ON THE SEQUENTIAL NUMBER AND SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

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ABSTRACT. A graph G is called edge-magic if there exists a bijective function $f: V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$. Also, G is called super edge-magic if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Moreover, the super edge-magic deficiency, $\mu_s(G)$, of a graph G is defined to be the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. In this paper, we introduce the notion of the sequential number, $\sigma(G)$, of a graph G without isolated vertices to be either the smallest positive integer n for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, \ldots, n\}$ in such a way that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels are $|E(G)|$ consecutive integers or $+\infty$ if there exists no such integer n. We prove that $\sigma(G) = \mu_s(G) + |V(G)| - 1$ for any graph G without isolated vertices, and $\sigma(K_{m,n}) = mn$ for every two positive integers m and n, which allows us to settle the conjecture that $\mu_s(K_{m,n}) = (m-1)(n-1)$ for every two positive integers m and n.

1. INTRODUCTION

To formalize this presentation, we introduce some necessary definitions and refer the reader to Chartrand and Lesniak [1] for all other graph theory terminology and notation not provided in this paper.

We let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. For the sake of brevity, we will denote $[a, b] \cap \mathbb{Z}$ by simply writing $[a, b]$, where $\mathbb{Z}$ denotes the set of integers.

In 1970, Kotzig and Rosa [11] initiated the study of magic valuations. These labelings are currently referred to as either edge-magic labelings or edge-magic total labelings; these terms were coined by Ringel and Lladó [13], and Wallis [15], respectively. In this paper, we will use the former for the sake of brevity. A graph G of order p and size q is called *edge-magic* if

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there exists a bijective function $f: V(G) \cup E(G) \to [1, p+q]$ such that $f(u) + f(v) + f(uv)$ is a constant (called the *valence*) for each $uv \in E(G)$. Such a function is called an *edge-magic labeling.*

In 1998, Enomoto et al. [2] defined an edge-magic labeling f of a graph G of order p to be a *super edge-magic labeling* if f has the additional property that $f(V(G)) = [1, p]$ (an alternative term exists for this kind of labeling, namely, strongly edge-magic labeling; see Wallis [15]). Thus, a graph that admits a super edge-magic labeling is called *super edge-magic.*

The following lemma from [3] gives us a necessary and sufficient condition for a graph to be super edge-magic, which is useful in this paper.

Lemma 1. *A graph G of order p and size q is super edge-magic if and only if there exists a bijective function $f: V(G) \to [1, p]$ such that the set*

$$S = \{f(u) + f(v) \,|uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min(S)$ and

$$S = [k - (p+q), k - (p+1)].$$

For every graph G, Kotzig and Rosa [11] proved that there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This motivated them to define the *edge-magic deficiency,* $\mu(G)$, of a graph G to be the smallest nonnegative integer n for which the graph $G \cup nK_1$ is edge-magic. Figueroa-Centeno et al. [5] analogously defined the notion of the *super edge-magic deficiency,* $\mu_s(G)$, of a graph G to be either the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. Thus, the super edge-magic deficiency of a graph G is a measure of how close G is to being super edge-magic.

Figueroa-Centeno et al. [5] showed that the super edge-magic deficiency of the complete bipartite graph satisfies that $\mu_s(K_{m,n}) \leq (m-1)(n-1)$ for every two positive integers m and n. They also conjectured that for all positive integers m and n, $\mu_s(K_{m,n}) = (m-1)(n-1)$, which they proved, with considerable effort, for $m = 2$. Furthermore, Hegde et al. [10] proved it to be true for $m = 3$, 4 and 5. In this paper, we prove this conjecture in general.

For further knowledge on the super edge magic deficiency of graphs, the authors suggest that the reader consult the results in [4, 9, 12, 14].

2. The Sequential Number and Super Edge-Magic Deficiency

In this section, we introduce the notion of the sequential number, $\sigma(G)$, of a graph G of order p without isolated vertices, and prove that $\sigma(G) = \mu_s(G) + p - 1$. To do this, we start with some definitions.

Harmonious graphs were first studied by Graham and Sloane [8]. A graph G of order p and size q with $q \geq p$ is *harmonious* if there exists an injective function $f : V(G) \to \mathbb{Z}_q$ such that each $uv \in E(G)$ is labeled $f(u) + f(v) \pmod q$ and the resulting edge labels are distinct. Such a function is called a *harmonious labeling.* If G is a tree (so that $q = p - 1$) exactly two vertices are labeled the same; otherwise, the definition is the same.

The notion of sequential graphs was introduced by Grace [6] who was inspired by the above definition of harmonious graphs. He defined a graph G of size q to be *sequential* if there exists an injective function $f : V(G) \to [0, q-1]$ (with the label q allowed if G is a tree) such that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting set of edge labels is $[m, m+q-1]$ for some positive integer m. Such a function is called a *sequential labeling.* Thus, every sequential graph is also harmonious. However, note that a harmonious labeling is not necessarily a sequential labeling. The existence of harmonious graphs admitting no sequential labeling is an open question.

We now provide the definition for the key concept to be discussed in this paper.

The *sequential number*, $\sigma(G)$, of a graph G of size q without isolated vertices is defined to be either the smallest positive integer n for which it is possible to label the vertices of G with distinct elements from the set $[0, n]$ in such a way that each $uv \in E(G)$ is labeled $f(u) + f(v)$ and the resulting edge labels are q consecutive integers or $+\infty$ if there exists no such integer n.

If a graph G of size q that is not a tree and has no isolated vertices satisfying $\sigma(G) \leq q - 1$, then G is sequential or, equivalently, if such a graph G is not sequential, then $\sigma(G) \geq q$.

Thus, the sequential number of a graph G is a measure of how close G is to being sequential. Moreover, for any graph G of order p without isolated vertices, it is immediate that $\sigma(G) \geq p - 1$ and the bound is sharp (for example, $\sigma(K_{1,n}) = n$).

With the above definition in hand, we have the following theorem.

Theorem 1. *If G is a graph of order p without isolated vertices, then*

$$\sigma(G) = \mu_s(G) + p - 1.$$

Proof. Let G be a graph of order p and size q without isolated vertices, and assume that $\sigma(G) = n \geq p - 1$ for some positive integer n. Then there exists an injective function $f : V(G) \to [0, n]$ such that

$$\{f(u) + f(v) \,|uv \in E(G)\} = [m, m+q-1]$$

for some positive integer m. Let

$$L = [0, n] - f(V(G)) = \{l_i | i \in [1, n-p+1]\}.$$

Then $|L| = n - p + 1$. If we let $H \cong G \cup (n-p+1) K_1$, and define the graph H with

$$V(H) = V(G) \cup \{w_i | i \in [1, n-p+1]\}$$

and $E(H) = E(G)$, then the bijective function $g : V(H) \to [1, n+1]$ such that

$$g(v) = \begin{cases} f(v) + 1 & \text{if } v \in V(G) \\ l_i + 1 & \text{if } v = w_i \text{ for some } i \in [1, n-p+1] \end{cases}$$

extends to a super edge-magic labeling with valence $m + n + q + 3$, since $|V(H)| = n + 1$, $|E(H)| = q$ and $\min\{f(u) + f(v) | uv \in E(H)\} = m + 2$. Therefore, $\mu_s(G) \leq n - p + 1$, implying that $\sigma(G) \geq \mu_s(G) + p - 1$. This indicates that if G is a graph such that $\sigma(G) < +\infty$, then $\mu_s(G) < +\infty$.

To show that $\sigma(G) \leq \mu_s(G) + p - 1$, let $H \cong G \cup nK_1$, where $n = \mu_s(G)$ for some nonnegative integer n. It follows from Lemma 1 that there exists a super edge-magic labeling $f : V(H) \to [1, n+p]$ with valence $p + q + s$ such that

$$\{f(u) + f(v) | uv \in E(H)\} = [s, s+q-1],$$

where $s = \min\{f(u) + f(v) | uv \in E(H)\}$. Now, define the bijective function $g : V(H) \to [0, n+p-1]$ such that $g(v) = f(v) - 1$ for all $v \in V(H)$. If we consider the restriction of g to $V(G)$, then we obtain

$$\{g(u) + g(v) | uv \in E(G)\} = [s-2, s+q-3], \text{ which}$$

is a set of q consecutive integers, and

$$\max\{g(v) | v \in V(G)\} \leq |V(H)| - 1 = n + p - 1.$$

Therefore, $\sigma(G) \leq n + p - 1$, implying that $\sigma(G) \leq \mu_s(G) + p - 1$. This indicates that if G is a graph such that $\mu_s(G) < +\infty$, then $\sigma(G) < +\infty$. □

From the above theorem, it follows that the problems of determining the sequential number and the super edge-magic deficiency are equivalent. It is also immediate from the same result that for any graph G, $\sigma(G) = +\infty$ if and only if $\mu_s(G) = +\infty$.

We conclude this section with three corollaries obtained from Theorem 1.

The contrapositive of the following corollary provides us with a sufficient condition for a graph to be super edge-magic.

Corollary 1. *If G is a graph of order p without isolated vertices that satisfies $\sigma(G) \geq p$, then G is not super edge-magic.*

The following corollary excludes certain graphs from the class of sequential graphs.

Corollary 2. *If G is a graph of order p and size q without isolated vertices that satisfies $\mu_s(G) \geq q - p + 1$, then G is not sequential.*

Figueroa-Centeno et al. [3] proved that every tree with an α-valuation is super edge-magic (see [7] for the definition and significance of α-valuations). Combining this fact with Theorem 1, we obtain the following corollary.

Corollary 3. *If T is a tree of order p with an α-valuation, then*

$$\sigma(T) = p - 1.$$

3. Complete Bipartite Graphs

Graham and Sloane [8] proved the following theorem.

Theorem 2. *The complete bipartite graph $K_{m,n}$ is harmonious if and only if $m = 1$ or $n = 1$.*

The above theorem allows us to compute the sequential numbers of all complete bipartite graphs.

Theorem 3. *For every two positive integers m and n,*

$$\sigma(K_{m,n}) = mn.$$

Proof. First, note that for every positive integer n, the star $K_{1,n}$ is clearly sequential by labeling the central vertex with 0 and the remaining vertices with 1 through n, implying that $\sigma(K_{1,n}) \leq n$; however, $\sigma(K_{1,n}) \geq n$, since $K_{1,n}$ has no isolated vertices. Thus, $\sigma(K_{1,n}) = n$.

Next, notice that for every two integers m and n with $m \geq 2$ and $n \geq 2$, the complete bipartite graph $K_{m,n}$ is not sequential (for otherwise it would be harmonious which would contradict Theorem 2), is not a tree and has no isolated vertices. Therefore, $\sigma(K_{m,n}) \geq mn$.

To show that $\sigma(K_{m,n}) \leq mn$ for every two integers m and n with $m \geq 2$ and $n \geq 2$, let $X = \{x_i | i \in [1, m]\}$ and $Y = \{y_j | j \in [1, n]\}$ be the partite sets of $K_{m,n}$, and define the injective function $f : V(K_{m,n}) \to [0, mn]$ such that

$$f(x_i) = (i-1)n \text{ for each } i \in [1, m]$$

and

$$f(y_j) = (m-1)n + j \text{ for each } j \in [1, n].$$

Then

$$\{f(x_i) | i \in [1, m]\} = \{0, n, 2n, \ldots, (m-1)n\}$$

is an arithmetic progression with m terms and common difference n, and

$$\{f(y_j) | j \in [1, n]\} = \{(m-1)n + 1, (m-1)n + 2, \ldots, mn\}$$

is a sequence of n consecutive integers. Therefore,

$$\{f(x) + f(y) | x \in X \text{ and } y \in Y\} = [(m-1)n + 1, (m-1)n + mn]$$

is a set of mn consecutive integers, implying that $\sigma(K_{m,n}) \leq mn$ and the proof is complete. □

The validity of the conjecture mentioned in introduction now follows readily from Theorems 1 and 3.

Corollary 4. *For every two positive integers* m *and* n,

$$\mu_s(K_{m,n}) = (m-1)(n-1).$$

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References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Wadsworth & Brook/Cole Advanced Books and Software, Monterey, Calif. 1986.
[2] H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.*, **34** (1998), 105–109.
[3] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.*, **231** (2001), 153–168.
[4] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, Some new results on the super edge-magic deficiency of graphs, *J. Combin. Math. Combin. Comput.*, **55** (2005) 17–33.
[5] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, On the super edge-magic deficiency of graphs, *Ars Combin.*, **78** (2006) 33–45.
[6] T. Grace, Graceful, Harmonious, and Sequential Graphs, Ph.D. Thesis, University of Illinois at Chicago Circle, 1982.
[7] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, (2010), #DS6.
[8] R.L. Graham and N.J. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Math.*, 1 (1980) 382–404.
[9] S.M. Hegde and S. Shetty, Strongly indexable graphs and applications, *Discrete Math.*, **309** (2009) 6160-6168.
[10] S.M. Hegde, S. Shetty and P. Shankran, Further results on super edge-magic deficiency of graphs, *Ars Combin.*, **99** (2011) 487–502.
Publications du Centre de Recheres Mathemátiques Université de Montréal **136**, (1971).
[11] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.*, **13** (1970), 451-461.
[12] A. Ngurah, E. Baskoro and R. Simamjuntak, On the super edge-magic deficiencies of graphs, *Australas. J. Combin.*, **40** (2008) 3–14.
[13] G. Ringel and A. Lladó, Another tree conjecture, *Bull. Inst. Combin. Appl.*, **18** (1996), 83–85.
[14] G. Santhosh and G. Singh, A note on the super edge-magic deficiency of graphs, *Far East J. Appl. Math.*, **24** (2006) 233–242.
[15] W.D. Wallis, *Magic Graphs*. Birkhäuser, Boston (2001).

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