Local restricted edge connectivity and restricted edge connectivity of graphs

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Abstract: Let G be a connected graph and $k \geq 1$ be an integer. Local k-restricted edge connectivity $\lambda_k(X,Y)$ of X,Y in G is the maximum number of the edge disjoint X-Y paths for $X,Y \subseteq V$ with $|X| = |Y| = k, X \cap Y = \emptyset$, $G[X]$ and $G[Y]$ are connected. k-restricted edge connectivity of G is defined as $\lambda_k(G) = \min\{\lambda_k(X,Y) : X,Y \subseteq V, |X| = |Y| = k, X \cap Y = \emptyset, G[X]$ and $G[Y]$ are connected$\}$. Then G is local optimal k-restricted edge connected, if $\lambda_k(X,Y) = \min\{\omega(X),\omega(Y)\}$ for all $X,Y \subseteq V$ with $|X| = |Y| = k$, $G[X]$ and $G[Y]$ are connected, where $\omega(X) = |[X,\overline{X}]|$. If $\lambda_k(G) = \xi_k(G)$ where $\xi_k(G) = \min\{\omega(X) : U \subset V, |U| = k$ and $G[U]$ is connected$\}$, then G is called λ_k-optimal. In this paper, we obtain several sufficient conditions for a graph to be λ_3-optimal (or local optimal k-restricted edge connected).

Key words: Local restricted edge connectivity, Restricted edge connectivity.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. Unless stated otherwise, we follow Bondy and Murty [1] for terminology and definitions.

Let $G = (V, E)$ be a connected graph, $d_G(v)$ be the degree of a vertex v in G (simply $d(v)$), and $\delta(G)$ be the minimum degree of G. Moreover, for $S \subset V$, $G[S]$ is the subgraph induced by S, $G - S$ denotes the subgraph of G induced by the vertex set of $V \setminus S$, and $\bar{S} = V - S$. We write K_n for the complete graph of order n. If $u, v \in V$, $d(u, v)$ denotes the length of a shortest (u, v)-path, $N(v)$ is the set of all vertices adjacent to v. And the diameter is $D(G)$ =max$\{d(u, v) : u, v \in V\}$. For $X, Y \subset V$, denote by $[X, Y]$ the set of edges with one end in X and the other in Y, P_n is a path with n vertices and C_n is a cycle with n vertices. The degree sequence of a graph G is defined as the non-increasing sequence of the degrees of the vertices of G.

The local edge connectivity $\lambda(u, v)$ of a connected graph G is the maximum number of edge disjoint u-v paths in G, and the edge connectivity $\lambda(G)$ =min$\{\lambda(u, v) : u, v \in V\}$. Clearly, $\lambda(u, v) \leq$min$\{d(u), d(v)\}$ for all pairs u and v of vertices in G. We call a graph G maximally local edge connected when $\lambda(u, v)$ =min$\{d(u), d(v)\}$ for all pairs u and v of vertices in G.

In 2000 Fricke, Oellermann and Swart have shown that some known sufficient conditions that guarantee $\lambda(G) = \delta(G)$ for a graph G also guarantee that G is maximally local edge connected. Hellwig and Volkmann obtained the next proposition:

Proposition 1.1. (*Hellwig, Volkmann* [4] 2004) *If a digraph or graph D is maximally local edge connected then $\lambda(D) = \delta(D)$.*

For further study, one can get that: let $X, Y \subseteq V$ with $|X| = |Y| = k \geq 1, X \cap Y = \varnothing$, $G[X]$ and $G[Y]$ be connected. Local k-restricted edge connectivity $\lambda_k(X, Y)$ is the maximum number of the edge disjoint X-Y paths. The k-restricted edge connectivity of G is defined as $\lambda_k(G)$ =min$\{\lambda_k(X, Y) :$ $X, Y \subseteq V, |X| = |Y| = k, X \cap Y = \varnothing, G[X]$ and $G[Y]$ are connected$\}$. If $\lambda_k(G)$ exists, then G is called λ_k-connected. If there is an edge cut

S such that $|S| = \omega(X) = \lambda_k(G)$, we call S a λ_k-cut, where $\omega(X) = |[X,\overline{X}]|$. Clearly, $\lambda_k(X,Y) \leq \min\{\omega(X),\omega(Y)\}$. G is local optimal k-restricted edge connected, if $\lambda_k(X,Y) = \min\{\omega(X),\omega(Y)\}$ for all $X, Y \subseteq V$ with $|X| = |Y| = k$, $G[X]$ and $G[Y]$ are connected. And $\xi_k(G) = \min \{\omega(X) : U \subset V, |U| = k$ and $G[U]$ is connected$\}$.

In this work we study the local 3-restricted edge connectivity and 3-restricted edge connectivity of graphs.

2 Local restricted edge connectivity of graphs

We will discuss the relation between the local restricted edge connectivity and restricted edge connectivity of graphs in the following proposition.

Proposition 2.1. *If G is a λ_k-connected graph and local optimal k-restricted edge connected graph, then $\lambda_k(G) = \xi_k(G)$.*

The proof is clearly.

By Menger's Theorem we get the following lemma.

Lemma 2.2. *Let G be a connected graph with disjoint sets $X, Y \subseteq V, |X| = |Y| = k$. $G[X]$ and $G[Y]$ are connected. Then $\lambda_k(X,Y) \geq q$ if and only if $|[S,\overline{S}]| \geq q$ for all subsets $S \subset V$ such that $X \subseteq S$ and $Y \subseteq \overline{S}$.*

Theorem 2.3. *Let G be a connected graph with diameter at most 2, then $\lambda_k(X,Y) = min\ \{\omega(X),\omega(Y)\}$ for all disjoint $X, Y \subseteq V$ with $|X| = |Y| = k, G[X]$ and $G[Y]$ are connected.*

Proof. Let $X, Y \subseteq V$ be disjoint sets with $|X| = |Y| = k, G[X]$ and $G[Y]$ be connected. Clearly, $\lambda_k(X,Y) \leq \min\{\omega(X),\omega(Y)\}$. Next we will show that $|[S,\overline{S}]| \geq \min\{\omega(X),\omega(Y)\}$ for all subsets $S \subset V$ such that $X \subseteq S$ and $Y \subseteq \overline{S}$. Let S be such a set.

Case 1. $|N(x) \cap \overline{S}| \geq 1$ for all $x \in S$.

This implies

$$min\{\omega(X),\omega(Y)\} \leq \omega(X) = \sum_{u\in X} |N(u) \cap S| + \sum_{u \in X} |N(u) \cap (S \setminus X)|$$

$$\leq \sum_{u\in X}|N(u)\cap S| + \sum_{x\in N(u)\cap(S\setminus X)}|N(x)\cap S|$$
$$\leq \sum_{x\in S}|N(x)\cap S| = |[S,S]|.$$

Case 2. There is a vertex $x \in S$ such that $|N(x)\cap S| = 0$.

Since $D(G) \leq 2$, it follows that $|N(x)\cap S| \geq 1$ for all $x \in S$. This leads to

$$\begin{aligned} min\{\omega(X),\omega(Y)\} &\leq \omega(Y) = \sum_{y\in Y}|N(y)\cap S| + \sum_{y\in Y}|N(y)\cap(S\setminus Y)| \\ &\leq \sum_{y\in Y}|N(y)\cap S| + \sum_{x\in N(y)\cap(S\setminus Y)}|N(x)\cap S| \\ &\leq \sum_{x\in S}|N(x)\cap S| = |[S,S]|. \end{aligned}$$

From Lemma 2.2, the proof is complete.

By Proposition 2.1 and Theorem 2.3 we can get the following results.

Corollary 2.4. (*Plesnik* [8] 1975) *If G is a graph of $D(G) \leq 2$, then $\lambda(G) = \delta(G)$.*

Corollary 2.5. (*Lesniak* [6] 1974) *If G is a graph of order n with $d(u) + d(v) \geq n-1$ for all distinct nonadjacent vertices u and v, then $\lambda(G) = \delta(G)$.*

Corollary 2.6. (*Chartrand* [2] 1966) *If G is a graph of order $n \leq 2\delta(G)+1$, then $\lambda(G) = \delta(G)$.*

Corollary 2.7. (*Fricke, Oellermann, Swart* [3] 2000) *Let G be a connected graph with diameter at most 2, then $\lambda(u,v)$ =min$\{d(u),d(v)\}$ for all pairs u and v of vertices in G.*

J. Liu obtained the following result about local optimal 2-restricted edge connected graphs.

Lemma 2.8. (*Juan Liu* [5] 2009) *Let G be a λ_2-connected graph of order n and minimum degree δ. If*

$$\delta \geq \left\lfloor \frac{n}{2} \right\rfloor + 1,$$

then G is local optimal 2-restricted edge connected.

Thus we consider the local optimal 3-restricted edge connectivity of graphs.

Theorem 2.9. *Let G be a λ_3-connected graph of order n and minimum degree δ. If*

$$\delta \geq \left\lfloor \frac{n}{2} \right\rfloor + 2,$$

then $\lambda_3(X,Y) = \min\{\omega(X), \omega(Y)\}$ for all disjoint $X, Y \subseteq V$ with $|X| = |Y| = 3, G[X]$ and $G[Y]$ are connected.

Proof. Let $X, Y \subseteq V$ be disjoint sets with $|X| = |Y| = 3, G[X]$ and $G[Y]$ be connected. Clearly, $\lambda_3(X,Y) \leq \min\{\omega(X), \omega(Y)\}$. Next we will show that $|[S, \overline{S}]| \geq \min\{\omega(X), \omega(Y)\}$ for all subsets $S \subset V$ such that $X \subseteq S$ and $Y \subseteq S$.

Without loss of generality, we assume that $|S| \leq |S|$, then $|S| \leq n/2$. Hence there exist three vertices $u, v, w \in S$ such that $H = G[X] = G[\{u, v, w\}]$ is connected. Because

$$d(u) + d(v) + d(w) = \omega(X) + \begin{cases} 4 & if\ H \cong P_3 \\ 6 & if\ H \cong C_3. \end{cases}$$

The hypothesis implies

$$\begin{aligned}
|[S, S]| &= \sum_{y \in S} d(y) - 2|E(G[S])| \\
&\geq \sum_{y \in S} d(y) - |S|(|S| - 1) + \begin{cases} 2 & if\ H \cong P_3 \\ 0 & if\ H \cong C_3 \end{cases} \\
&\geq d(u) + d(v) + d(w) + |S - 3|\delta - |S|(|S| - 1) + \begin{cases} 2 & if\ H \cong P_3 \\ 0 & if\ H \cong C_3 \end{cases} \\
&= \omega(X) + 6 + |S - 3|\delta - |S|(|S| - 1) \\
&= \omega(X) + |S - 3|(\delta - 2 - |S|) \\
&\geq \omega(X) + |S - 3|(n/2 - |S|) \\
&\geq \omega(X).
\end{aligned}$$

Therefore the proof is complete.

3 Restricted edge connectivity of graphs

We start this section with a useful lemma.

Lemma 3.1. *(Meng, Ji [2]) If G is a λ_3-connected graph, then $\lambda_3(G) \leq \xi_3(G)$.*

Lemma 3.2. *Let G be a λ_3-connected graph of order $n \geq 6$. If there is a λ_3-cut S with the vertex sets X and $\overline{X}$ of the two components of $G - S$ such that each vertex in X has at least three neighbors in $\overline{X}$, then G is λ_3-optimal.*

Proof. Let $[X, X]$ be an edge-cut satisfying the assumption. Note that any $u \in N(X) \cup N(z) \cup N(y) \setminus \{x, y, z\}$ has at least three neighbors and each vertex in $u \in N(X) \cup N(z) \cup N(y) \setminus \{x, y, z\}$ has at most three neighbors in $\{x, y, z\}$. Then $\xi_3(G) \leq d(x) + d(y) + d(z) - 2|E(G[\{x, y, z\}])| \leq |[\{x, y, z\}, X]| + |X \setminus [\{x, y, z\}, X]| = \lambda_3(G)$. Thus, we complete the proof.

Theorem 3.3. *Let G be a λ_3-connected graph of order $n \geq 6$ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. If*

$$\sum_{i=1}^{max\{1,\delta-4\}} d_{n-i} \geq max\{1, \delta - 4\}(\left\lfloor \frac{n}{2} \right\rfloor + 2 - \frac{2}{n-7}),$$

then G is λ_3-optimal.

Proof. Let $S = [X, \overline{X}]$ be an arbitrary λ_3-cut. Assume, without loss of generality that $|X| \leq |\overline{X}|$, then $|X| \leq \lfloor n/2 \rfloor$. If $|X| = 3$, then we are done. Now let $|X| \geq 4$. If $|X| \leq \delta - 2$, then every vertex in X has at least three neighbor in X, Lemma 3.2 leads to the desired result in this case. For $|X| \geq \delta - 1$, let $x \in X$ such that $d(x) = \min\{d(u) : u \in X\}$. Choose three vertices x, y, z in $G[X]$ such that $H = G[\{x, y, z\}]$ is connected, using the inequality $\max\{1, \delta - 4\} \leq |X| - 3$, the hypothesis yields

$$
\begin{aligned}
|[X,\overline{X}]| &\geq \sum_{u\in X} d(u) - |X|(|X|-1) + \begin{cases} 2 & if\ H \cong P_3 \\ 0 & if\ H \cong C_3 \end{cases} \\
&= d(x)+d(y)+d(z)-4+4+\sum_{u\in X\setminus\{x,y,z\}} d(u) - |X|(|X|-1) \\
&\quad + \begin{cases} 2 & if\ H \cong P_3 \\ 0 & if\ H \cong C_3 \end{cases} \\
&\geq \xi_3(G) + 6 + \sum_{i=1}^{max\{1,\delta-4\}} d_{n-i} + \sum_{i=max\{2,\delta-3\}}^{|X|-3} d_{n-i} \\
&\quad -|X|(|X|-3) - 2|X| \\
&\geq \xi_3(G) + (|X|-3)(\lfloor \frac{n}{2} \rfloor + 2 - \frac{2}{n-7}) - |X|(|X|-3) \\
&\quad -2(|X|-3) \\
&\geq \xi_3(G) + (|X|-3)(\lfloor \frac{n}{2} \rfloor + 2 - \frac{2}{n-7} - |X| - 2) \\
&= \xi_3(G) - \frac{2(|X|-3)}{n-7}.
\end{aligned}
$$

Since $\lambda_3(G) = |[X,\overline{X}]|$ and $\xi_3(G)$ are integers and $2(|X|-3)/(n-7) < 1$, it follows that $\lambda_3(G) \geq \xi_3(G)$ and thus G is λ_3-optimal. □

Corollary 3.4. *Let G be a λ_3-connected graph of order $n \geq 6$. If*

$$d(x) \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$$

for all vertices x in G with at most one exception, then G is λ_3-optimal.

The following example shows that the above lower bound is sharp.

Let $H_i, i = 1,2$ be a copy of the complete graph K_p and $K_{p+1} (p \geq 4)$ with $V(H_1) = \{x_1, x_2, \cdots, x_p\}$ and $V(H_2) = \{v_1, v_2, \cdots, v_{p+1}\}$. The graph G is defined as the disjoint union of H_1 and H_2 together with the $2p$ edges $x_1v_1, x_2v_2, \cdots, x_pv_p$ and $x_1v_{p+1}, x_2v_{p+1}, \cdots, x_pv_{p+1}$. Then, $n(G) = 2p+1$, $\delta(G) = p+1 = \lfloor n(G)/2 \rfloor + 1$, and $\xi_3(G) = 3p-3$. However, G is not λ_3-optimal.

Theorem 3.5. *Let G be a λ_3-connected graph. If $|N(u) \cap N(v)| \geq 5$ for all pairs u, v of nonadjacent vertices, then G is λ_3-optimal.*

Proof. By Proposition 2.1 and Theorem 2.3, we can get the result. □

Acknowledgements

We would like to thank the referees for kind help and valuable suggestions. The project is supported by NSFC (No. 11301440, 11171279), Xiamen University of Technology (YKJ12030R), the Foundation to the Educational Committee of Fujian (JA13240).

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