Edge Friendly Labelings of Graph - I *†

Deepa Sinha ‡ and Jaspreet Kaur §

ABSTRACT

Let $G = (V, E)$ be a graph, an edge labeling $f : E \to \mathbb{Z}_2$ induces a vertex labeling $f^* : V \to \mathbb{Z}_2$ defined by $f^*(v) \equiv \sum_{uv \in E} f(uv) \pmod 2$. For each, $i \in \mathbb{Z}_2$ define $E_f(i) = |f^{-1}(i)|$ and $V_f(i) = |f^{*-1}(i)|$. We call f edge friendly if $|E_f(1) - E_f(0)| \leq 1$. The edge friendly index $I_f(G)$, is defined as $V_f(1) - V_f(0)$ and the full edge-friendly index set $FEFI(G)$, is defined as $\{I_f(G) : f \text{ is an edge friendly labeling}\}$. Further, the edge friendly index set $EFI(G)$, is defined as $\{|I_f(G)| : f \text{ is an edge friendly labeling}\}$. In this paper, we study the full edge-friendly index set of the star $K_{1,n}$, 2-regular graph, wheel W_n, m copies of path mP_n, $m \geq 1$.

AMS Subject Classification : 05C78, 05C25.

Keywords : Labeling, cordial labeling, friendly labeling, edge friendly labeling, full edge-friendly index set.

*Presented in 76^{th} IMS annual conference held at S.V.N.I.T, Surat on Dec27-30, 2010.

†Research is supported by the Department of Science and Technology (Govt. of India), New Delhi, India, under the project SR/S4/MS:409/06.

‡South Asian University, Akbar Bhawan, Chanakyapuri, New Delhi-110021, India (*deepa_sinha2001@yahoo.com*).

§Centre for Mathematical Sciences, Banasthali University, Banasthali-304022, Rajasthan, India (*bagga.jaspreetkaur@gmail.com*).

1 Introduction

For the undefined concepts and notation used in this paper, we refer the reader to the books [2, 8]. All graphs considered in this paper are finite, simple, and undirected. The graph with all vertices of even (odd) degree will be called *even* (*odd*) *degree graph.* The graph with each vertex of finite degree will be called *locally finite graph.*

Let $G = (V, E)$ be a graph, and Γ be an abelian group and $f : V \rightarrow \Gamma$ be a vertex labeling of G with the elements of Γ. Let $f^* : E \rightarrow \Gamma$ be defined by

$$f^*(xy) = f(x) \circ f(y), \ \forall xy \in E,$$

where '$\circ$' is the group operation. For each $g \in \Gamma$, let $v_f(g) = |f^{-1}(g)|$ and $e_f(g) = |f^{*-1}(g)|$. We call f a Γ-*friendly labeling* if

$$|v_f(g_1) - v_f(g_2)| \leq 1 \, \forall \, g_1, \, g_2 \in \Gamma, \tag{1}$$

Γ-*edge friendly labeling* if

$$|e_f(g_1) - e_f(g_2)| \leq 1 \, \forall \, g_1, \, g_2 \in \Gamma \tag{2}$$

and Γ-*cordial* if both (1) and (2) hold (cf.:[12]).

A graph G is said to be *cordial* if it admits a $\mathbb{Z}_2$-cordial labeling, where $\mathbb{Z}_2 = \{0, 1\}$ is the additive group with respect to modulo 2 addition (cf.:[3]). Cordial labelings of graphs have been studied extensively in [1, 4, 5, 6, 11, 13, 14, 18, 24].

Lee and Ng [21] have defined the dual concept of cordial labeling as edge cordial labeling. It was further redefined by Collins and Hovey [9] as:
Let $f : E \rightarrow \mathbb{Z}_2$ be an edge labeling of G. Define vertex labeling $f^* : V \rightarrow \mathbb{Z}_2$ by $f^*(v) \equiv \sum_{uv \in E} f(uv) \pmod 2$. Then G is *edge cordial* if and only if

1. $|V| \equiv 0, \ 1, \ 3 \pmod 4$ and there exists an edge labeling f such that $|e_f(1) - e_f(0)| \leq 1$ and $|v_f(1) - v_f(0)| \leq 1$ or
2. $|V| \equiv 2 \pmod 4$ and there exists an edge labeling f such that $|e_f(1) - e_f(0)| \leq 1$ and $|v_f(1) - v_f(0)| = 2$.

$\mathbb{Z}_2$-friendly labelings are simply called *friendly labelings* (cf.:[7]). Friendly labelings of graphs have been studied extensively in [15, 16, 17, 19, 22, 23, 25, 26, 27, 28, 29, 30, 31, 32] and $\mathbb{Z}_2$-edge friendly labelings are called *edge friendly labelings.* Let $f : E(G) \rightarrow \mathbb{Z}_2$ be an edge friendly labeling, which induces a vertex labeling $f^* : V(G) \rightarrow \mathbb{Z}_2$ defined by $f^*(v) \equiv \sum_{uv \in E} f(uv) \pmod 2$.

For $i \in \mathbb{Z}_2$, we define $V_f(i) = \left|f^{*-1}(i)\right|$ and $E_f(i) = \left|f^{-1}(i)\right|$. The number $I_f(G) = V_f(1) - V_f(0)$ is called the *edge friendly index* of f.

If F' denotes the set of all edge friendly labeling, then we have following definitions:

The *maximum edge friendly index* is denoted by $\varphi(G)$ and is defined as: $\varphi(G) = max\{I_f(G) : f \in F'\}$ and the corresponding edge labeling f is called *maximum edge friendly labeling* of G.

The *minimum edge friendly index* is denoted by $\varphi'(G)$ and is defined as: $\varphi'(G) = min\{I_f(G) : f \in F'\}$ and the corresponding edge labeling f is called *minimum edge friendly labeling* of G.

The *maximum modulus edge friendly index* is denoted by $\phi(G)$ and is defined as: $\phi(G) = max\{|I_f(G)| : f \in F'\}$ and the corresponding edge labeling f is called *maximum modulus edge friendly labeling* of G.

The *minimum modulus edge friendly index* is denoted by $\phi'(G)$ and is defined as: $\phi'(G) = min\{|I_f(G)| : f \in F'\}$ and the corresponding edge labeling f is called *minimum modulus edge friendly labeling* of G.

The *edge friendly index set* of the graph G, denoted by $EFI(G)$ and is defined as:

$$EFI(G) = \{|I_f(G)| : f \in F'\}.$$

We extend the concept to *full edge-friendly index set* of the graph G denoted by $FEFI(G)$ and is defined as:

$$FEFI(G) = \{I_f(G) : f \in F'\}.$$

In this paper, we determine the full edge-friendly index set of star $K_{1,n}$, 2-regular graph, wheel W_n, m copies of path mP_n, $m \geq 1$. Further, edge friendly index set of a graph can be calculated from the known full edge-friendly index set.

2 Some basic properties

Theorem 1. *Let $G = (V,\ E)$ be any $(p,\ q)$-graph, then*

$$FEFI(G) \subseteq \{2i - p :\ 0 \leq i \leq p\}.$$

Proof. Let f be an edge friendly labeling of the graph G. If $V_f(1) = i$, then $V_f(0) = p - i$. Hence $I_f(G) = 2i - p$. Thus, $FEFI(G) \subseteq \{2i - p :\ 0 \leq i \leq p\}$. □

Theorem 2. [21] *For any connected graph $G = (V, E)$, if f is an edge friendly labeling of G then $V_f(1)$ is even.*

The *parity* of an object states whether it is even or odd. Recall that if f is a friendly labeling then its inverse labeling $g : V(G) \to \mathbb{Z}_2$ defined by $g(v) = 1 - f(v)$ is friendly. Moreover, $i_f(G) = i_g(G)$. In general the similar result do not hold good for edge friendly labeling. However, results for some special cases are discussed.

Theorem 3. *Let $G = (V, E)$ be any locally finite graph and $f : E \to \mathbb{Z}_2$ be any labeling. Let $g : E \to \mathbb{Z}_2$ be the inverse of f defined by $g(e) = 1 - f(e)$ for every $e \in E$. Then for every vertex v in G,* $g(v) = \begin{cases} f(v) & if\ deg(v)\ is\ even, \\ 1 - f(v) & if\ deg(v)\ is\ odd. \end{cases}$

Proof. It follows easily by the fact that sum of any two numbers are even if and only if both the numbers are of the same parity. □

Corollary 4. *Let G be an even degree graph. If f is an edge friendly labeling and g be its inverse, then $I_g(G) = I_f(G)$.*

Corollary 5. *Let G be an odd degree graph. If f is an edge friendly labeling and g be its inverse, then $I_g(G) = -I_f(G)$.*

Theorem 6. *Let $f : E \to \mathbb{Z}_2$ is an edge friendly labeling and g be its inverse labeling then $I_f(G) = I_g(G)$ $(-I_g(G))$ if and only if exactly half of its odd (even) degree vertices induce label 1 by the edge labeling f.*

Proof. Let f be an edge friendly labeling of G such that exactly half of its odd (even) degree vertices induce label 1. Then by theorem 3, $I_f(G) = I_g(G)$ $(-I_g(G))$.

Conversely, let f be an edge friendly labeling of a graph $G = (V, E)$ and g be its inverse. We will use the following notation: p_{odd} and p_{even} denote the total numbers of odd and even degree vertices respectively. Let p_{odd0} and p_{odd1} be the total numbers of odd degree vertices receiving label 0 and 1 under f, respectively. Similarly, let p_{even0} and p_{even1} be the total numbers of even degree vertices receiving label 0 and 1 under f, respectively.

Then $V_f(1) = p_{odd1} + p_{even1}$ and $V_f(0) = p_{odd0} + p_{even0}$. Therefore, $I_f(G) = V_f(1) - V_f(0) = p_{odd1} + p_{even1} - p_{odd0} - p_{even0}$. By Theorem 3,
the total number of odd degree vertices receiving label 0 under g is p_{odd1},
the total number of odd degree vertices receiving label 1 under g is p_{odd0},
the total number of even degree vertices receiving label 1 under g is p_{even1},
the total number of even degree vertices receiving label 0 under g is p_{even0}.
So, $I_g(G) = p_{odd0} + p_{even1} - p_{odd1} - p_{even0}$. Thus, we have

$$I_f(G) - I_g(G) = 2(p_{odd1} - p_{odd0}) \tag{3}$$

$$I_f(G) + I_g(G) = 2(p_{even1} - p_{even0}) \quad (4)$$

By using (3) and (4) we have, if $I_f(G) = I_g(G)$ $(-I_g(G))$, then exactly half of its odd(even) degree vertices induce label 1 by the edge labeling f. □

3 Full edge-friendly index set of star

A star is the complete bipartite graph $K_{1,n}$, a tree on n vertices with one vertex having vertex degree $n-1$ and other $n-1$ vertices of vertex degree 1.

Theorem 7. *The full edge-friendly index set of star graph $K_{1,n}$ is*

$$FEFI(K_{1,n}) = \begin{cases} \{-1\} & n \equiv 0 \pmod 4, \\ \{-2,\ 2\} & n \equiv 1 \pmod 4, \\ \{1\} & n \equiv 2 \pmod 4, \\ \{0\} & n \equiv 3 \pmod 4. \end{cases}$$

Proof. The number of edges can be of the form $4m,\ 4m+1,\ 4m+2,\ 4m+3$.

no. of edges	number of edges labeled by 0	number of edges labeled by 1	label received by center vertex	number of end vertices receiving label 0	number of end vertices receiving label 1	$I_f(G)$ values
$4m$	$2m$	$2m$	0	$2m$	$2m$	-1
$4m+1$	$2m+1$	$2m$	0	$2m+1$	$2m$	-2
$4m+1$	$2m$	$2m+1$	1	$2m$	$2m+1$	2
$4m+2$	$2m+1$	$2m+1$	1	$2m+1$	$2m+1$	1
$4m+3$	$2m+2$	$2m+1$	1	$2m+2$	$2m+1$	0
$4m+3$	$2m+1$	$2m+2$	0	$2m+1$	$2m+2$	0

Thus,

$$FEFI(K_{1,n}) = \begin{cases} \{-1\} & n \equiv 0 \pmod 4, \\ \{-2,\ 2\} & n \equiv 1 \pmod 4, \\ \{1\} & n \equiv 2 \pmod 4, \\ \{0\} & n \equiv 3 \pmod 4. \end{cases}$$

□

4 Edge-friendly index set of 2-regular graph

First, we will evaluate the full edge-friendly index set of cycle C_n. A cycle $C_n = e_1 e_2 \ldots e_n$ is a connected graph with n vertices and n edges and every vertex is of degree 2.

Observation 1. *There exists no edge friendly labeling of cycle C_n such that $V_f(1) = 0$.*

Theorem 8. *The full edge-friendly index set of the cycle C_n is*

$$FEFI(C_n) = \{4i - n : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}.$$

Proof. By Theorem 1, Theorem 2 and Observation 1, we get

$$FEFI(C_n) \subseteq \{4i - n : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}. \tag{5}$$

For a fixed i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, define an edge labeling f of C_n as:

$$f(e_j) = \begin{cases} 0 & j = 1,\, 2,\, \ldots,\, \lfloor \frac{n}{2} \rfloor - i + 1; \\ 0 & \lfloor \frac{n}{2} \rfloor - i + 3,\, \lfloor \frac{n}{2} \rfloor - i + 5,\, \ldots,\, \lfloor \frac{n}{2} \rfloor + i - 1 \\ & \\ 1 & i = \lfloor \frac{n}{2} \rfloor - i + 2,\, \lfloor \frac{n}{2} \rfloor - i + 4,\, \ldots,\, \lfloor \frac{n}{2} \rfloor + i; \\ 1 & \lfloor \frac{n}{2} \rfloor + i + 1,\, \lfloor \frac{n}{2} \rfloor + i + 2,\, \ldots,\, n. \end{cases}$$

Thus,

$$\{4i - n : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\} \subseteq FEFI(C_n). \tag{6}$$

From (5) and (6), we get
$FEFI(C_n) = \{4i - n : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$. □

Remark: For a 2-regular graph G, it is known that G is a disjoint union of some cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ for some $k \geq 1$. We shall write $G = C(n_1, n_2, \ldots, n_k)$. It is clear that G is isomorphic to its line graph [20]. So the full friendly index set of G and the full edge-friendly index set of G are the same.

Theorem 9. *Suppose $G = C(n_1, n_2, \ldots, n_k)$. Let $n = \sum_{i=1}^{k} n_i$ and l be the total number of odd cycles among the k cycles. First we let*

$$S = \begin{cases} \{-(n-4),\, -(n-8),\, \ldots,\, n - 4\lfloor \frac{l}{2} \rfloor\} & \text{if } n \text{ is even}, \\ \{-(n-4),\, -(n-8),\, \ldots,\, n - 4\lfloor \frac{l}{2} \rfloor - 2\} & \text{if } n \text{ is odd}. \end{cases}$$

Next we modify S as follows:

1. *Remove $-(n-4)$ from S if k is even and $n_1 = n_2 = \cdots = n_k$.*
2. *Add $-n$ to S if there exists a partition of $\{n_1, n_2, \ldots, n_k\}$ into two subsets X and Y such that $\left|\sum_{n_i \in X} n_i - \sum_{n_j \in Y} n_j\right| \leq 1$.*

Then $FEFI(G) = S$.

Proof. It follows from the above remark and the result of Kwong, Lee and Ng [16]. □

5 Full edge-friendly index set of wheel

A wheel W_n is a graph with n vertices formed by connecting a single vertex to all vertices of an $(n-1)$- cycle i.e. $W_n = K_1 + C_{n-1}$. We will refer the edges of W_n by $u_1, u_2, \ldots, u_{n-1}, u_n, \ldots, u_{2n-1}$(as shown in Figure 1).

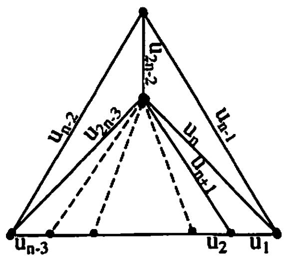

Figure 1: Wheel W_n

Theorem 10. *The full edge-friendly index set of wheel W_n is*

$$FEFI(W_n) = \{4i - n : 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\}.$$

Proof. By Theorem 1 and Theorem 2, we get

$$FEFI(W_n) \subseteq \{4i - n : 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\}. \tag{7}$$

Define the edge labeling g as:

$$g(u_j) = \begin{cases} 0 & j = 1,\ 2,\ n+1,\ n+3,\ n+4,\ \ldots,\ 2n-2, \\ 1 & j = 3,\ 4,\ \ldots,\ n,\ n+2. \end{cases}$$

Clearly g is an edge friendly labeling of the graph W_n and gives $V_g(1) = 0$. Also for $1 \le i \le \lfloor \frac{n}{2} \rfloor$, define the edge labeling f as:

$$f(u_j) = \begin{cases} 1 & i \le j \le n + i - 2, \\ 0 & otherwise. \end{cases}$$

Clearly f is an edge friendly labeling of the graph W_n and gives $V_f(1) = 2i,\ 1 \le i \le \lfloor \frac{n}{2} \rfloor$ i.e.

$$\{4i - n : 0 \le i \le \left\lfloor \frac{n}{2} \right\rfloor\} \subseteq FEFI(W_n). \tag{8}$$

From (7) and (8), we get
$FEFI(W_n) = \{4i - n : 0 \le i \le \lfloor \frac{n}{2} \rfloor\}$. □

6 Full edge-friendly index set of m copies of path

We refer the graph mP_n as disjoint union of m copies of path P_n. The following important corollaries of Theorem 2 come as a ready reference for study of full edge-friendly index set of mP_n.

Corollary 11. *Let f be an edge labeling of $P_n = x_1x_2\dots x_n$. If $E_f(1) > 0$, then $V_f(1) \geq 2$.*

Proof. Let i be the smallest index such that $f(x_ix_{i+1}) = 1$. Then x_i is a 1-vertex. By Theorem 2, there are at least two 1-vertices. □

Corollary 12. *For $n \geq 2$,*
$V_{max}(1) = max\{V_f(1) : f\ is\ an\ edge\ friendly\ labeling\ of\ P_n\}$

$$= \begin{cases} n & n\ is\ even, \\ n-1 & n\ is\ odd. \end{cases}$$

Proof. Let f be an edge friendly labeling of $P_n = x_1x_2\dots x_n$. By Theorem 2, we know that n or $n-1$ is an upper bound of $V_f(1)$ for n is even or odd, respectively. The bound is attained by the following edge labeling g which is defined by

$$g(x_ix_{i+1}) = \begin{cases} 1 & i \text{ is odd}, \\ 0 & i \text{ is even}, \end{cases}$$

where $1 \leq i \leq n-1$. □

Further, we have following Lemmas regarding full edge-friendly index set of mP_n:

Lemma 13. *Let f be an edge friendly labeling of mP_n, where $m \geq 1$ and $n \geq 3$. Then*

$$V_f(1) \geq \begin{cases} m & m\ is\ even, \\ m+1 & m\ is\ odd. \end{cases}$$

Proof. Since f is edge friendly, by pigeon hole principle, there are at least $\lceil \frac{m}{2} \rceil$ paths P_n containing 1-edge. By Theorem 2, $V_f(1) \geq 2\lceil \frac{m}{2} \rceil$. The result follows. □

Following we will use the notation:
$V_{min}(1) = min\{V_f(1) : f$ is an edge friendly labeling of $mP_n\}$ and
$V_{max}(1) = max\{V_f(1) : f$ is an edge friendly labeling of $mP_n\}$.

Corollary 14. *For $m \geq 1$ and $n \geq 3$, we have*

$$V_{min}(1) = \begin{cases} m & m\ is\ even, \\ m+1 & m\ is\ odd. \end{cases}$$

Proof. For $m = 2k$, we label all the edges of the first k paths P_n by 1 and the other by 0. Then the number of 1-vertices is m.
For $m = 2k + 1$, we label all the edges of the first k paths P_n by 1 and those of the next k paths P_n by 0. For the last path, we label the first $\lceil \frac{n}{2} \rceil$ edges of the P_n by 1 and the others by 0. Then the number of 1-vertices is $m + 1$. Combining with Lemma 13, we have the corollary. □

Lemma 15. *For $1 \leq r \leq n - 1$ where $n \geq 2$, there are edge friendly labelings f and f' of P_{2n} with $E_f(1) - E_f(0) = 1$ and $E_{f'}(0) - E_{f'}(1) = 1$ such that $V_f(1) = V_{f'}(1) = 2r$. Moreover, $I_f(P_{2n}) = I_{f'}(P_{2n})$.*

Proof. Let $P_{2n} = x_1 x_2 \ldots x_{2n}$. For $1 \leq r \leq n - 1$, first take an alternating sequence of $1's$ and $0's$ of length $2r$. Add $n - r$ $1's$ adjacent to any 1 of the sequence in all or in parts and $n - r - 1$ $0's$ adjacent to any 0 of the sequence in all or in parts. Call the resultant sequence as $a_1, a_2, \ldots, a_{2n-1}$. Now, define f as $f(x_i x_{i+1}) = a_i$ for $1 \leq i \leq 2n - 1$. Clearly $E_f(1) = n$, $E_f(0) = n - 1$ and $V_f(1) = 2r$.
Similarly again for $1 \leq r \leq n - 1$, first take an alternating sequence of $1's$ and $0's$ of length $2r$. Add $n - 1 - r$ $1's$ adjacent to any 1 of the sequence in all or in parts and $n - r$ $0's$ adjacent to any 0 of the sequence in all or in parts. Call the resultant sequence as $a_1, a_2, \ldots, a_{2n-1}$. Now, define f' as $f'(x_i x_{i+1}) = a_i$ for $1 \leq i \leq 2n - 1$. Clearly $E_{f'}(1) = n - 1$, $E_{f'}(0) = n$ and $V_{f'}(1) = 2r$.
Thus, for $1 \leq r \leq n - 1$ where $n \geq 2$, there exists edge friendly labelings f and f' of P_{2n} with $E_f(1) - E_f(0) = 1$ and $E_{f'}(0) - E_{f'}(1) = 1$ such that $V_f(1) = V_{f'}(1) = 2r$. Moreover, $I_f(P_{2n}) = I_{f'}(P_{2n})$. □

Lemma 16. *For $m \geq 1$ and even $n \geq 4$,*

$$V_{max}(1) = \begin{cases} m(n-1) & m \text{ is even}, \\ m(n-1)+1 & m \text{ is odd}. \end{cases}$$

Proof. Note that mP_n contains totally $m(n-1)$ edges. Let f be an edge friendly labeling of mP_n. Each 1-edge induces at most two 1-vertices. There are at most $\lceil \frac{m(n-1)}{2} \rceil$ 1-edges. Thus, f induces at most $2\lceil \frac{m(n-1)}{2} \rceil$ 1-vertices. That is,

$$V_f(1) \leq \begin{cases} m(n-1) & m \text{ is even}, \\ m(n-1)+1 & m \text{ is odd}. \end{cases}$$

The bound is attained by the following labeling. Let g be the edge labeling defined in the proof of Corollary 12. By Lemma 15, there exists edge friendly labeling h of P_n such that $E_h(0) - E_h(1) = 1$ and $V_h(1) = n - 2$. Let $x_{i,j}$ denote the i-th vertex of the j-th path in mP_n. Define the edge labeling f of mP_n by

$$f(x_{i,j} x_{i+1,j}) = \begin{cases} g(x_i x_{i+1}) & 1 \leq i \leq n-1, 1 \leq j \leq \lceil \frac{m}{2} \rceil, \\ h(x_i x_{i+1}) & 1 \leq i \leq n-1, \lceil \frac{m}{2} \rceil + 1 \leq j \leq m. \end{cases}$$

Clearly f is friendly and

$$V_f(1) \le \begin{cases} m(n-1) & m \text{ is even}, \\ m(n-1)+1 & m \text{ is odd}. \end{cases}$$

Hence the lemma follows. □

Lemma 17. *For $m \geq 1$ and odd $n \geq 3$, $V_{max}(1) = m(n-1)$.*

Proof. From the proof of Lemma 16, we know that $m(n-1)$ is an upper bound for the maximum number of 1-vertices. The bound is attained by labeling all P_n's by g which is defined in Corollary 12. □

The *sum* of two number sets A and B is defined by $A+B = \{a+b : a \in A, b \in B\}$. The *difference* of two number sets A and B is defined by $A \setminus B = \{a : a \in A$ and $a \notin B\}$. We have the following lemma.

Lemma 18. *If P_{n_1} and P_{n_2} are two paths, then*

$$\{FEFI(P_{n_1}) \setminus \{x\}\} + FEFI(P_{n_2}) \subseteq FEFI(P_{n_1} \cup P_{n_2})$$

where $x = \max FEFI(P_{n_1})$

Proof. Let $X = \{FEFI(P_{n_1}) \setminus \{x\}\} + FEFI(P_{n_2})$. If $s \in X$, then $s = a + b$ such that $a \in FEFI(P_{n_1})$, $b \in FEFI(P_{n_2})$ and $a \neq x$. Now, there arises four cases:

Case 1: *Let both n_1 and n_2 be odd.* There exist edge friendly labelings f and f' of P_{n_1} and P_{n_2} respectively such that $E_f(1) - E_f(0) = 0$, $V_f(1) - V_f(0) = a$ and $E_{f'}(1) - E_{f'}(0) = 0$, $V_{f'}(1) - V_{f'}(0) = b$.

Define an edge labeling g of $P_{n_1} \cup P_{n_2}$ by $g(e) = \begin{cases} f(e) & e \in E(P_{n_1}), \\ f'(e) & e \in E(P_{n_2}). \end{cases}$

Clearly, $E_g(1) = E_f(1) + E_{f'}(1)$ and $E_g(0) = E_f(0) + E_{f'}(0)$ *i.e.* $E_g(1) - E_g(0) = 0$. Thus, g is an edge friendly labeling of $P_{n_1} \cup P_{n_2}$. Also, $V_g(1) = V_f(1) + V_{f'}(1)$, $V_g(0) = V_f(0) + V_{f'}(0)$. This implies that $V_g(1) - V_g(0) = a+b$. Thus, in this case, for every $a \in FEFI(P_{n_1})$, $b \in FEFI(P_{n_2})$ and $a \neq x$, there exist an edge labeling g such that $a + b \in FEFI(P_{n_1} \cup P_{n_2})$.

Case 2: *Let n_1 be even and n_2 be odd.* There exist edge friendly labelings f and f' of P_{n_1} and P_{n_2} respectively such that $E_f(1) - E_f(0) = 0$, $V_f(1) - V_f(0) = a$ and $E_{f'}(1) - E_{f'}(0) = 1$, $V_{f'}(1) - V_{f'}(0) = b$.

Define an edge labeling g of $P_{n_1} \cup P_{n_2}$ by $g(e) = \begin{cases} f(e) & e \in E(P_{n_1}), \\ f'(e) & e \in E(P_{n_2}). \end{cases}$

Clearly, $E_g(1) = E_f(1) + E_{f'}(1)$ and $E_g(0) = E_f(0) + E_{f'}(0)$ *i.e.* $E_g(1) - E_g(0) = 1$. Thus, g is an edge friendly labeling of $P_{n_1} \cup P_{n_2}$. Also, $V_g(1) - V_g(0) = a + b$. Thus, in this case, for every $a \in FEFI(P_{n_1})$, $b \in FEFI(P_{n_2})$ and $a \neq x$, there exist an edge labeling g such that $a + b \in FEFI(P_{n_1} \cup P_{n_2})$.

Case 3: *Let n_1 be odd and n_2 be even.* The proof is similar to case 2.

Case 4: *Let both n_1 and n_2 be even.* Let f' be an edge friendly labeling of P_{n_2} such that $V_{f'}(1) - V_{f'}(0) = b$. For the case $E_{f'}(0) - E_{f'}(1) = 1$, by Lemma 15 there exist an edge friendly labeling f of P_{n_1} such that $E_f(1) - E_f(0) = 1$ and $V_f(1) - V_f(0) = a$. And for the case $E_{f'}(1) - E_{f'}(0) = 1$, by Lemma 15 there exists edge friendly labeling f of P_{n_1} such that $E_f(0) - E_f(1) = 1$ and $V_f(1) - V_f(0) = a$.

Define an edge labeling g of $P_{n_1} \cup P_{n_2}$ by $g(e) = \begin{cases} f(e) & e \in P_{n_1}, \\ f'(e) & e \in P_{n_2}. \end{cases}$

Clearly, $E_g(1) = E_f(1) + E_{f'}(1)$ and $E_g(0) = E_f(0) + E_{f'}(0)$ *i.e.* $E_g(1) - E_g(0) = E_f(1) - E_f(0) + E_{f'}(1) - E_{f'}(0) = 0$. Thus, g is an edge friendly labeling of $P_{n_1} \cup P_{n_2}$. Also, $V_g(1) - V_g(0) = a + b$. Thus, in this case, for every $a \in FEFI(P_{n_1})$, $b \in FEFI(P_{n_2})$ and $a \neq x$, there exist an edge labeling g such that $a + b \in FEFI(P_{n_1} \cup P_{n_2})$. □

This Lemma can be easily generalized as:

$\lceil \frac{m}{2} \rceil \{FEFI(P_n) \setminus max(FEFI(P_n)\} + \lfloor \frac{m}{2} \rfloor FEFI(P_n) \subseteq FEFI(mP_n)$

First we will evaluate the full edge-friendly index set of $P_n = x_1, x_2, \ldots, x_n$ for $n \geq 3$. We let $e_j = x_j x_{j+1}$ for $1 \leq j \leq n-1$.

Theorem 19. *The full edge-friendly index set of path P_n, $n \geq 3$ is*

$$FEFI(P_n) = \{4i - n : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}.$$

Proof. By Theorem 1, Theorem 2, Corollary 12, Corollary 14, we get

$$FEFI(P_n) \subseteq \{4i - n : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}. \quad (9)$$

For a fixed i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, define the edge friendly labeling f of P_{2n+1} as:

$$f(e_j) = \begin{cases} 0 & j = 1, 2, \ldots, n-i-1, n-i+1, n-i+3, \ldots, n+i-1, \\ 1 & j = n-i+2, n-i+4, \ldots, n+i-2, n+i, n+i+1, \ldots, 2n. \end{cases}$$

Also, for a fixed i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, define the edge friendly labeling g of P_{2n} as:

$$g(e_j) = \begin{cases} 0 & j = 2, 3, \ldots, n-i+2, n-i+4, n-i+6, \ldots, n+i-2, \\ 1 & j = 1, n-i+3, n-i+5, \ldots, n+i-3, n+i-1, n+i, \ldots, 2n-1. \end{cases}$$

Thus,

$$\{4i - n : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\} \subseteq FEFI(P_n). \quad (10)$$

From (9) and (10), we get

$$FEFI(P_n) = \{4i - n : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\}.$$

□

Evaluation of full edge-friendly index set of mP_n

There arises two cases:

Case 1: *Let n be even.* By Theorem 1 and Theorem 2, we get

$FEFI(mP_n) \subseteq \{4i - mn : 0 \leq i \leq \frac{mn}{2}\}$.

Subcase 1.1: *Let m be even.* By Lemma 16 and Corollary 14, we get

$$FEFI(mP_n) \subseteq \{4i - mn : \frac{m}{2} \leq i \leq \frac{mn - m}{2}\}. \quad (11)$$

We have, $FEFI(P_n) = \{4i - n : 1 \leq i \leq \frac{n}{2}\} = \{4 - n, 8 - n, \ldots, n\}$. Using Lemma 18, we get

$\{4-n, 8-n, \ldots, n-4\} + \cdots + \{4-n, 8-n, \ldots, n-4\}(\frac{m}{2} \text{ times}) + \{4-n, 8-n, \ldots, n-4, n\} + \cdots + \{4-n, 8-n, \ldots, n-4, n\}(\frac{m}{2} \text{ times}) \subseteq FEFI(mP_n)$

which implies that $\{4m - mn, 4(m+1) - mn, \ldots, mn - 2m\} \subseteq FEFI(mP_n)$.

Let $A = [e_{jk}]^0_{m \times (n-1)}$ be the initial matrix representation of an edge labeling of mP_n where, e_{jk} is label of the k-th edge of the j-th path of mP_n.

$$[e_{jk}]^0 = \begin{array}{c} \\ P_1 \\ P_2 \\ \cdots \\ P_{m-1} \\ P_m \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_{n-1} \end{array} \\ \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 \end{pmatrix} \end{array}$$

Clearly, $[e_{jk}]^0$ defines an edge friendly labeling f of mP_n. Clearly $V_f(1)$ is m. In the first iteration $[e_{jk}]^1$, we interchange $e_{1(n-1)}$ with $e_{2(n-1)}$ and keep the remaining entries fixed. In the second iteration $[e_{jk}]^2$, we interchange the elements $e_{3(n-1)}$ with $e_{4(n-1)}$ in the matrix obtained $[e_{jk}]^1$ and keep the remaining entries fixed. In general, in x-th iteration $[e_{jk}]^x$ we interchange the elements $e_{(2x-1)(n-1)}$ with $e_{(2x)(n-1)}$ in the matrix obtained $[e_{jk}]^{x-1}$ and keep the remaining entries fixed. We repeat this procedure $\frac{m}{2}$ times and each time we obtain edge friendly labeling in $[e_{jk}]^0$, $[e_{jk}]^1$, $\ldots$, $[e_{jk}]^{\frac{m}{2}}$ which gives the value of $V_f(1)$ as m, $m+2$, $\ldots$, $2m$ respectively. Hence,

$$\{4i - mn : \frac{m}{2} \leq i \leq \frac{mn-m}{2}\} \subseteq FEFI(mP_n). \tag{12}$$

From (11) and (12), we get

$$FEFI(mP_n) = \{4i - mn : \frac{m}{2} \leq i \leq \frac{mn-m}{2}\}, \ m, \ n \text{ is even.}$$

Subcase 1.2 *Let m be odd.* By Lemma 16 and Corollary 14, we get

$$FEFI(mP_n) \subseteq \{4i - mn : \frac{m+1}{2} \leq i \leq \frac{mn-m+1}{2}\}. \tag{13}$$

Now, $FEFI(P_n) = \{4i - n : 1 \leq i \leq \frac{n}{2}\} = \{4-n, \ 8-n, \ \ldots, \ n\}$. Thus, using Lemma 18, we get

$\{4-n, \ 8-n, \ \ldots, \ n-4\} + \cdots + \{4-n, \ 8-n, \ \ldots, \ n-4\}(\frac{m-1}{2} \text{ times}) + \{4-\text{n}, \ 8-\text{n}, \ \ldots, \ \text{n}-4, \ \text{n}\} + \cdots + \{4-\text{n}, \ 8-\text{n}, \ \ldots, \ \text{n}-4, \ \text{n}\}(\frac{\text{m}+1}{2} \text{ times}) \subseteq \text{FEFI}(\text{mP}_\text{n})$.

This implies that $\{4m-mn, \ 4(m+1)-mn, \ \ldots, \ mn-2m+2\} \subseteq FEFI(mP_n)$.

Let $A = [e_{jk}]^0_{m\times(n-1)}$ be the initial matrix representation of an edge labeling of mP_n where, e_{jk} is fixed label of the k-th edge of the j-th path of mP_n.

$$[e_{jk}]^0 = \begin{array}{c} \\ P_1 \\ P_2 \\ \cdots \\ P_{m-2} \\ P_{m-1} \\ P_m \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & \cdots & e_{n-2} & e_{n-1} \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & m\ldots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \end{array}$$

Clearly, $[e_{jk}]^0$ defines an edge friendly labeling f of mP_n. Clearly, $V_f(1)$ is $m+1$. In the first iteration $[e_{jk}]^1$, we interchange $e_{1(n-1)}$ with $e_{2(n-1)}$ and keep the remaining entries fixed. In the second iteration $[e_{jk}]^2$, we interchange the elements $e_{3(n-1)}$ with $e_{4(n-1)}$ in the matrix obtained $[e_{jk}]^1$ and keep the remaining entries fixed. In general, in x-th iteration $[e_{jk}]^x$ we interchange the elements $e_{(2x-1)(n-1)}$ with $e_{(2x)(n-1)}$ in the matrix obtained $[e_{jk}]^{x-1}$ and keep the remaining entries fixed. We repeat this procedure $\frac{m-1}{2}$ times and each time we obtain edge friendly labeling in $[e_{jk}]^0, [e_{jk}]^1, \ldots, [e_{jk}]^{\frac{m-1}{2}}$ which gives the value of $V_f(1)$ as $m+1, m+3, \ldots, 2m$ respectively. Hence,

$$\{4i - mn : \frac{m+1}{2} \leq i \leq \frac{mn-m+1}{2}\} \subseteq FEFI(mP_n). \quad (14)$$

From (13) and (14), when m is odd and n is even, we get

$$FEFI(mP_n) = \{4i - mn : \frac{m+1}{2} \leq i \leq \frac{mn-m+1}{2}\}.$$

Case 2: *Let n be odd.* By Theorem 1 and Theorem 2, we get $FEFI(mP_n) \subseteq \{4i - mn : 0 \leq i \leq \frac{mn}{2}\}$.

By Lemma 17 and Corollary 14, we get

$$FEFI(mP_n) \subseteq \{4i - mn : z \leq i \leq \frac{m(n-1)}{2}\}. \quad (15)$$

where, $z = \begin{cases} \frac{m}{2} & m \text{ is even,} \\ \frac{m+1}{2} & m \text{ is odd.} \end{cases}$

Now, $FEFI(P_n) = \{4i - n : 1 \leq i \leq \frac{n}{2}\} = \{4-n, 8-n, \ldots, n\}$. Thus, using Lemma 18, we get

$$\{4-n,\ 8-n,\ \ldots,\ n\} + \cdots + \{4-n,\ 8-n,\ \ldots,\ n\} (m\, times) \subseteq FEFI(mP_n).$$

This implies that $\{4m - mn,\ 4(m+1) - mn,\ \ldots,\ mn\} \subseteq FEFI(mP_n)$.

Let $A = [e_{jk}]^0_{m \times (n-1)}$ be the initial matrix representation of an edge labeling of mP_n where, e_{jk} is fixed label of the k-th edge of the j-th path of mP_n.

Subcase 2.1: *Let m be even.*
Let

$$[e_{jk}]^0 = \begin{array}{c} \\ P_1 \\ P_2 \\ \ldots \\ P_{m-1} \\ P_m \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \ldots & e_{n-1} \end{array} \\ \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 \end{pmatrix} \end{array}$$

Clearly, $[e_{jk}]^0$ defines an edge friendly labeling f of mP_n. Clearly, $V_f(1)$ is m. In the first iteration $[e_{jk}]^1$, we interchange $e_{1(n-1)}$ with $e_{2(n-1)}$ and keep the remaining entries fixed. In the second iteration $[e_{jk}]^2$, we interchange the elements $e_{3(n-1)}$ with $e_{4(n-1)}$ in the matrix obtained $[e_{jk}]^1$ and keep the remaining entries fixed. In general, in x-th iteration $[e_{jk}]^x$ we interchange the elements $e_{(2x-1)(n-1)}$ with $e_{(2x)(n-1)}$ in the matrix obtained $[e_{jk}]^{x-1}$ and keep the remaining entries fixed. We repeat this procedure $\frac{m}{2}$ times and each time we obtain edge friendly labeling in $[e_{jk}]^0$, $[e_{jk}]^1$, $\ldots$, $[e_{jk}]^{\frac{m}{2}}$ which gives the value of $V_f(1)$ as m, $m+2$, $\ldots$, $2m$ respectively. Hence,

Subcase 2.2: *Let m be odd.*
Let

$$[e_{jk}]^0 = \begin{array}{c} \\ P_1 \\ P_2 \\ \cdots \\ P_{m-2} \\ P_{m-1} \\ P_m \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & \cdots & e_{n-2} & e_{n-1} \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \end{array}$$

Clearly, $[e_{jk}]^0$ defines an edge friendly labeling f of mP_n. Clearly, $V_f(1)$ is $m+1$. In the first iteration $[e_{jk}]^1$, we interchange $e_{1(n-1)}$ with $e_{2(n-1)}$ and keep the remaining entries fixed. In the second iteration $[e_{jk}]^2$, we interchange the elements $e_{3(n-1)}$ with $e_{4(n-1)}$ in the matrix obtained $[e_{jk}]^1$ and keep the remaining entries fixed. In general, in x-th iteration $[e_{jk}]^x$ we interchange the elements $e_{(2x-1)(n-1)}$ with $e_{(2x)(n-1)}$ in the matrix obtained $[e_{jk}]^{x-1}$ and keep the remaining entries fixed. We repeat this procedure $\frac{m-1}{2}$ times and each time we obtain edge friendly labeling in $[e_{jk}]^0$, $[e_{jk}]^1$, $\ldots$, $[e_{jk}]^{\frac{m}{2}}$ which gives the value of $V_f(1)$ as $m+1$, $m+3$, $\ldots$, $2m$ respectively. Hence,

$$\{4i - mn : z \leq i \leq \frac{m(n-1)}{2}\} \subseteq FEFI(mP_n), \ n \text{ is even.} \tag{16}$$

where, $z = \begin{cases} \frac{m}{2} & m \text{ is even,} \\ \frac{m+1}{2} & m \text{ is odd.} \end{cases}$

From (15) and (16), we get

$FEFI(mP_n) = \{4i - mn : \frac{m}{2} \leq i \leq \frac{m(n-1)}{2}\}$, m is even and n is odd.

$FEFI(mP_n) = \{4i - mn : \frac{m+1}{2} \leq i \leq \frac{m(n-1)}{2}\}$, m, n is odd.

Thus,
$FEFI(mP_n) = \{4i - mn : \frac{m}{2} \leq i \leq \frac{mn-m}{2}\}$, m, n is even.
$FEFI(mP_n) = \{4i - mn : \frac{m+1}{2} \leq i \leq \frac{mn-m+1}{2}\}$, m is odd, n is even.
$FEFI(mP_n) = \{4i - mn : \frac{m}{2} \leq i \leq \frac{m(n-1)}{2}\}$, m is even and n is odd.

$FEFI(mP_n) = \{44i - mn : \frac{m+1}{2} \leq i \leq \frac{m(n-1)}{2}\}$, m, n is odd.

□

7 A comparative study of full edge-friendly index set and edge friendly index set of some graphs

Graph	Configuration	$\varphi(G)$	$\varphi'(G)$	$\phi(G)$	$\phi'(G)$
$K_{1,n}$	$n \equiv 0 \pmod 4$	-1	-1	1	1
$K_{1,n}$	$n \equiv 1 \pmod 4$	2	-2	2	2
$K_{1,n}$	$n \equiv 2 \pmod 4$	1	1	1	1
$K_{1,n}$	$n \equiv 3 \pmod 4$	0	0	0	0
C_n	n be even	n	$4-n$	n	0
C_n	n be odd	$n-2$	$4-n$	$n-2$	1
W_n	n be even	$2n-2$	$2-2n$	$2n-2$	2
W_n	n be odd	$2n-2$	$2-2n$	$2n-2$	0
P_n	n be even	n	$4-n$	n	0
P_n	n be odd	$n-2$	$4-n$	$n-2$	1

8 Conclusion

In this paper, we have worked out full edge-friendly index sets of the star $K_{1,n}$, 2-regular graph, wheel W_n, m copies of path mP_n, $m \geq 1$. Further in literature A-cordial labeling exists which gives insight to find A-friendly and A-edge friendly labelings of graphs.

9 Acknowledgement

The authors express their gratitude to B.D. Acharya and M. Acharya for encouraging them to obtain more results on edge friendly labeling as also for their constant readiness for discussions, which almost invariably yielded deeper insight. We are thankful to S.M. Hegde who visited Centre for Mathematical Sciences, Bansthali University for intensive interaction in graph labelings. Also, we are deeply indebted to the refree who made extensive and constructively critical comments on the previous versions of the paper.

References

[1] M. Benson and S.M. Lee, *On cordialness of regular windmill graphs*, Congr. Numer., **68**(1989), 49-58.

[2] J.A. Bondy and U.S.R. Murty, *Graph theory and applications*, New York: Macmillan Ltd. Press (1976).

[3] I. Cahit, *Cordial graphs: a weaker version of graceful and harmonious graphs*, Ars Combin., **23**(1987), 201-207.

[4] I. Cahit, *Recent results and open problems on cordial graphs*, Contemporary Methods in Graph Theory, Bibligraphisches Inst. Mannhiem (1990), 209-230.

[5] I. Cahit, *On cordial and 3-equitable graphs*, Util. Math., **37**(1990), 189-198.

[6] N. Cairnie and K. Edwards, *The computational complexity of cordial and equitable labellings*, Discrete Math., **216**(2000), 29-34.

[7] G. Chartrand, S.M. Lee and P. Zhang, *Uniformly cordial graphs*, Discrete Math., **306**(2006), 726-737.

[8] Gary Chartrand and Ping Zhang, *Introduction to graph theory*, Tata McGraw-Hill Ltd, 2004.

[9] Karen L. Collins and Mark Hovey, *Most graphs are edge cordial*, Ars Combin., **30**(1990), 289-295.

[10] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin., **18**(2011), #DS6, 1-256.

[11] Y.S. Ho, S.M. Lee, and S.C. Shee, *Cordial Labellings of the Cartesian Product and Composition of Graphs*, Ars Combin., **29**(1990), 169-180.

[12] M. Hovay, *A-cordial graphs*, Discrete Math., **93**(1991), 183-194.

[13] W.W. Kirchherr, *On the Cordiality of Certain Specific Graphs*, Ars Combin., **31**(1991), 127-138.

[14] S. Kuo, G.J. Chang and Y.H.H. Kwong, *Cordial Labeling of* mK_n, Discrete Math., **169**(1997), 121-131.

[15] H. Kwong and S.M. Lee, *On friendly index sets of generalized books*, JCMCC, **66**(2008), 43-58.

[16] H. Kwong, S.M. Lee and H.K. Ng, *On friendly index set of 2-regular graphs*, Discrete Math., **308**(2008), 5522-5532.

[17] H. Kwong, S.M. Lee and Y-C. Wang, *On friendly index sets of (p, p + 1)-graphs*, JCMCC, **78**(2011), 3-14.

[18] S.M. Lee and A. Liu, *A Construction of Cordial Graphs from Smaller Cordial Graphs*, Ars Combin., **32**(1991), 209-214.

[19] S.M. Lee and H.K. Ng, *On friendly index sets of bipartite graphs*, Ars Combin., **86**(2008), 257-271.

[20] V.V. Menon, *On repeated interchange graphs*, Amer. Math. Monthly, **13**(1966), 986-989.

[21] H.K. Ng and S.M. Lee, *A conjecture on edge cordial trees*, Abstracts Amer. Math. Soc., **9**(1988), 286-287.

[22] E. Salehi and S. De, *On a conjecture concerning the friendly index sets of trees*, Ars Combin., **90**(2009), 371-381.

[23] E. Salehi and S.M. Lee, *On friendly index sets of trees*, Congr. Numer., **178**(2006), 173-183.

[24] M.A. Seoud and A.E.I. Abdel Ma.qsoud, *On Cordial and Balanced Labelings of Graphs*, J. Egyptian Math. Soc., **1**(1999), 127-135.

[25] W.C. Shiu and H. Kwong, *Full friendly index sets of* $P_2 \times P_n$, Discrete Math., **308**(2007), 3688-3693.

[26] W.C. Shiu and H. Kwong, *Product-cordial index and friendly index of regular graphs*, Transactions on Combinatorics, **1**(2012), 15-20.

[27] W.C. Shiu and M.H. Ling, *Extreme friendly indices of* $C_m \times C_n$, Congr. Numer., **188**(2007), 175-182.

[28] W.C. Shiu and M.H. Ling, *Full friendly index sets of Cartesian products of two cycles*, Acta Math. Sinica, **26**(2010), 1233-1244.

[29] W.C. Shiu and F.S. Wong, *Extreme friendly indices of* $C_m \times P_n$, Congr. Numer., **197**(2009), 65-75.

[30] W.C. Shiu and F.S. Wong, *Full friendly index sets of cylinder graphs*, Australas. J. Combin.,**52**(2012), 141-162.

[31] D. Sinha and J. Kaur, *Full Friendly Index Set - I*,(presented at the Jubilee Conference on Discrete Mathematics, Banasthali University, Banasthali, Jan 11-13, 2009), to appear in Discrete Appl. Math.

[32] D. Sinha and J. Kaur, *Full friendly index set - II*, (presented at ICMCS-2010, Loyala College, Chennai, Feb 5-6, 2010), JCMCC, **79**(2011), 65-75.

****