ON TOTAL EDGE IRREGULARITY STRENGTH OF GRAPH

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Abstract

In this paper we discuss how the addition of a new edge affects the total edge irregularity strength of a graph.

1 Introduction

In this paper we consider simple undirected and connected graphs. Let $G = (V, E)$ be a graph. We define a labeling $f : V \cup E \rightarrow \{1, 2, ..., k\}$ to be a total edge irregular k-labeling of the graph G if for every two different edges e_i and e_j for G, $\omega(e_i) \neq \omega(e_j)$ where the weight of an edge $e = uv$ in the labeling f is $\omega(e) = f(u) + f(v) + f(e)$. The minimum k for which the graph G has an edge irregular total k-labeling is called the total edge irregularity strength of G, $tes(G)$.

The irregularity strength of a graph was introduced by Chartrand et al.[1]. The total vertex-irregularity strength and total edge-irregularity strength were introduced by Baca et al.[2]. Also they have determined the total vertex and edge-irregularity strengths of some classes of graphs, namely cycles, paths, stars and wheels. Kathiresan et al.[6, 7] discussed the stability of irregularity strength of a graph by adding an edge. Also they discussed the stability of total vertex irregularity strength in [8].

In this paper we discuss the stability of total edge irregularity strength by adding an edge.

Definition 1.1. Let $G = (V, E)$ be any graph which is not complete. Let e be any edge of $\overline{G}$. Then e is called a positive edge of G, if $tes(G + e) > tes(G)$, and stable edge of G if $tes(G + e) = tes(G)$.

Example 1.2. In P_4, path of length 4, the edge joining any two non adjacent vertices is a positive edge of P_4. In P_3, the edge joining the two non adjacent vertices is a stable edge of P_3.

Theorem 1.3. *Addition of any edge to a graph G from its complement will not decrease the total edge irregularity.*

Proof. Let f be a total irregular s−labeling of G. Let u and v be any two non adjacent vertices of G. Suppose $tes(G + uv) < s$. Then there exists an irregular total labeling f' of $G+uv$ such that its maximum label is less than or equal to $s - 1$. Now by deleting the edge uv we get an irregular total labeling of G and $tes(G) \leq s - 1$, which is a contradiction to $tes(G) = s$. Hence, addition of any edge will not decrease the total edge irregularity strength. □

Definition 1.4. If all the edges of $\overline{G}$ are positive(stable) edges of G, then G is called a positive(stable) graph.

Example 1.5. P_7 a path on 7 edges is a positive graph. C_5 a cycle on 5 edges is a stable graph.

2 Positive, Stable edges of certain families of graphs

Theorem 2.1. *Star graph $K_{1,n}$ with $n \geq 2$ pendant vertices is a stable graph.*

Proof. Let x be the vertex of degree n and let $v_1, v_2, v_3, ..., v_n$ be the pendant vertices of $K_{1,n}$. Let $e_i = xv_i$ for $i = 1, 2, ..., n$ be the edges of $K_{1,n}$. For any $e \in \overline{K}_{1,n}$, $K_{1,n} + e$ is isomorphic to $K_{1,n} + v_{n-1}v_n$. Define the total labeling $f : E \cup V(K_{1,n} + e) \to Z^+$ as follows.

$$f(x) = 1$$

$$f(v_i) = \lceil \tfrac{i+1}{2} \rceil \quad for \quad i = 1, 2, ..., n$$

$$f(e_i) = \lfloor \tfrac{i+1}{2} \rfloor \quad for \quad i = 1, 2, ..., n$$

$f(v_{n-1}v_n) = \lceil \frac{n+1}{2} \rceil$

It is easy to see that the labeling f makes $K_{1,n} + v_{n-1}v_n$ irregular. Therefore $tes(K_{1,n} + v_{n-1}v_n) \leq \lceil \frac{n+1}{2} \rceil$. $tes(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$ was proved by Baca et al.[2]. By Theorem 1.3, $tes(K_{1,n} + v_{n-1}v_n) = \lceil \frac{n+1}{2} \rceil = tes(K_{1,n})$. This concludes the proof. □

Theorem 2.2. *Path $P_n, n > 1$ with $n+1$ vertices is a stable graph if $n \not\equiv 1(mod\ 3)$ and positive graph if $n \equiv 1(mod\ 3)$.*

Proof. Let $v_1e_1v_2e_2v_3...e_{n-1}v_ne_nv_{n+1}$ be the path P_n on $n+1$ vertices. $tes(P_n) = \lceil \frac{n+2}{3} \rceil$ was proved by Baca et al.[2]. We define the labeling for P_n as follows,

$$f(e_i) = \lceil \frac{i}{3} \rceil, 1 \leq i \leq n$$

$$f(v_i) = \lceil \frac{i+1}{3} \rceil, 1 \leq i \leq n+1$$

Then the weight of the edges are $\omega(e_1) = 3, \omega(e_2) = 4, \omega(e_3) = 5, ..., \omega(e_n) = n+2$.

Case 1: If $n \not\equiv 1(mod\ 3)$, then $n = 3m$ or $n = 3m+2$ for some m. Add the edge $e = v_iv_j$, $1 \leq i < i+1 < j \leq n+1$, and assign label $m+1$ to the edge e.

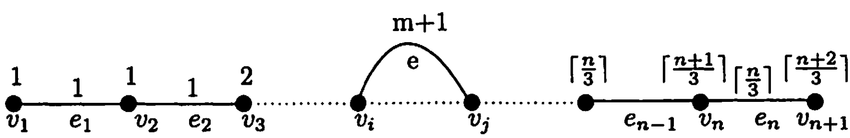

Figure.1

Now the weight of the edge e is $\omega(e) = m+1+f(v_i)+f(v_j) = m+1+\lceil \frac{i+1}{3} \rceil + \lceil \frac{j+1}{3} \rceil \leq m+1+\lceil \frac{n}{3} \rceil + \lceil \frac{n+2}{3} \rceil = n+2 = \omega(e_n)$. Therefore there exists an edge e_k such that $\omega(e) = \omega(e_k)$. Now increase the label of the edges $e_k, e_{k+1}, ..., e_n$ by 1, to increase their weight by 1. Thus, there exists an irregular total labeling for $P_n + e$ with maximum label $\lceil \frac{n+2}{3} \rceil$. Therefore $tes(P_n + e) \leq \lceil \frac{n+2}{3} \rceil$. Further it is not possible to obtain the irregular total labeling with fewer than $m+1$ label. Therefore $tes(P_n + e) = tes(P_n)$. Hence the edge e is stable edge.

Case 2: If $n \equiv 1(mod\ 3)$, then $n = 3m+1$ for some m. Add the edge $e = v_iv_j$, $1 \leq i < i+1 < j \leq n+1$, assign the label $m+1$ to the edge e.

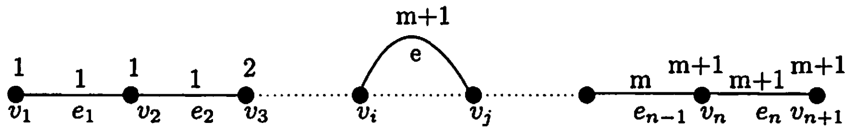

Figure.2

Now the weight of the edge e is $\omega(e) = m + 1 + f(v_i) + f(v_j) = m + 1 + \lceil\frac{i+1}{3}\rceil + \lceil\frac{j+1}{3}\rceil \leq m + 1 + \lceil\frac{n}{3}\rceil + \lceil\frac{n+2}{3}\rceil = n + 2 = \omega(e_n)$. Therefore there exists an edge e_k such that $\omega(e) = \omega(e_k)$. Now increase the labelings of the edges $e_k, e_{k+1}, ..., e_n$ by 1, to increase their weight by 1. Thus there exists an irregular total labeling for $P_n + e$ with maximum label $m + 2$. Therefore $tes(P_n + e) \leq m + 2$. Since $P_n + e$ has $n + 1$ edges and the optimum weights are $3, 4, 5, ..., n + 3$, it is not possible to obtain an irregular total labeling with fewer than $m + 2$ label. Therefore $tes(P_n + e) \geq m + 2$. Therefore $tes(P_n + e) = m + 2 > tes(P_n)$. Hence the edge e is positive edge. □

Theorem 2.3. *For $n > 3$, cycle C_n with n vertices, is a stable graph if $n \not\equiv 1(mod\ 3)$ and a positive graph if $n \equiv 1(mod\ 3)$.*

Proof. Let $C_n, n > 3$ be the cycle $v_1e_1v_2e_2v_3...e_{n-1}v_ne_nv_1$. Baca et al.[2] determined that $tes(C_n) = \lceil\frac{n+2}{3}\rceil$. They define the labeling of C_n by induction method as follows. The optimal irregular total labeling of C_4 and C_5 are as follows.

Figure.3

Suppose we have an irregular labeling of C_n for $n = 3(k-1) + 2$, $k \geq 2$ with the edge e_{n-1} labeled as $f(e_{n-1}) = k + 1, f(v_{n-1}) = f(v_n) = k$.

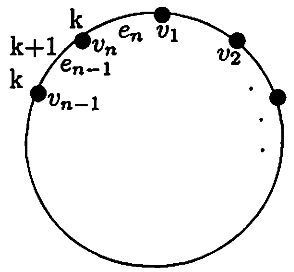

Figure.4

For C_{n+1}, we split the edge e_{n-1} into two edges by adding a new vertex x_1 and we label the new vertex and edges as, $f(v_{n-1}x_1) = k+1, f(x_1) = k+1, f(x_1v_n) = k$.

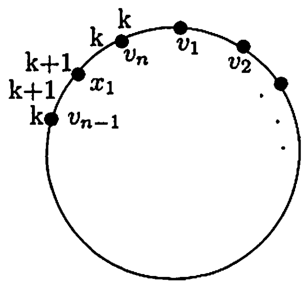

Figure.5

For C_{n+2}, we split the edge e_{n-1} into three edges by adding new vertices x_1 and x_2 and we label the new vertices and edges as, $f(x_1) = f(x_2) = k+1, f(v_{n-1}x_1) = f(x_1x_2) = k+1, f(x_2v_n) = k$.

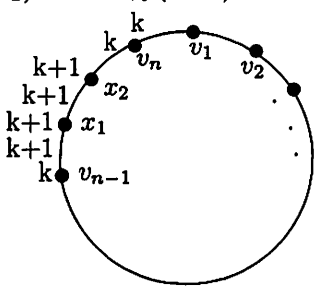

Figure.6

For C_{n+3}, we split the edge e_{n-1} into four edges by adding new vertices x_1,x_2 and x_3 and we label the new vertices and edges as, $f(x_1) = f(x_2) = f(x_3) = k+1, f(v_{n-1}x_1) = f(x_2x_3) = k+1, f(x_1x_2) = k+2, f(x_3v_n) = k$.

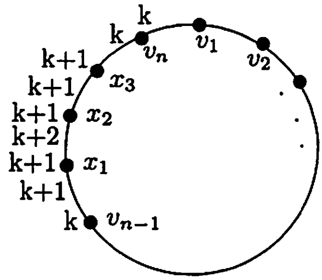

Figure.7

In C_{n+3} we reorder the vertices and edges so that the edge x_1x_2 will play the role of e_{n-1} of the above construction.

Case 1: If $n \not\equiv 1(mod\ 3)$, then $n = 3k$ or $n = 3k+2$ for some k. Therefore we have the labeling for C_n as in Figure.5 and Figure.7. Now let us rename the edges as $e_1, e_2, e_3, ..., e_n$ such that their weights are $3, 4, 5, ..., n+2$ respectively.

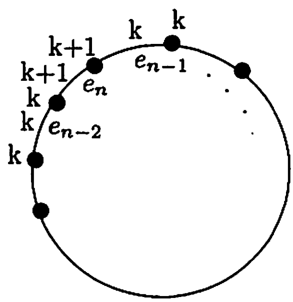

Figure.8 (n=3k)

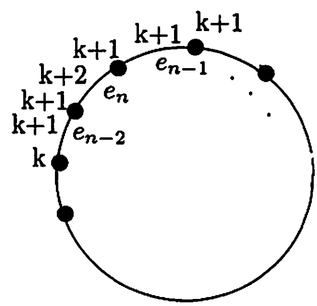

Figure.9 (n=3k+2)

Add the edge $e = v_iv_j$, $1 \leq i < i+1 < j \leq n$, and label the edge e as $f(e) = \lceil\frac{n+2}{3}\rceil$. Now the weight of the edge e is $\omega(e) = \lceil\frac{n+2}{3}\rceil + f(v_i) + f(v_j) \leq n+2 = \omega(e_n)$. Therefore there exists an edge e_m such that $\omega(e)$=$\omega(e_m)$. Now to increase the weight of the edges e_m, e_{m+1}, $e_{m+2}, ...,$ e_{n-1}, e_n by 1, we increase the label of the edges $e_m, e_{m+1}, ..., e_{n-3}, e_{n-1}$ by 1. Further increase the label of the common vertex of e_{n-2} and e_n by 1. If the common vertex of e_{n-2} and e_n is v_i or v_j then change the edge e as

$v_{i+1}v_{j+1}(v_{i+1}v_1$ if $j = n)$, since $C + v_iv_j$ is isomorphic to $C + v_{i+1}v_{j+1}$. Now we have an irregular labeling for $C_n + e$ with maximum label $\lceil \frac{n+2}{3} \rceil$. Therefore $tes(C_n) = tes(C_n + e) = \lceil \frac{n+2}{3} \rceil$. Hence e is a stable edge.

Case 2: If $n \equiv 1(mod\ 3)$, then $n = 3k + 1$ for some k. Therefore we have the labeling for C_n as in Figure.6. Now let us rename the edges as $e_1, e_2, e_3, ..., e_n$ such that their weights are $3, 4, 5, ..., 3k + 3$.

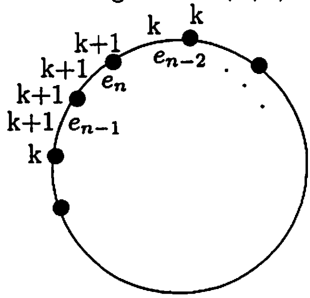

Figure.10

Add the edge $e = v_iv_j$, $1 \leq i < i + 1 < j \leq n$, and label the edge e as $f(e) = k+1$. Now the weight of the edge e is $\omega(e) = k+1+f(v_i)+f(v_j) \leq 3k + 3 = \omega(e_n)$. Therefore there exists an edge e_m such that $\omega(e)=\omega(e_m)$. Now we increase the weight of the edges $e_m, e_{m+1}, e_{m+2}, ..., e_{n-1}, e_n$ by 1. But to increase the weight of e_n by 1, we have to increase the label of e_n or one of the vertices incident with e_n. Therefore we get an optimal irregular labeling for $C_n + e$ with maximum label $k + 2$. Therefore $tes(C_n) = k + 1$ and $tes(C_n + e) = k + 2$. Hence e is a positive edge. □

Theorem 2.4. *For $n \geq 4$, wheel W_n with $n + 1$ vertices is a stable graph if $n \not\equiv 2(mod\ 3)$ and a positive graph if $n \equiv 2(mod\ 3)$.*

Proof. Let W_n be the wheel with $V(W_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$ and $E(W_n) = \{vv_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_nv_1\}$.

Baca et al.[2], determined that $tes(W_n) = \lceil \frac{2n+2}{3} \rceil$. Define the total labeling of W_n as follows.

For W_4: $f(v) = 4, f(v_1) = f(v_2) = 1, f(v_3) = 2, f(v_4) = 3, f(vv_1) = f(v_1v_2) = f(v_1v_4) = f(v_2v_3) = 1, f(vv_2) = 2, f(vv_3) = f(vv_4) = f(v_3v_4) = 3$.

For W_5: $f(v) = f(v_4) = 4, f(v_1) = f(v_2) = 1, f(v_3) = 2, f(v_5) = 3, f(vv_1) = f(v_1v_2) = f(v_1v_5) = f(v_2v_3) = 1, f(vv_2) = f(vv_3) = f(vv_5) = 2, f(vv_4) = f(v_3v_4) = f(v_4v_5) = 4$.

For W_6: $f(v) = f(v_5) = 5, f(v_1) = f(v_2) = f(v_3) = 1, f(v_4) = 3, f(v_6) = 4, f(vv_1) = f(v_1v_2) = f(v_1v_6) = f(v_3v_4) = 1, f(vv_2) = f(vv_4) = f(v_2v_3) = f(vv_6) = 2, f(vv_3) = 3, f(vv_5) = f(v_4v_5) = f(v_5v_6) = 4.$

For $n > 6$,

$f(v) = \lceil \frac{2n+2}{3} \rceil$

$$f(v_i) = \begin{cases} 1 & if\ \ 1 \le i \le \lceil \frac{2n+2}{3} \rceil - 2, \\ \lceil \frac{2n+2}{3} \rceil - 2 & if\ \ i = \lceil \frac{2n+2}{3} \rceil - 1\ \ and\ \ i = n, \\ \lceil \frac{2n+2}{3} \rceil & if\ \ \lceil \frac{2n+2}{3} \rceil \le i \le n-1. \end{cases}$$

$$f(v_iv_{i+1}) = \begin{cases} i & if\ \ 1 \le i \le \lceil \frac{2n+2}{3} \rceil - 3, \\ 1 & if\ \ i = \lceil \frac{2n+2}{3} \rceil - 2, \\ 5 & if\ \ i = \lceil \frac{2n+2}{3} \rceil - 1, \\ 2n+2-\lceil \frac{2n+2}{3} \rceil - i & if\ \ \lceil \frac{2n+2}{3} \rceil \le i \le n-2, \\ 4 & if\ \ i = n-1, \\ 2 & if\ \ i = n. \end{cases}$$

$$f(vv_i) = \begin{cases} i & if\ \ 1 \le i \le \lceil \frac{2n+2}{3} \rceil - 2, \\ 2 & if\ \ i = \lceil \frac{2n+2}{3} \rceil - 1, \\ i - \lceil \frac{2n+2}{3} \rceil + 4 & if\ \ \lceil \frac{2n+2}{3} \rceil \le i \le n-1, \\ 3 & if\ \ i = n. \end{cases}$$

Case 1: If $n \not\equiv 2(mod\ 3)$, then $n = 3m$ or $n = 3m+1$ for some m. Name the edges of W_n such that $\omega(e_1) = 3, \omega(e_2) = 4, ..., \omega(e_{2n-1}) = 2n+1, \omega(e_{2n}) = 2n+2$.

Add the edge $e = v_iv_j, 1 \le i < i+1 < j \le n$ and assign the label $2m$ to the edge e. Since, $\omega(e) = 2m + f(v_i) + f(v_j) \le 2n+2 = \omega(e_{2n})$, there exists an edge e_k such that $\omega(e) = \omega(e_k)$. Now increase the label of $e_k, e_{k+1}, ..., e_{2n}$ by 1, to get irregular labeling. Thus we have an optimal irregular labeling for $W_n + e$ with the maximum label $\lceil \frac{2n+2}{3} \rceil$. Therefore $tes(W_n) = tes(W_n + e) = \lceil \frac{2n+2}{3} \rceil$. Hence e is stable edge.

Case 2: If $n \equiv 2(mod\ 3)$, then $n = 3m+2$, for some m. Name the edges of W_n such that $\omega(e_1) = 3, \omega(e_2) = 4, ..., \omega(e_{2n-1}) = 6m+5, \omega(e_{2n}) = 6m+6$. Edge e_{2n} receives the label $2m+2$ and the corresponding end vertices receive labels $2m+2$.

Add the edge $e = v_iv_j, 1 \le i < i+1 < j \le n$ and assign the label $2m$ to the edge e. Since, $\omega(e) = 2m + f(v_i) + f(v_j) \le 2m + 2m + 2 + 2m + 2 =$

$6m+4 \leq \omega(e_{2n})$, there exists an edge e_k such that $\omega(e) = \omega(e_k)$. Now increase the label of $e_k, e_{k+1}, ..., e_{2n}$ by 1, to get irregular labeling. Thus we have an optimal irregular labeling for $W_n + e$ with the maximum label $2m+3$. Hence e is positive. □

Theorem 2.5. *The friendship graph $F_n, n \geq 2$ is a stable graph.*

Proof. Let F_n be the friendship graph with $V(F_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\}$ and $E(F_n) = \{vv_i : 1 \leq i \leq 2n\} \cup \{v_iv_{i+1} : i = 1, 3, 5, ..., 2n-1\}$.

Baca et al.[2], determined that $tes(F_n) = \lceil \frac{3n+2}{3} \rceil = n+1$. Since $F_n + v_{2n-2}v_{2n-1} \cong F_n + e$, for any $e \in \overline{F_n}$, it enough to prove that $v_{2n-2}v_{2n-1}$ is a stable edge. Let us define the labeling for $F_n + v_{2n-2}v_{2n-1}$ as follows. Let $s_i = vv_i$ for $1 \leq i \leq 2n$ and $r_1 = v_1v_2, r_2 = v_3v_4, r_3 = v_5v_6, ..., r_{n-1} = v_{2n-3}v_{2n-2}, r_n = v_{2n-2}v_{2n-1}, r_{n+1} = v_{2n-1}v_{2n}$.

For F_2: $f(v_1) = f(v_2) = 1, f(v_3) = f(v_4) = 3, f(v) = 2, f(s_1) = 1, f(s_2) = f(s_3) = 2, f(s_4) = 3, f(r_1) = 1, f(r_2) = 2, f(r_3) = 3$.

For F_3: $f(v_1) = f(v_2) = f(v_3) = 1, f(v_4) = f(v_5) = f(v_6) = 4, f(v) = 2, f(s_1) = 1, f(s_2) = f(s_4) = 2, f(s_3) = f(s_5) = f(s_6) = 4, f(r_1) = 1, f(r_2) = 2, f(r_3) = 3, f(r_4) = 4$.

For $n \geq 4$,

Case 1: If n is odd.

$f(v) = \frac{n+3}{2}$,
$f(v_i) = 1 \quad for \quad 1 \leq i \leq n+1$,
$f(v_i) = n+1 \quad for \quad n+2 \leq i \leq 2n$,
$f(s_i) = i \quad for \quad 1 \leq i \leq n+1$,
$f(s_i) = i-n \quad for \quad n+2 \leq i \leq 2n$,
$f(r_i) = i \quad for \quad 1 \leq i \leq n+1$.

Case 2: If n is even.

$f(v) = \frac{n}{2} + 1$,
$f(v_i) = 1 \quad for \quad 1 \leq i \leq n$,
$f(v_i) = n+1 \quad for \quad n+1 \leq i \leq 2n$,
$f(s_i) = i \quad for \quad 1 \leq i \leq n$,
$f(s_i) = i-n \quad for \quad n+1 \leq i \leq 2n$,
$f(r_i) = i \quad for \quad 1 \leq i \leq n+1$.

By the above labeling we have, $tes(F_n+e) \leq n+1$. Since $tes(F_n) = n+1$ and by Theorem 1.3, $tes(F_n+e) \geq n+1$. Therefore $tes(F_n+e) = n+1$. Therefore e is a stable edge, and hence F_n is a stable graph. □

Theorem 2.6. *Let $G = P_n \times K_2, n \geq 2$ be the cartesian product of a path on n vertices with a complete graph on two vertices. Then $tes(G) = n$.*

Proof. G consists of two paths $P : v_1e_1v_2e_2v_3...v_{n-1}e_{n-1}v_n$ and $P' : v'_1e'_1v'_2$ $e'_2v'_3...v'_{n-1}e'_{n-1}v'_n$ which are connected by the edges $v_iv'_i, i = 1, 2, 3, ..., n$. Now we define the irregular total labeling for G as follows.
For $i = 1, 2, 3, ..., n$
$f(v_i) = f(v'_i) = i$
$f(e_i) = i, f(e'_i) = i + 1, f(v_iv'_i) = i$
By the above labeling, $tes(G) \leq n$. Since G has $3n - 2$ edges, the maximum weight of the edge is at least $3n$. Since it is not possible to obtain the weight $3n$ with fewer than n labels, $tes(G) \geq n$. Hence $tes(G) = n$. □

Corollary 2.7. *The graph* $G = P_n \times K_2, n \geq 2$ *is a positive graph. If we add any edge e to G then the maximum weight will be at least* $3n + 1$*. Since it is not possible to obtain the weight* $3n + 1$ *with fewer than n labels,* $tes(G + e) > n$.

In this paper we discussed some families of positive and stable graphs. In our experience we found no graph having positive as well as stable edge. Thus we propose the following open problem.

Open problem: There is no graph which contains both positive edge and stable edge.

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