \ge 4$ *and* $g(G) \ge 6$;
(4) $\Delta \ge 3$ *and* $g(G) \ge 14$.

Proof. Since it is trivial that $\chi''_a(G) \ge \Delta + 1$ for all graphs, we only prove that $\chi''_a(G) \le \Delta + 1$. Assume by a contradiction that $\chi''_a(G) > \Delta + 1$. Let G be a $(\Delta + 1)$-minimal plane graph with maximum degree $\Delta \ge 3$. By Lemma 2.1, $\delta(G) \ge 2$.

If $\Delta \ge 4$, then let G_2 be the subgraph induced by the edges incident with 2-vertices of G. By Lemma 2.2 and Lemma 2.5, each 2-vertex is adjacent to Δ-vertices and we have no even cycle $G_2 \supseteq C = v_1v_2 \cdots v_{2n}$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$. Thus G_2 is a forest. Hence, one can find a matching M in G saturating all 2-vertices. If $uv \in M$ and $d(u) = 2$, v is called the 2-*master* of u. Each 2-vertex has a 2-master and each vertex of degree Δ can be the 2-master of at most one 2-vertex.

Case 1: $\Delta \geq 9$ and $g(G) \geq 4$.

A 3-vertex v with $N(v) = \{x, y, z\}$ and $d(x) \leq d(y) \leq d(z)$ is called *bad* if the followings hold: (1): v is not incident with any 5^+-face; (2): $d(z) = \Delta$; (3): x is a 3-vertex. For a vertex v, we use $n_k(v)$ to denote the number of k-vertices adjacent to v.

In the beginning, we assign a weight $w(v) = d(v) - 4$ to each vertex v and a weight $w(f) = d(f) - 4$ to each face f. By applying Euler's formula $|V| + |F| - |E| = 2$ for plane graphs, we have $\Sigma_{x \in V(G) \cup F(G)} w(x) = -8$. If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum w^*(x)=-8$. Hence, if $w^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and Case 1 is proved.

The new weight w^* is obtained by the following discharging rules.

($R_{1,1}$) Each 2-vertex receives 2 from its 2-master.

($R_{1,2}$) Each Δ-vertex transfers $\frac{1}{2}$ to each adjacent bad 3-vertex, $\frac{3}{8}$ to each adjacent non-bad 3-vertex.

($R_{1,3}$) Each vertex v with $6 \leq d(v) \leq \Delta - 1$ transfers $\frac{d(v)-4}{d(v)}$ to each adjacent 3-vertex.

($R_{1,4}$) Each vertex v with $d(v) \in \{4, 5\}$ transfers $\frac{1}{4}$ to each adjacent 3-vertex.

($R_{1,5}$) Each $(\Delta - 2)^+$-vertex v transfers $\frac{1}{4}$ to each adjacent 4-vertex.

($R_{1,6}$) Each 5^+-face f transfers its positive charge to each incident 3-vertex equally.

Since $g(G) \geq 4$, $w^*(f) \geq 0$ for each face. Let v be a k-vertex. We have $k \geq 2$, since G has no 1-vertex.

If $k = 2$, then $w(v) = -2$. By ($R_{1,1}$), $w^*(v) = -2 + 2 = 0$.

If $k = 3$, then $w(v) = -1$. Assume that $N(v) = \{x, y, z\}$ with $d(x) \leq d(y) \leq d(z)$. By Lemma 2.2, $d(x) \geq 3$. First, assume that $d(x) = 3$, then by Lemma 2.7, $d(z) \geq d(y) \geq 9$. If $d(z) < \Delta$, then by ($R_{1,3}$), $w^*(v) \geq -1+2\times\frac{5}{9} \geq 0$. Now, assume that $d(z) = \Delta$. If v is incident with a 5^+-face f, then by Lemma 2.7, f is incident with at most three 3-vertices. By ($R_{1,2}$), ($R_{1,3}$) and ($R_{1,6}$), $w^*(v) \geq -1 + \frac{3}{8}+\min\{\frac{3}{8}, \frac{5}{9}\} + \frac{1}{3} \geq 0$. So we assume that v is incident with three 4-faces. Then v is a bad 3-vertex. By ($R_{1,2}$) and ($R_{1,3}$), $w^*(v) \geq -1+\frac{1}{2}+\min\{\frac{1}{2}, \frac{5}{9}\} = 0$. If $d(x) = 4$, then $d(z) \geq d(y) \geq 8$. By ($R_{1,2}$), ($R_{1,3}$) and ($R_{1,4}$), we have $w^*(v) \geq -1 + 2\times\min\{\frac{3}{8}, \frac{1}{2}\} + \frac{1}{4} \geq 0$. If $d(x) = 5$, then $d(z) \geq d(y) \geq 7$. By

$(R_{1,2})$, $(R_{1,3})$ and $(R_{1,4})$, $w^*(v) \geq -1 + \frac{1}{4}+\min\{\frac{1}{2}, \frac{3}{8}, \frac{3}{7}\} \times 2 \geq 0$. Otherwise, if $d(x) \geq 6$, then by $(R_{1,2})$ and $(R_{1,3})$, $w^*(v) \geq -1+\min\{\frac{1}{2}, \frac{3}{8}, \frac{1}{3}\} \times 3 \geq 0$.

If $k = 4$, then $w(v) = 0$. Note that, by Lemma 2.8, $n_3(v) \leq 2$ and v is adjacent to at least two $(\Delta - 2)^+$-vertices. Then by $(R_{1,4})$ and $(R_{1,5})$, $w^*(v) \geq 2 \times \frac{1}{4} - 2 \times \frac{1}{4} \geq 0$.

If $k = 5$, then $w(v) = 1$. Lemma 2.9 implies that v is adjacent to at most four 3-vertices. Hence, by $(R_{1,4})$, $w^*(v) \geq 1 - 4 \times \frac{1}{4} \geq 0$.

If $6 \leq k \leq \Delta - 1$, then $w(v) = k - 4$. The facts that $\Delta - 2 \geq 7$, $\frac{k-4}{k} \geq \frac{3}{7}$, for $k \geq 7$, $\max\{\frac{3}{7}, \frac{1}{4}\} = \frac{3}{7}$, and rules $(R_{1,3})$ and $(R_{1,4})$ imply that $w^*(v) \geq k - 4 - k \times \frac{k-4}{k} \geq 0$.

Suppose $k = \Delta \geq 9$. Let $N(v) = \{v_i | i = 1, 2, \cdots, \Delta\}$. We first prove the following claims.

Claim 3.1 *If v_i, v_{i+1} and v_{i+2} are the three consecutive 3-vertices adjacent to v, then v_{i+1} is not a bad 3-vertex.*

Proof. Suppose v_{i+1} is a bad 3-vertex, then v_{i+1} is incident with three 4-faces. Therefore, there exist two vertices x and y such that $x \in N(v_i) \cap N(v_{i+1})$ and $y \in N(v_{i+1}) \cap N(v_{i+2})$. By Lemma 2.7, $d(x) \geq 4$ and $d(y) \geq 4$. A contradiction to the definition of a bad vertex.

Claim 3.2 *Let v_i, v_{i+1} and v_{i+2} are the three consecutive vertices adjacent to v. If $2 \leq d(v_i) \leq 3$, $d(v_{i+1}) = 3$ and v_{i+1} is a bad 3-vertex, then $d(v_{i+2}) = \Delta$.*

Proof. Suppose v_{i+1} is a bad 3-vertex, then v_{i+1} is incident with three 4-faces. Therefore, there exist two vertices x and y such that $x \in N(v_i) \cap N(v_{i+1})$ and $y \in N(v_{i+1}) \cap N(v_{i+2})$. If $d(v_i) = 2$, then by Lemma 2.2, $d(x) = \Delta$. We have $d(y) = 3$. By Lemma 2.7 and the definition of bad 3-vertex, $d(v_{i+2}) = \Delta$. Otherwise, if $d(v_i) = 3$, then by Lemma 2.7, $d(x) \geq 4$. We have $d(y) = 3$. Hence, by Lemma 2.7, it follows that $d(v_{i+2}) = \Delta$.

Now, let us check the final charge of v. If v is not a master of some 2-vertex, then $w^*(v) \geq \Delta - 4 - \Delta \times \frac{1}{2} \geq 0$ by $(R_{1,2})$. So we assume that v is a master. We have the following cases.

First, we assume that $n_3(v) \leq \Delta - 4$. Here, we can only use rules $(R_{1,1})$, $(R_{1,2})$ and $(R_{1,5})$. Since $\frac{1}{4} \leq \frac{3}{8} \leq \frac{1}{2}$, the worst case is when $n_3(v) = \Delta - 4$ and $n_4(v) = 3$. If $\Delta \geq 10$, then $w^*(v) \geq \Delta - 4 - 2 - (\Delta - 4) \times \frac{1}{2} - 3 \times \frac{1}{4} \geq 0$. Hence, we assume that $\Delta = 9$. If v is adjacent to at least one 5^+-vertex, then $w^*(v) \geq 9 - 4 - 2 - 5 \times \frac{1}{2} - 2 \times \frac{1}{4} \geq 0$. Otherwise, there exist four consecutive neighbors of v, named v_i, v_j, v_k and v_l, such that $d(v_j) = d(v_k) = 3$

and $\{d(v_i), d(v_l)\} \subseteq \{2,3,4\}$. By our claims, at least two 3-vertices adjacent to v are not bad 3-vertices. We have $w^*(v) \geq 9-4-2-3\times\frac{1}{2}-3\times\frac{1}{4}-2\times\frac{3}{8} \geq 0$.

Let $n_3(v) = \Delta - 3$. Let v_k and v_l be the other two neighbors of v. If $d(v_k) \geq 5$ and $d(v_l) \geq 5$, then $w^*(v) \geq \Delta - 4 - 2 - (\Delta-3)\times\frac{1}{2} \geq 0$. Otherwise, v is adjacent to at most two bad 3-vertices, $w^*(v) \geq \Delta - 4 - 2 - 2\times\frac{1}{2} - 2\times\frac{1}{4} - (\Delta-5)\times\frac{3}{8} \geq 0$.

Now, we assume that $n_3(v) = \Delta - 2$. Let v_k be the other adjacent vertex of v, $d(v_k) = 2$ or $d(v_k) \geq 4$. If $d(v_k) \geq 5$, then at most v_{k-1} and v_{k+1} may be bad 3-vertices, we have $w^*(v) \geq \Delta-4-2-(\Delta-4)\times\frac{3}{8}-2\times\frac{1}{2} \geq 0$. Otherwise, $w^*(v) \geq \Delta - 4 - 2 - (\Delta-2)\times\frac{3}{8} - \frac{1}{4} \geq 0$.

Finally, we assume that $n_3(v) = \Delta - 1$. By Claim 3.1 and Claim 3.2, v is not adjacent to any bad 3-vertices. Hence, $w^*(v) \geq \Delta-4-2-(\Delta-1)\times\frac{3}{8} \geq 0$.

Observe that for any vertex v, $w^*(v) \geq 0$, $0 \leq \sum_{x\in V\cup F} w(x) = -8$. This contradiction completes the proof of this case.

Case 2: $\Delta \geq 6$ and $g(G) \geq 5$.

In the beginning, we assign a weight $w(v) = 3d(v) - 10$ to each vertex v and a weight $w(f) = 2d(f) - 10$ to each face f. By applying Euler's formula $|V| + |F| - |E| = 2$ for plane graphs, we have $\Sigma_{x\in V(G)\cup F(G)} w(x) = -20$. If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum w^*(x) = -20$. Hence, if $w^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and this case is proved.

The new weight w^* is obtained by the following discharging rules.

($R_{2,1}$) Each 2-vertex receives 4 from its 2-master.

($R_{2,2}$) Each 3-vertex receives $\frac{1}{2}$ from each adjacent 4^+-vertices.

Since $g(G) \geq 5$, $w^*(f) \geq 0$ for each face. Let v be a k-vertex. We have $k \geq 2$, since G has no 1-vertex.

If $k = 2$, then $w(v) = -4$. By ($R_{2,1}$), $w^*(v) = -4 + 4 = 0$.

If $k = 3$, then $w(v) = -1$. Assume that $N(v) = \{x, y, z\}$ with $d(x) \leq d(y) \leq d(z)$. By Lemma 2.2, $d(x) \geq 3$. If $d(x) = 3$, then by Lemma 2.6, $d(z) \geq d(y) \geq \Delta - 2 \geq 4$. By ($R_{2,2}$), $w^*(v) = -1 + 2\times\frac{1}{2} \geq 0$. If $d(x) \geq 4$, then $w^*(v) \geq -1 + 3\times\frac{1}{2} \geq 0$ by ($R_{2,2}$).

If $4 \leq k \leq \Delta - 1$, then by ($R_{2,2}$), $w^*(v) \geq 3k - 10 - \frac{1}{2}k \geq 0$.

Suppose $k = \Delta \geq 6$. Then, by ($R_{2,1}$) and ($R_{2,2}$), $w^*(v) \geq 3k - 10 - 4 - \frac{1}{2}(k-1) \geq 0$.

Observe that for any vertex v, $w^*(v) \geq 0$, $0 \leq \sum_{x\in V\cup F} w(x) = -20$. This contradiction completes the proof of this case.

Case 3: $\Delta \geq 4$ and $g(G) \geq 6$.

In the beginning, we assign a weight $w(v) = 2d(v) - 6$ to each vertex v and a weight $w(f) = d(f) - 6$ to each face f. By applying Euler's formula $|V| + |F| - |E| = 2$ for plane graphs, we have $\Sigma_{x \in V(G) \cup F(G)} w(x) = -12$. If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum w^*(x) = -12$. Hence, if $w^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and this case is proved.

The new weight w^* is obtained by the following discharging rule.

($R_{3,1}$) Each 2-vertex receives 2 from its 2-master.

Since $g(G) \geq 6$, $w^*(f) \geq 0$ for each face. Let v be a k-vertex. We have $k \geq 2$, since G has no 1-vertex.

If $k = 2$, then $w(v) = -2$. By ($R_{3,1}$), $w^*(v) = -2 + 2 = 0$.

If $3 \leq k \leq \Delta - 1$, then $w^*(v) = w(v) \geq 0$.

Suppose $k = \Delta \geq 4$. By ($R_{3,1}$), $w^*(v) \geq w(v) - 2 \geq 0$.

Observe that for any vertex v, $w^*(v) \geq 0$, $0 \leq \sum_{x \in V \cup F} w(x) = -12$. This contradiction completes the proof of this case.

Case 4: $\Delta \geq 3$ and $g(G) \geq 14$.

If $\Delta \geq 4$, then this case is solved. Hence, we assume that $\Delta = 3$.

In the beginning, we assign a weight $w(v) = 6d(v) - 14$ to each vertex v and a weight $w(f) = d(f) - 14$ to each face f. By applying Euler's formula $|V| + |F| - |E| = 2$ for plane graphs, we have $\Sigma_{x \in V(G) \cup F(G)} w(x) = -28$. If we obtain a new weight $w^*(x)$ for all $x \in V \cup F$ by transferring weights from one element to another, then we also have $\sum w^*(x) = -28$. Hence, if $w^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$, then we get a contradiction and this case is proved.

The new weight w^* is obtained by the following discharging rule.

($R_{4,1}$) Let v be a 2-vertex. If v is adjacent to exactly one 3-vertex, then v receives 2 from the adjacent 3-vertex. Otherwise, if v is adjacent to two 3-vertices, then v receives 1 from each adjacent 3-vertex.

Since $g(G) \geq 14$, $w^*(f) \geq 0$ for each face. Let v be a k-vertex. We have $k \geq 2$, since G has no 1-vertex.

If $k = 2$, then $w(v) = -2$. Lemma 2.3 implies that v is adjacent to at least one 3-vertex. Then by ($R_{4,1}$) we have $w^*(v) = -2 + 2 = 0$.

If $k = 3$, then by Lemma 2.4 and ($R_{4,1}$), $w^*(v) \geq 4 - 2 - 2 \times 1 \geq 0$.

Observe that for any vertex v, $w^*(v) \geq 0$, $0 \leq \sum_{x \in V \cup F} w(x) = -28$. This contradiction completes the proof of this case.

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