G-Designs Having a Prescribed Number of Blocks in Common

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Abstract

Two G-designs $(X,\mathcal{A}_1)$ and $(X,\mathcal{A}_2)$ are said to intersect in m blocks if $|\mathcal{A}_1 \cap \mathcal{A}_2| = m$. In this paper, we complete the solution of intersection problem for G-designs, where G is a connected graph of size five which contains a cycle.

Keywords: G-design, Group divisible design, Graph decomposition.

1 Introduction

Let H be a simple graph and G be a subgraph of H. A G-design of H is a pair $(V,\mathcal{A})$, where V is the vertex set of H and $\mathcal{A}$ is a collection of the edge-disjoint decomposition of H into isomorphic copies of G (called *blocks*). If H is the complete graph of order n, we refer to such a G-design as one of order n. In this paper, we concentrate on a connected graph G with five edges and containing a cycle. There are six different types of graphs G (See Figure 1). We begin with some notation to describe these six graphs.

$T_1 = (\{a,b,c,d,e\},\{ab,bc,ac,bd,ce\})$, denote this graph by $(a,b,c,d,e)_1$;
$T_2 = (\{a,b,c,d,e\},\{ab,bc,ac,cd,ce\})$, denote this graph by $(a,b,c,d,e)_2$;
$T_3 = (\{a,b,c,d,e\},\{ab,bc,ac,cd,de\})$, denote this graph by $(a,b,c,d,e)_3$;
$T_4 = (\{a,b,c,d,e\},\{ab,bc,cd,da,de\})$, denote this graph by $(a,b,c,d,e)_4$;
$T_5 = (\{a,b,c,d,e\},\{ab,bc,cd,de,ea\})$;

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$T_6 = (\{a, b, c, d\}, \{ab, bc, ac, da, dc\})$.

In fact, T_5 graph is a 5-cycle C_5 and T_6 is called a $K_4 - e$ graph, meaning a complete graph of order 4 with the vertex set $\{a, b, c, d\}$ minus an edge bd.

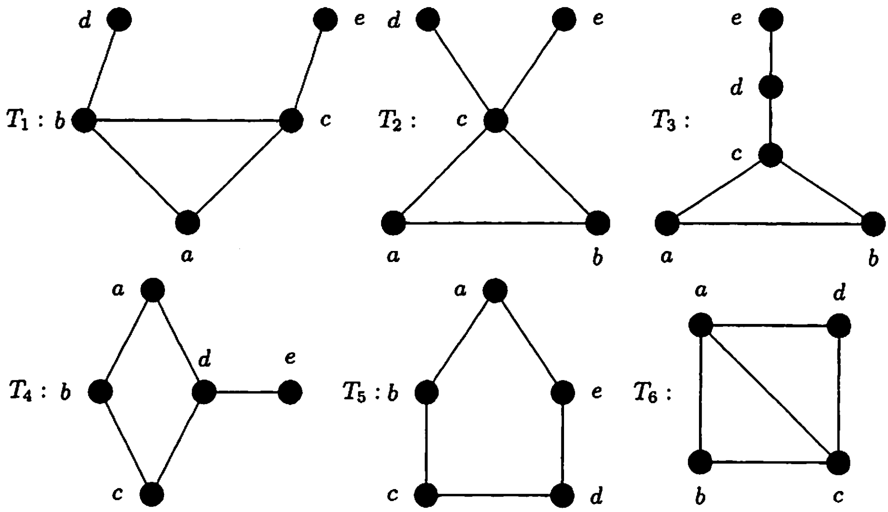

Figure 1: Graphs T_1, T_2, T_3, T_4, T_5 and T_6

Two T_i-designs $(X, \mathcal{A}_1)$ and $(X, \mathcal{A}_2)$ of order n are said to intersect in m blocks if $|\mathcal{A}_1 \cap \mathcal{A}_2| = m$. Recently, many authors deal with "variations" to the classic intersection problem, such as intersection problem for triple systems in [12], disjoint intersection problem in [7] and [10], the triangle intersection in [2], [3] and [4], the fine triangle intersection in [5] and [6].

Concerning the intersection problem for T_5 and T_6-designs, in 1988, C. M. Fu [9] obtained the solution for the intersection numbers of pentagon systems (i.e. T_5-designs) of order n, where $n \equiv 0, 1 \pmod 5$ $n \geq 6$:

$$\{0, 1, \ldots, n(n-1)/10 - 2, n(n-1)/10\};$$

in 1997, E. J. Billington, M. Gionfriddo and C. C. Lindner [1] obtained the solution for the intersection numbers of $K_4 - e$ designs (i.e. T_6-designs) of order n, where $n \equiv 0, 1 \pmod 5$ $n \geq 6$:

$$\{0, 1, \ldots, n(n-1)/10 - 3, n(n-1)/10\}.$$

The purpose of this paper is to solve the intersection problem for T_i-designs for $i = 1, 2, 3, 4$.

Since a necessary condition for the existence of a T_i-design of order n is that $\binom{n}{2}/5$ is an integer, $n \equiv 0$ or $1 \pmod 5$ satisfies this requirement. In what follows, let $I_{T_i}(n)$ denote the set of all integers k for which there exists two T_i-designs of order n, $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$, such that $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$. For $n \equiv 0, 1 \pmod 5$, let $J_{T_i}(n) = \{0, 1, 2, \ldots, n(n-1)/10 - 2, n(n-1)/10\}$. In other words, $J_{T_i}(n)$ denotes the intersection numbers one expects to achieve with a T_i-design of order n. Modifying this notation slightly, let $I_{T_i}(H)$ and $J_{T_i}(H)$ denote the achievable and expected intersection numbers for T_i-design of graph H, respectively. It is clear that $I_{T_i}(H) \subseteq J_{T_i}(H)$ for $i = 1, 2, 3, 4$. Here we deal with for the reverse containment.

Main Theorem $I_{T_i}(n) = J_{T_i}(n)$ for $n \equiv 0, 1 \pmod 5$ and $n \geq 10$; $I_{T_i}(n) = J_{T_i}(n)$ for $(i, n) = (1, 5), (1, 6), (2, 6), (3, 6)$.

Let A and B be two sets of integers and k a positive integer. Define $A + B = \{a + b \mid a \in A, b \in B\}$, $k + A = \{k + a \mid a \in A\}$ and $k \cdot A = \underbrace{A + A + \ldots + A}_{k}$.

We quote the following known result for later use.

Lemma 1.1 [8] *Let g, t and u be nonnegative integers. There exists a 3-GDD of type $g^t u^1$ if and only if the following conditions are all satisfied:*

(1) if $g > 0$ then $t \geq 3$, or $t = 2$ and $u = g$, or $t = 1$ and $u = 0$, or $t = 0$;

(2) $u \leq g(t-1)$ or $gt = 0$;

(3) $g(t-1) + u \equiv 0 \pmod 2$ or $gt = 0$;

(4) $gt \equiv 0 \pmod 2$ or $u = 0$;

(5) $g^2 t(t-1)/2 + gtu \equiv 0 \pmod 3$.

2 Ingredients

For convenience, let $V(K_n) = \{1, 2, \ldots, n\}$; $V(K_n \setminus K_v) = \{1, 2, \ldots, n\}$, where $V(K_v) = \{1, 2, \ldots, v\}$; $V(K_{i,j,k,l}) = X_1 \cup X_2 \cup X_3 \cup X_4$, where $X_1 = \{1, 2, \ldots, i\}$, $X_2 = i + \{1, 2, \ldots, j\}$, $X_3 = (i + j) + \{1, 2, \ldots, k\}$ and $X_4 = (i + j + k) + \{1, 2, \ldots, l\}$. Similarly, $V(K_{i,j})$ and $V(K_{i,j,k})$. When T_i-design of graph H exists, we have $b \in I_{T_i}(H)$, where b is the number of blocks in the T_i-design. It is easy to see that T_i-design of order n does not exist for $(i, n) = (2, 5), (3, 5), (4, 5), (4, 6)$.

2.1 Small orders of T_j-designs for $j = 1, 2, 3$

Lemma 2.1 $I_{T_1}(5) = J_{T_1}(5)$.

Proof: Let $\mathcal{A}$ and $\mathcal{B}$ be two T_1-designs of order 5, where $\mathcal{A} = \{(2,1,3,5,4)_1,(2,5,4,3,1)_1\}$ and $\mathcal{B} = \{(2,1,3,4,5)_1,(2,5,4,1,3)_1\}$. Then $|\mathcal{A} \cap \mathcal{B}| = 0$. From the result, we obtain $I_{T_1}(5) = J_{T_1}(5)$. ■

Lemma 2.2 *$I_{T_j}(6) = J_{T_j}(6)$ for $j = 1,2,3$.*

Proof: $j = 1$: Let $\mathcal{B}$ be a T_1-design of order 6, where $\mathcal{B} = \{(3,2,1,4,5)_1,(2,5,6,4,1)_1,(6,4,3,1,5)_1\}$. Then $|\mathcal{B} \cap \pi_i\mathcal{B}| = i$ for $i = 0,1$, where $\pi_0 = (6,5)$ and $\pi_1 = (3,5,4,6,2)$.

$j = 2$: Let $\mathcal{B}$ be a T_2-design of order 6, where $\mathcal{B} = \{(6,5,1,2,3)_2,(2,6,4,1,5)_2,(2,5,3,4,6)_2\}$. Then $|\mathcal{B} \cap \pi_i\mathcal{B}| = i$ for $i = 0,1$, where $\pi_0 = (4,5)$ and $\pi_1 = (6,5)$.

$j = 3$: Let $\mathcal{B}$ be a T_3-design of order 6, where $\mathcal{B} = \{(6,3,1,4,2)_3,(2,1,5,3,4)_3,(4,5,6,2,3)_3\}$. Then $|\mathcal{B} \cap \pi_i\mathcal{B}| = i$ for $i = 0,1$, where $\pi_0 = (6,5)$ and $\pi_1 = (5,4)$.

From the above results, we obtain $I_{T_j}(6) = J_{T_j}(6)$ for $j = 1,2,3$.■

Lemma 2.3 $0,1,8,9,12,15 \in I_{T_1}(K_{5,5,5})$ *and* $0,8,15 \in I_{T_j}(K_{5,5,5})$ *for* $j = 2,3$.

Proof: $j = 1$: Let $\mathcal{B}_1$ be a T_1-design of $K_{5,5,5}$, where $\mathcal{B}_1 = \{(11,9,3,14,12)_1, (13,9,5,12,14)_1, (15,6,5,4,8)_1, (15,7,4,3,9)_1, (15,8,3,2,10)_1, (15,9,2,1,6)_1, (15,10,1,5,7)_1\} \cup \{b_1, b_2, \ldots, b_8\}$, $b_1 = (11,6,1,14,2)_1$, $b_2 = (11,10,2,14,12)_1$, $b_3 = (11,8,4,14,12)_1$, $b_4 = (13,10,4,12,14)_1$, $b_5 = (11,7,5,14,12)_1$, $b_6 = (13,6,3,12,14)_1$, $b_7 = (13,7,2,12,14)_1$, $b_8 = (13,8,1,12,14)_1$. Then $|\mathcal{B}_1 \cap \pi_i\mathcal{B}_1| = i$ for $i = 0,1,8$, where $\pi_0 = (14,15)$, $\pi_1 = (1,2)(3,4)(6,7)$ and $\pi_8 = (1,2)$. Now, $\mathcal{B}_2$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_6, b_8\}$ and replacing them with $\{(1,6,12,11,8)_1, (6,3,14,13,1)_1, (8,1,13,11,6)_1\}$; $\mathcal{B}_3$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_2, b_3, b_4, b_5, b_7, b_8\}$ and replacing them with $\{(12,10,2,13,11)_1, (14,10,4,11,13)_1, (12,8,4,13,11)_1, (14,8,1,11,13)_1, (12,7,5,13,11)_1, (14,7,2,11,13)_1\}$. Then $|\mathcal{B}_1 \cap \mathcal{B}_2| = 12$ and $|\mathcal{B}_1 \cap \mathcal{B}_3| = 9$.

$j = 2$: Let $\mathcal{B}_1$ be a T_2-design of $K_{5,5,5}$, where $\mathcal{B}_1 = \{(11,8,4,12,9)_2, (11,7,5,12,8)_2, (13,8,1,14,9)_2, (13,10,4,14,6)_2, (13,9,5,14,10)_2, (15,1,10,12,14)_2, (15,2,9,12,14)_2, (15,3,8,12,14)_2\} \cup \{b_1, b_2, \ldots, b_7\}$, $b_1 = (11,10,2,12,6)_2$, $b_2 = (13,7,2,14,8)_2$, $b_3 = (11,6,1,12,7)_2$, $b_4 = (15,5,6,12,14)_2$, $b_5 = (15,4,7,12,14)_2$, $b_6 = (11,9,3,12,10)_2$, $b_7 = (13,6,3,14,7)_2$. Then $|\mathcal{B}_1 \cap \pi_0\mathcal{B}_1| = 0$, where $\pi_0 = (13,12)$. Now, $\mathcal{B}_2$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ and replacing them with $\{(11,1,6,12,14)_2, (5,6,15,7,4)_2, (1,11,7,4,14)_2, (11,10,2,6,8)_2, (13,7,2,12,14)_2, (11,9,3,7,10)_2, (13,6,3,12,14)_2\}$. Then $|\mathcal{B}_1 \cap \mathcal{B}_2| = 8$.

$j=3$: Let $\mathcal{B}$ be a T_3-design of $K_{5,5,5}$, where $\mathcal{B}=\{(11, 8, 4, 9, 12)_3, (13, 7, 2, 14, 8)_3, (15, 1, 10, 5, 12)_3, (15, 2, 9, 1, 12)_3, (15, 3, 8, 2, 12)_3, (15, 4, 7, 3, 12)_3, (11, 6, 1, 7, 12)_3, (13, 8, 1, 14, 9)_3, (11, 10, 2, 6, 12)_3, (13, 10, 4, 14, 6)_3, (15, 5, 6, 4, 12)_3, (11, 9, 3, 10, 12)_3, (13, 6, 3, 14, 7)_3, (11, 7, 5, 8, 12)_3, (13, 9, 5, 14, 10)_3\}$. Then $|\mathcal{B}\cap\pi_i\mathcal{B}|=i$ for $i=0,8$, where $\pi_0=(13,12)$ and $\pi_8=(5,4)$.

From above results, we have $0,1,8,9,12,15\in I_{T_1}(K_{5,5,5})$ and $0,8,15\in I_{T_j}(K_{5,5,5})$ for $j=2,3$.■

Lemma 2.4 $0,1,30\in I_{T_j}(K_{5,5,5,5})$ *for* $j=1,2,3$.

Proof: $j=1$: Let $\mathcal{B}$ be a T_1-design of $K_{5,5,5,5}$, where $\mathcal{B}=\{(10, 11, 2, 4, 20)_1, (7, 13, 2, 5, 18)_1, (8, 16, 4, 2, 20)_1, (10, 18, 4, 6, 19)_1, (9, 18, 5, 12, 19)_1, (17, 10, 13, 12, 19)_1, (14, 3, 16, 18, 10)_1, (7, 4, 17, 14, 11)_1, (16, 13, 9, 8, 3)_1, (7, 18, 15, 13, 19)_1, (10, 5, 14, 8, 9)_1, (6, 16, 15, 12, 20)_1, (15, 1, 10, 6, 19)_1, (3, 20, 7, 10, 14)_1, (5, 20, 6, 12, 19)_1, (8, 17, 12, 5, 2)_1, (11, 9, 20, 4, 8)_1, (6, 13, 3, 4, 10)_1, (9, 15, 2, 5, 17)_1, (14, 19, 1, 7, 16)_1, (3, 19, 8, 2, 15)_1, (3, 15, 17, 4, 1)_1, (13, 20, 1, 14, 18)_1, (16, 11, 7, 5, 1)_1, (4, 6, 12, 2, 1)_1, (7, 5, 12, 16, 3)_1, (1, 11, 8, 3, 2)_1, (12, 19, 9, 11, 1)_1, (8, 18, 14, 11, 2)_1, (14, 17, 6, 9, 11)_1\}$. Then $|\mathcal{B}\cap\pi_i\mathcal{B}|=i$ for $i=0,1$, where $\pi_0=(8,7)(13,12)(18,17)(20,19)$ and $\pi_1=(9,8)(13,12)(18,17)$ $(20,19)$.

$j=2$: Let $\mathcal{B}$ be a T_2-design of $K_{5,5,5,5}$, where $\mathcal{B}=\{(6, 11, 1, 12, 8)_2, (10, 11, 2, 6, 18)_2, (7, 13, 2, 14, 8)_2, (8, 16, 4, 9, 20)_2, (10, 18, 4, 6, 19)_2, (9, 18, 5, 10, 19)_2, (18, 8, 11, 4, 19)_2, (19, 9, 12, 16, 5)_2, (17, 10, 13, 4, 19)_2, (14, 3, 16, 2, 10)_2, (7, 4, 17, 5, 11)_2, (16, 13, 9, 17, 3)_2, (6, 17, 14, 4, 18)_2, (7, 18, 15, 4, 19)_2, (10, 20, 14, 5, 9)_2, (6, 16, 15, 5, 20)_2, (1, 15, 10, 12, 19)_2, (3, 20, 7, 19, 14)_2, (5, 20, 6, 12, 19)_2, (3, 12, 18, 13, 6)_2, (8, 17, 12, 4, 2)_2, (11, 9, 20, 12, 8)_2, (6, 13, 3, 11, 10)_2, (15, 9, 2, 19, 20)_2, (14, 19, 1, 9, 16)_2, (3, 19, 8, 14, 15)_2, (3, 15, 17, 1, 2)_2, (13, 20, 1, 7, 18)_2, (8, 13, 5, 11, 16)_2, (16, 11, 7, 5, 12)_2\}$. Then $|\mathcal{B}\cap\pi_i\mathcal{B}|=i$ for $i=0,1$, where $\pi_0=(13,12)(15,14)(18,17)(20,19)$ and $\pi_1=(13,12)(15,14)(19,20,18)$.

$j=3$: Let $\mathcal{B}$ be a T_3-design of $K_{5,5,5,5}$, where $\mathcal{B}=\{(6, 11, 1, 7, 12)_3, (10, 11, 2, 17, 5)_3, (7, 13, 2, 14, 5)_3, (8, 16, 4, 11, 7)_3, (10, 18, 4, 20, 15)_3, (9, 5, 18, 6, 4)_3, (19, 9, 12, 2, 6)_3, (17, 10, 13, 4, 9)_3, (7, 4, 17, 1, 8)_3, (16, 13, 9, 1, 12)_3, (14, 17, 6, 12, 4)_3, (7, 18, 15, 19, 4)_3, (10, 20, 14, 18, 1)_3, (6, 16, 15, 5, 11)_3, (1, 15, 10, 3, 9)_3, (3, 20, 7, 19, 2)_3, (5, 20, 6, 19, 13)_3, (3, 18, 12, 16, 10)_3, (12, 17, 8, 14, 4)_3, (11, 9, 20, 12, 10)_3, (3, 6, 13, 18, 2)_3, (15, 9, 2, 20, 8)_3, (1, 19, 14, 9, 17)_3, (3, 19, 8, 15, 4)_3, (3, 15, 17, 11, 3)_3, (13, 20, 1, 16, 7)_3, (8, 13, 5, 7, 14)_3, (5, 10, 19, 11, 16)_3, (18, 11, 8, 2, 16$)_3$, (3, 14, 16, 5, 12$)_3$}. Then $|\mathcal{B} \cap \pi_i\mathcal{B}| = i$ for $i = 0, 1$, where $\pi_0 = (20,1)(19,2)(18,3)(17,4)(16,5)(15,6)(14,7)(13,8)(12,9)(11,10)$ and $\pi_1 = (9,10,8)(15,14)(19,20,18)$.

From above results, we have $0, 1, 30 \in I_{T_j}(K_{5,5,5,5})$ for $j = 1, 2, 3$.■

Lemma 2.5 $0, 7 \in I_{T_j}(K_{10} \setminus K_5)$ *for* $j = 1, 2, 3$.

Proof: $j = 1$: Let $\mathcal{B}$ be a T_1-design of $K_{10} \setminus K_5$, where $\mathcal{B} = \{(8,6,1,9,7)_1, (2,9,10,4,7)_1, (3,6,10,4,5)_1, (4,8,10,7,1)_1, (9,7,3,4,8)_1, (8,9,5,1,6)_1, (6,7,2,5,8)_1\}$ Then $|\mathcal{B} \cap \pi_0\mathcal{B}| = 0$, where $\pi_0 = (1,2)(4,5)$.

$j = 2$: Let $\mathcal{B}$ be a T_2-design of $K_{10} \setminus K_5$, where $\mathcal{B} = \{$(9, 10, 1, 6, 8$)_2$, (2, 9, 7, 10, 1$)_2$, (3, 10, 8, 7, 4$)_2$, (4, 6, 7, 3, 5$)_2$, (5, 6, 8, 9, 2$)_2$, (2, 6, 10, 4, 5$)_2$, (3, 6, 9, 4, 5$)_2$}. Then $|\mathcal{B} \cap \pi_0\mathcal{B}_1| = 0$, where $\pi_0 = (8,7)(10,9)$.

$j = 3$: Let $\mathcal{B}$ be a T_3-design of $K_{10} \setminus K_5$, where $\mathcal{B} = \{$(7, 1, 6, 3, 10$)_3$, (2, 8, 7, 9, 1$)_3$, (3, 8, 9, 5, 7$)_3$, (4, 9, 10, 8, 5$)_3$, (5, 6, 10, 1, 8$)_3$, (6, 9, 2, 10, 7$)_3$, (8, 6, 4, 7, 3$)_3$}. Then $|\mathcal{B} \cap \pi_0\mathcal{B}| = 0$, where $\pi_0 = (9,10,8)$.

From above results, we have $0, 7 \in I_{T_j}(K_{10} \setminus K_5)$ for $j = 1, 2, 3$.■

Lemma 2.6 $0, 2, 3, 6 \in I_{T_j}(K_{10} \setminus K_6)$ *for* $j = 1, 2, 3$.

Proof: $j = 1$: Let $\mathcal{B}$ be a T_1-design of $K_{10} \setminus K_6$, where $\mathcal{B} = \{(10,1,7,9,3)_1, (7,2,8,10,4)_1, (9,3,8,10,1)_1, (10,4,9,7,5)_1, (8,5,10,7,6)_1, (7,6,9,8,2)_1\}$. Then $|\mathcal{B} \cap \pi_i\mathcal{B}| = i$ for $i = 0, 2, 3$, where $\pi_0 = (10,9)$, $\pi_2 = (5,6,4)$ and $\pi_3 = (6,5)$.

$j = 2$: Let $\mathcal{B}$ be a T_2-design of $K_{10} \setminus K_6$, where $\mathcal{B} = \{(6,9,10,5,4)_2, (5,8,9,4,3)_2, (4,7,8,3,6)_2, (3,10,7,6,5)_2, (10,8,2,9,7)_2, (9,7,1,10,8)_2\}$, Then $|\mathcal{B} \cap \pi_i\mathcal{B}| = i$ for $i = 0, 2, 3$, where $\pi_0 = (9,10,8)$, $\pi_2 = (10,8)$ and $\pi_3 = (6,5)$.

$j = 3$: Let $\mathcal{B}_1$ be a T_3-design of $K_{10} \setminus K_6$, where $\mathcal{B}_1 = \{(6,9,10,5,7)_3, (4,7,8,3,9)_3, (3,10,7,2,9)_3, (1,7,9,4,10)_3, (5,9,8,1,10)_3, (2,10,8,6,7)_3\}$. Then $|\mathcal{B}_1 \cap \pi_i\mathcal{B}_1| = i$ for $i = 0, 2, 3$, where $\pi_0 = (10,9)$, $\pi_2 = (5,4)$ and $\pi_3 = (6,5)$.

From above results, we have $0, 2, 3, 6 \in I_{T_j}(K_{10} \setminus K_6)$ for $j = 1, 2, 3$.■

Lemma 2.7 $I_{T_j}(10) = J_{T_j}(10)$ *for* $j = 1, 2, 3$.

Proof: The graph K_{10} can be regarded as a union of a copy of $K_{10} \setminus K_6$, and a copy K_6. Therefore, we have

$$I_{T_j}(K_{10}) \supseteq I_{T_j}(K_{10} \setminus K_6) + I_{T_j}(K_6) \supseteq \{0, 2, 3, 6\} + J_{T_j}(K_6) = J_{T_j}(K_{10}).$$■

Lemma 2.8 $0, 1, 3, 4, 8 \in I_{T_j}(K_{11} \setminus K_6)$ *for* $j = 1, 2, 3$.

Proof: $j = 1$: Let $\mathcal{B}_1$ be a T_1-design of $K_{11} \setminus K_6$, where $\mathcal{B}_1 = \{(1,10,11,5,6)_1, (10,8,2,3,9)_1, (11,9,3,5,10)_1, (9,10,4,6,7)_1, (7,8,1,4,9)_1, (11,8,5,9,7)_1, (9,7,6,10,8)_1, (2,7,11,3,4)_1\}$. Then $|\mathcal{B}_1 \cap \pi_i \mathcal{B}_1| = i$ for $i = 0,1,3,4$, where $\pi_0 = (6,1)(4,5,3)$, $\pi_1 = (4,5,3)$, $\pi_3 = (4,3)$ and $\pi_4 = (4,2)$.

$j = 2$: Let $\mathcal{B}_1$ be a T_2-design of $K_{11} \setminus K_6$, where $\mathcal{B}_1 = \{(4,8,9,6,7)_2, (11,2,9,3,1)_2, (5,11,7,6,4)_2, (8,3,11,4,1)_2, (10,1,8,6,5)_2, (2,8,7,3,1)_2, (11,6,10,4,2)_2, (9,5,10,7,3)_2\}$. Then $|\mathcal{B}_1 \cap \pi_i \mathcal{B}_1| = i$ for $i = 0,1,3,4$, where $\pi_0 = (9,8)(11,10)$, $\pi_1 = (10,11,9)$, $\pi_3 = (10,9)$ and $\pi_4 = (6,5)$.

$j = 3$: Let $\mathcal{B}_1$ be a T_3-design of $K_{11} \setminus K_6$, where $\mathcal{B}_1 = \{(11,10,6,7,4)_3, (10,9,5,8,6)_3, (9,8,4,11,1)_3, (11,8,3,7,1)_3, (11,9,2,10,4)_3, (10,8,1,9,3)_3, (5,11,7,9,6)_3, (2,8,7,10,3)_3\}$. Then $|\mathcal{B}_1 \cap \pi_i \mathcal{B}_1| = i$ for $i = 0,1,3,4$, where $\pi_0 = (10,11,9)$, $\pi_1 = (11,10)$, $\pi_3 = (5,4)$ and $\pi_4 = (3,2)$.

From above results, we have $0,1,3,4,8 \in I_{T_j}(K_{11} \setminus K_6)$ for $j = 1,2,3$.■

Lemma 2.9 $I_{T_j}(11) = J_{T_j}(11)$ *for* $j = 1,2,3$.

Proof: The graph K_{11} can be regarded as a union of a copy of $K_{11} \setminus K_6$ and a copy of K_6. Therefore, we have

$$I_{T_j}(K_{11}) \supseteq I_{T_j}(K_{11} \setminus K_6) + I_{T_j}(K_6) \supseteq \{0,1,3,4,8\} + J_{T_j}(K_6) = J_{T_j}(K_{11}).\blacksquare$$

Lemma 2.10 $0,19 \in I_{T_j}(K_{15} \setminus K_5)$ *for* $j = 1,2,3$.

Proof: $j = 1$: The graph $K_{15} \setminus K_5$ can be regarded as a union of a copy of tripartite graph $K_{5,5,5}$ and 2 copies of K_5. Therefore, we have

$$I_{T_1}(K_{15} \setminus K_5) \supseteq I_{T_1}(K_{5,5,5}) + 2 \cdot I_{T_1}(K_5) \supseteq \{0,15\} + 2 \cdot \{0,2\} \supseteq \{0,19\}.$$

$j = 2$: Let $\mathcal{B}$ be a T_2-design of $K_{15} \setminus K_5$, where $\mathcal{B} = \{(11, 1, 6, 2, 12)_2, (6, 3, 13, 14, 11)_2, (15, 8, 3, 12, 10)_2, (12, 9, 2, 15, 14)_2, (12, 13, 15, 1, 10)_2, (6, 8, 7, 10, 9)_2, (11, 15, 14, 9, 3)_2, (4, 9, 6, 10, 14)_2, (1, 14, 12, 10, 11)_2, (11, 5, 7, 1, 12)_2, (13, 2, 7, 14, 3)_2, (11, 4, 8, 5, 12)_2, (15, 6, 5, 12, 14)_2, (15, 7, 4, 12, 14)_2, (11, 3, 9, 15, 8)_2, (13, 5, 9, 10, 1)_2, (11, 2, 10, 1, 8)_2, (13, 4, 10, 14, 5)_2, (13, 1, 8, 14, 2)_2\}$. Then $|\mathcal{B} \cap \pi_0 \mathcal{B}| = 0$, where $\pi_0 = (11,10)(14,13)$.

$j = 3$: Let $\mathcal{B}$ be a T_3-design of $K_{15} \setminus K_5$, where $\mathcal{B} = \{(11, 5, 7, 1, 12)_3, (13, 4, 10, 6, 9)_3, (6, 5, 15, 11, 13)_3, (15, 7, 4, 6, 12)_3, (15, 9, 2, 8, 12)_3, (10, 1, 15, 12, 9)_3, (6, 8, 7, 9, 10)_3, (15, 14, 13, 12, 11)_3, (9, 8, 5, 14, 11)_3, (12, 5, 10, 14, 2)_3, (11, 1, 6, 2, 12)_3, (13, 3, 6, 14, 4)_3, (11, 2, 10, 3, 12)_3, (13, 2, 7, 14, 3)_3, (15, 8, 3, 7, 12)_3, (11, 4, 8, 10, 7)_3, (13, 1, 8, 14, 12)_3, (11, 3, 9, 4, 12)_3, (14, 1, 9, 13, 5)_3\}$. Then $|\mathcal{B} \cap \pi_0 \mathcal{B}| = 0$, where $\pi_0 = (15,1)(14,2)(13,3)(12,4)(11,5)(10,6)(9,7)$.

From above results, we have $0,19 \in I_{T_j}(K_{15} \setminus K_5)$ for $j = 1,2,3$.■

Lemma 2.11 $I_{T_j}(15) = J_{T_j}(15)$ *for* $j = 1, 2, 3$.

Proof: $j = 1$: The graph K_{15} can be regarded as a union of a copy of $K_{5,5,5}$ and 3 copies of K_5. Therefore, we have

$$I_{T_1}(15) \supseteq I_{T_1}(K_{5,5,5}) + 3 \cdot I_{T_1}(K_5) \supseteq \{0, 1, 8, 9, 12, 15\} + 3 \cdot \{0, 2\} = J_{T_1}(15).$$

$j = 2$: Let $\mathcal{B}_1$ be a T_2-design of K_{15}, where $\mathcal{B}_1 = \{(12, 8, 5, 11, 7)_2, (13, 9, 5, 14, 10)_2, (3, 10, 1, 15, 5)_2, (8, 11, 9, 2, 15)_2, (8, 15, 3, 2, 12)_2, (12, 14, 13, 11, 15)_2, (15, 12, 11, 6, 14)_2, (1, 4, 2, 5, 15)_2, (7, 8, 10, 11, 14)_2, (6, 10, 9, 7, 14)_2\} \cup \{b_1, b_2, \ldots, b_{11}\}$, $b_1 = (7, 11, 1, 12, 6)_2$, $b_2 = (13, 8, 1, 14, 9)_2$, $b_3 = (15, 4, 7, 12, 14)_2$, $b_4 = (14, 8, 6, 7, 12)_2$, $b_5 = (5, 6, 15, 14, 10)_2$, $b_6 = (12, 10, 2, 6, 11)_2$, $b_7 = (13, 7, 2, 14, 8)_2$, $b_8 = (4, 5, 3, 11, 9)_2$, $b_9 = (13, 6, 3, 14, 7)_2$, $b_{10} = (12, 9, 4, 11, 8)_2$ and $b_{11} = (13, 10, 4, 6, 14)$. Then $|\mathcal{B}_1 \cap \pi_i \mathcal{B}_1| = i$ for $i = 0, 1, \ldots, 12$, where $\pi_0 = (12, 11)(15, 14)$, $\pi_1 = (13, 12)(15, 14)$, $\pi_2 = (12, 11)(14, 13)$, $\pi_3 = (13, 15, 12)$, $\pi_4 = (13, 14, 12)$, $\pi_5 = (15, 14)$, $\pi_6 = (14, 12)$, $\pi_7 = (13, 12)$, $\pi_8 = (15, 13)$, $\pi_9 = (14, 13)$, $\pi_{10} = (10, 9)$, $\pi_{11} = (13, 10)$ and $\pi_{12} = (13, 9)$. Now, $\mathcal{B}_2$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_2\}$ and replacing them with $\{(11, 7, 1, 6, 9)_2, (13, 8, 1, 12, 14)_2\}$; $\mathcal{B}_3$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_3, b_4, b_5\}$ and replacing them with $\{(4, 7, 15, 5, 10)_2, (8, 6, 14, 7, 15)_2, (12, 7, 6, 5, 15)_2\}$; $\mathcal{B}_4$ comes from $\mathcal{B}_2$ by removing the blocks $\{b_6, b_7\}$ and replacing them with $\{(12, 10, 2, 6, 8)_2, (13, 7, 2, 11, 14)_2\}$; $\mathcal{B}_5$ comes from $\mathcal{B}_3$ by removing the blocks $\{b_6, b_7\}$ and replacing them with $\{(12, 10, 2, 6, 8)_2, (13, 7, 2, 11, 14)_2\}$; $\mathcal{B}_6$ comes from $\mathcal{B}_4$ by removing the blocks $\{b_8, b_9\}$ and replacing them with $\{(4, 5, 3, 9, 7)_2, (13, 6, 3, 11, 14)_2\}$; $\mathcal{B}_7$ comes from $\mathcal{B}_5$ by removing the blocks $\{b_8, b_9\}$ and replacing them with $\{(4, 5, 3, 9, 7)_2, (13, 6, 3, 11, 14)_2\}$; $\mathcal{B}_8$ comes from $\mathcal{B}_6$ by removing the blocks $\{b_{10}, b_{11}\}$ and replacing them with $\{(12, 9, 4, 8, 6)_2, (13, 10, 4, 11, 14)_2\}$. Then $|\mathcal{B}_1 \cap \mathcal{B}_i| = 21 - i$ for $i = 2, 3, \ldots, 8$.

$j = 3$: Let $\mathcal{B}_1$ be a T_3-design of K_{15}, where $\mathcal{B}_1 = \{(11, 10, 2, 6, 12)_3, (11, 8, 4, 9, 7)_3, (14, 4, 6, 8, 10)_3, (12, 1, 9, 2, 15)_3, (15, 4, 7, 3, 12)_3, (1, 3, 2, 4, 5)_3, (15, 14, 13, 12, 11)_3, (3, 10, 14, 12, 15)_3, (6, 10, 7, 8, 9)_3, (13, 4, 10, 9, 14)_3\} \cup \{b_1, b_2, \ldots, b_{11}\}$, $b_1 = (11, 6, 1, 7, 12)_3$, $b_2 = (13, 8, 1, 14, 11)_3$, $b_3 = (11, 7, 5, 8, 12)_3$, $b_4 = (13, 7, 2, 14, 8)_3$, $b_5 = (15, 3, 8, 2, 12)_3$, $b_6 = (11, 9, 3, 5, 2)_3$, $b_7 = (13, 6, 3, 4, 12)_3$, $b_8 = (13, 9, 5, 1, 4)_3$, $b_9 = (12, 10, 5, 14, 7)_3$, $b_{10} = (1, 10, 15, 11, 13)_3$ and $b_{11} = (5, 6, 15, 9, 6)_3$. Then $|\mathcal{B}_1 \cap \pi_i \mathcal{B}_1| = i$ for $i = 0, 1, \ldots, 11$, where $\pi_0 = (15, 1)(14, 2)(13, 3)(12, 4)(11, 5)(10, 6)(9, 7)$, $\pi_1 = (11, 10)(13, 12)(15, 14)$, $\pi_2 = (12, 11)(14, 15, 13)$, $\pi_3 = (12, 11)(15, 14)$, $\pi_4 = (13, 12)(15, 14)$, $\pi_5 = (13, 14, 12)$, $\pi_6 = (13, 12)$, $\pi_7 = (14, 15, 13)$, $\pi_8 = (14, 11)$, $\pi_9 = (15, 13)$, $\pi_{10} = (15, 14)$, and $\pi_{11} = (14, 13)$. Now, $\mathcal{B}_2$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_2\}$ and replacing them

with $\{(11,6,1,14,11),(13,8,1,7,12)\}$; $\mathcal{B}_3$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_3,b_4,b_5\}$ and replacing them with $\{(12,2,8,5,7)_3,\ (13,2,7,11,5)_3,\ (15,3,8,14,2)_3\}$; $\mathcal{B}_4$ comes from $\mathcal{B}_2$ by removing the blocks $\{b_6,b_7\}$ and replacing them with $\{(11,9,3,4,12)_3,(13,6,3,5,2)_3\}$; $\mathcal{B}_5$ comes from $\mathcal{B}_3$ by removing the blocks $\{b_1,b_2\}$ and replacing them with $\{(11,6,1,14,11)_3,(13,8,1,7,12)_3\}$; $\mathcal{B}_6$ comes from $\mathcal{B}_4$ by removing the blocks $\{b_8,b_9\}$ and replacing them with $\{(13,9,5,14,7)_3,(12,10,5,1,4)_3\}$; $\mathcal{B}_7$ comes from $\mathcal{B}_5$ by removing the blocks $\{b_6,b_7\}$ and replacing them with $\{(11,9,3,4,12)_3,(13,6,3,5,2)_3\}$; $\mathcal{B}_8$ comes from $\mathcal{B}_6$ by removing the blocks $\{b_{10},b_{11}\}$ and replacing them with $\{(1,10,15,9,6)_3,(5,6,15,11,13)_3\}$; $\mathcal{B}_9$ comes from $\mathcal{B}_7$ by removing the blocks $\{b_8,b_9\}$ and replacing them with $\{(13,9,5,14,7)_3,(12,10,5,1,4)_3\}$. Then $|\mathcal{B}_1 \cap \mathcal{B}_i| = 21 - i$ for $i = 2,3,\ldots,9$.
From above results, we have $I_{T_j}(K_{15}) = J_{T_j}(K_{15})$ for $j = 1,2,3$.■

Lemma 2.12 $0,21 \in I_{T_j}(K_{16} \setminus K_6)$ *for* $j = 1,2,3$.

Proof: The graph $K_{16} \setminus K_6$ can be regarded as a union of a copy of $K_{5,5,5}$ and 2 copies of K_6. Therefore, we have

$$I_{T_j}(K_{16} \setminus K_6) \supseteq I_{T_j}(K_{5,5,5}) + 2 \cdot I_{T_j}(K_6) \supseteq \{0,15\} + 2 \cdot J_{T_j}(K_6) \supseteq \{0,21\}.■$$

Lemma 2.13 $I_{T_j}(16) = J_{T_j}(16)$ *for* $j = 1,2,3$.

Proof: The graph K_{16} can be regarded as a union of a copy of $K_{5,5,5}$ and 3 copies of K_6. Therefore, we have

$$I_{T_j}(K_{16}) \supseteq I_{T_j}(K_{5,5,5}) + 3 \cdot I_{T_j}(K_6) \supseteq \{0,8,15\} + 3 \cdot J_{T_j}(K_6) = J_{T_j}(K_{16}).■$$

Lemma 2.14 $I_{T_j}(20) = J_{T_j}(20)$ *and* $I_{T_j}(21) = J_{T_j}(21)$ *for* $j = 1,2,3$.

Proof: $I_{T_j}(K_{20}) \supseteq I_{T_j}(K_{5,5,5}) + 2 \cdot I_{T_j}(K_{10} \setminus K_5) + I_{T_j}(K_{10}) \supseteq \{0,15\} + 2 \cdot \{0,7\} + J_{T_j}(K_{10}) = J_{T_j}(20)$.
$I_{T_j}(21) \supseteq I_{T_j}(K_{5,5,5}) + 2 \cdot I_{T_j}(K_{11} \setminus K_6) + I_{T_j}(K_{11}) \supseteq \{0,15\} + 2 \cdot \{0,8\} + J_{T_j}(K_{11}) = J_{T_j}(21)$.■

Lemma 2.15 $I_{T_j}(25) = J_{T_j}(25)$ *and* $I_{T_j}(26) = J_{T_j}(26)$ *for* $j = 1,2,3$.

Proof: $I_{T_j}(25) \supseteq I_{T_j}(K_{5,5,5,5}) + 3 \cdot I_{T_j}(K_{10} \setminus K_5) + I_{T_j}(K_{10}) \supseteq \{0,1,30\} + 3 \cdot \{0,7\} + J_{T_j}(K_{10}) = J_{T_j}(25)$.
$I_{T_j}(26) \supseteq I_{T_j}(K_{5,5,5,5}) + 3 \cdot I_{T_j}(K_{11} \setminus K_6) + I_{T_j}(K_{11}) \supseteq \{0,30\} + 3 \cdot \{0,8\} + J_{T_j}(K_{11}) = J_{T_j}(26)$.■

2.2 Small orders of T_4-design

Lemma 2.16 $0,5 \in I_{T_4}(K_{5,5})$, $0,6 \in I_{T_4}(K_{5,6})$ *and* $0,15 \in I_{T_4}(K_{5,5,5})$.

Proof:

[$K_{5,5}$] Let $\mathcal{B}$ be a T_4-design of $K_{5,5}$, where $\mathcal{B} = \{(6,2,9,3,10)_4, (7,3,8,4,6)_4, (10,1,8,2,7)_4, (10,4,9,5,8)_4, (7,5,6,1,9)_4\}$. Then $|\mathcal{B} \cap \pi_0\mathcal{B}| = 0$, where $\pi_0 = (9,8)$.

[$K_{5,6}$] Let $\mathcal{B}$ be a T_4-design of $K_{5,6}$, where $\mathcal{B} = \{(1,7,2,6,3)_4, (3,8,4,9,1)_4, (6,5,7,4,10)_4, (11,2,10,3,7)_4, (2,9,5,8,1)_4, (1,10,5,11,4)_4\}$. Then $|\mathcal{B} \cap \pi_0\mathcal{B}| = 0$, where $\pi_0 = (8,7)(11,10)$.

[$K_{5,5,5}$] The graph $K_{5,5,5}$ can be regarded as a union of 3 copies of $K_{5,5}$. Therefore, we have

$$I_{T_4}(K_{5,5,5}) \supseteq 3 \cdot I_{T_4}(K_{5,5}) \supseteq 3 \cdot \{0,5\} \supseteq \{0,15\}. \blacksquare$$

Lemma 2.17 $0,7 \in I_{T_4}(K_{10} \setminus K_5)$.

Proof: Let $\mathcal{B}$ be a T_4-design of $K_{10} \setminus K_5$, where $\mathcal{B} = \{(6,2,9,3,10)_4, (10,8,3,7,4)_4, (10,1,8,2,7)_4, (10,4,9,5,8)_4, (7,5,6,1,9)_4, (10,9,7,6,8)_4, (4,6,9,8,7)_4\}$. Then $|\mathcal{B} \cap \pi_0\mathcal{B}| = 0$, where $\pi_0 = (10,9)$. ■

Lemma 2.18 $I_{T_4}(10) = J_{T_4}(10)$.

Proof: Let $\mathcal{B}_1$ be a T_4-design of K_{10}, where $\mathcal{B}_1 = \{(4,6,9,5,10)_4, (6,8,2,7,10)_4, (7,9,3,8,10)_4, (8,1,4,9,10)_4\} \cup \{b_1, b_2, \ldots, b_5\}$, $b_1 = (1,3,6,2,10)_4$, $b_2 = (2,4,7,3,10)_4$, $b_3 = (3,5,8,4,10)_4$, $b_4 = (5,7,1,6,10)_4$ and $b_5 = (9,2,5,1,10)_4$. Then $|\mathcal{B}_1 \cap \pi_i\mathcal{B}_1| = i$ for $i = 0,1,2,3$, where $\pi_0 = (10,9)$, $\pi_1 = (8,9,7)$, $\pi_2 = (9,7)$ and $\pi_3 = (9,8)$. Now, $\mathcal{B}_2$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_5\}$ and replacing them with $\{(3,6,2,1,9)_4, (5,1,10,2,9)_4\}$; $\mathcal{B}_3$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_4, b_5\}$ and replacing them with $\{(2,9,1,6,10)_4, (1,7,5,2,10)_4, (3,6,5,1,10)_4\}$; $\mathcal{B}_4(\mathcal{B}_5)$ comes from $\mathcal{B}_2(\mathcal{B}_3)$ by removing the blocks $\{b_2, b_3\}$ and replacing them with $\{(7,3,10,4,2)_4, (5,8,4,3,2)_4\}$. Then $|\mathcal{B}_1 \cap \mathcal{B}_i| = 9-i$ for $i = 2,3,4,5$. From those results, we have $I_{T_4}(10) = J_{T_4}(10)$. ■

Lemma 2.19 $I_{T_4}(11) = J_{T_4}(11)$.

Proof: Let $\mathcal{B}_1$ be a T_4-design of K_{11}, where $\mathcal{B}_1 = \{(8,10,2,9,3)_4, (9,11,3,10,4)_4, (10,1,4,11,5)_4\} \cup \{b_1, b_2, \ldots, b_8\}$, $b_1 = (1,3,6,2,7)_4$, $b_2 = (2,4,7,3,$

8)$_4$, $b_3 = (3,5,8,4,9)_4$, $b_4 = (4,6,9,5,10)_4$, $b_5 = (5,7,10,6,11)_4$, $b_6 = (6,8,11,7,\ 1)_4$, $b_7 = (7,9,1,8,2)_4$ and $b_8 = (11,2,5,1,6)_4$. Then $|\mathcal{B}_1 \cap \pi_i\mathcal{B}_1| = i$ for $i = 0,1,2,3,4$, where $\pi_0 = (8,7)(11,10)$, $\pi_1 = (9,8)(11,10)$, $\pi_2 = (10,11,9)$, $\pi_3 = (11,8)$ and $\pi_4 = (11,10)$. Now, $\mathcal{B}_2$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_5\}$ and replacing them with $\{(3,\ 1,\ 2,\ 6,\ 11)_4, (10,\ 6,\ 5,\ 7,\ 2)_4\}$; $\mathcal{B}_3$ comes from $\mathcal{B}_1$ by removing the blocks $\{b_1, b_6, b_8\}$ and replacing them with $\{(1,3,6,2,11)_4, (1,5,2,7,11)_4, (1,11,8,6,7)_4\}$; $\mathcal{B}_4$ comes from $\mathcal{B}_2$ by removing the blocks $\{b_2, b_6\}$ and replacing them with $\{(4,2,3,7,1)_4, (11,7,6,\ 8,3)_4\}$; $\mathcal{B}_5(\mathcal{B}_6)$ comes from $\mathcal{B}_3(\mathcal{B}_4)$ by removing the blocks $\{b_3, b_7\}$ and replacing them with $\{(5,3,4,8,2)_4, (1,8,7,9,4)_4\}$. Then $|\mathcal{B}_1 \cap \mathcal{B}_i| = 11 - i$ for $i = 2,3,\ldots,6$. From those results, we have $I_{T_4}(11) = J_{T_4}(11)$. ■

Lemma 2.20 $0, 19 \in I_{T_4}(K_{15} \setminus K_5)$ *and* $0, 21 \in I_{T_4}(K_{16} \setminus K_6)$.

Proof: The graph $K_{15} \setminus K_5$ can be regarded as a union of 2 copies of $K_{5,5}$ and a copy of K_{10}. Therefore, we have

$$I_{T_4}(K_{15} \setminus K_5) \supseteq 2 \cdot I_{T_4}(K_{5,5}) + I_{T_4}(K_{10}) \supseteq 2 \cdot \{0,5\} + \{0,9\} \supseteq \{0,19\}.$$

The graph $K_{16} \setminus K_6$ can be regarded as a union of 2 copies of $K_{5,6}$ and a copy of K_{10}. Therefore, we have

$$I_{T_4}(K_{16} \setminus K_6) \supseteq 2 \cdot I_{T_4}(K_{5,6}) + I_{T_4}(K_{10}) \supseteq 2 \cdot \{0,6\} + \{0,9\} \supseteq \{0,21\}. ■$$

Lemma 2.21 $I_{T_4}(15) = J_{T_4}(15)$ *and* $I_{T_4}(16) = J_{T_4}(16)$.

Proof: The graph K_{15} can be regarded as a union of a copy of $K_{10} \setminus K_5$, a copy of $K_{5,5}$ and and a copy of K_{10}. Therefore, we have $I_{T_4}(K_{15}) \supseteq I_{T_4}(K_{5,5}) + I_{T_4}(K_{10} \setminus K_5) + I_{T_4}(K_{10}) \supseteq \{0,5\} + \{0,7\} + J_{T_4}(K_{10}) = J_{T_4}(K_{15})$.

The graph K_{16} can be regarded as a union of a copy of $K_{10} \setminus K_5$, a copy of $K_{5,6}$ and a copy of K_{11}. Therefore, we have $I_{T_4}(K_{16}) \supseteq I_{T_4}(K_{5,6}) + I_{T_4}(K_{10} \setminus K_5) + I_{T_4}(K_{11}) \supseteq \{0,6\} + \{0,7\} + J_{T_4}(K_{11}) = J_{T_4}(K_{16})$. ■

Lemma 2.22 $I_{T_4}(20) = J_{T_4}(20)$ *and* $I_{T_4}(21) = J_{T_4}(21)$.

Proof: The graph K_{20} can be regarded as a union of 4 copies of $K_{5,5}$ and 2 copies of K_{10}. Therefore, we have

$$I_{T_4}(K_{20}) \supseteq 4 \cdot I_{T_4}(K_{5,5}) + 2 \cdot I_{T_4}(K_{10}) \supseteq 4 \cdot \{0,5\} + 2 \cdot J_{T_4}(K_{10}) = J_{T_4}(K_{20}).$$

The graph K_{21} can be regarded as a union of 4 copies of $K_{5,5}$ and 2 copies of K_{11}. Therefore, we have

$$I_{T_4}(K_{20}) \supseteq 4 \cdot I_{T_4}(K_{5,5}) + 2 \cdot I_{T_4}(K_{11}) \supseteq 4 \cdot \{0,5\} + 2 \cdot J_{T_4}(K_{11}) = J_{T_4}(K_{21}). ■$$

Lemma 2.23 *$I_{T_4}(25) = J_{T_4}(25)$ and $I_{T_4}(26) = J_{T_4}(26)$.*

Proof: The graph K_{25} can be regarded as a union of 6 copies of $K_{5,5}$, a copy of K_{10} and a copy of K_{15}. Therefore, we have $I_{T_4}(K_{25}) \supseteq 6 \cdot I_{T_4}(K_{5,5}) + I_{T_4}(K_{10}) + I_{T_4}(K_{15}) \supseteq 6 \cdot \{0,5\} + J_{T_4}(K_{10}) + J_{T_4}(K_{15}) = J_{T_4}(K_{25})$.

The graph K_{26} can be regarded as a union of 4 copies of $K_{5,5}$, 2 copies of $K_{5,6}$, a copy of K_{10} and a copy of K_{16}. Therefore, we have $I_{T_4}(K_{26}) \supseteq 4 \cdot I_{T_4}(K_{5,5}) + 2 \cdot I_{T_4}(K_{5,6}) + I_{T_4}(K_{10}) + I_{T_4}(K_{16}) \supseteq 4 \cdot \{0,5\} + 2 \cdot \{0,6\} + J_{T_4}(K_{10}) + J_{T_4}(K_{16}) = J_{T_4}(K_{26})$. ■

3 Main Results

For proving the Main Theorem, the Wilson's Fundamental Construction of GDDs is an extremely useful tool.

Let K be a set of positive integers. A group divisible design (GDD), K-GDD, is a triple $(X, \mathcal{G}, \mathcal{B})$ such that the following properties are satisfied: (1) $\mathcal{G}$ is a partition of a finite set X into subsets (called *groups*); (2) $\mathcal{B}$ is a set of subset of X (called *blocks*), each of cardinality from K, such that every 2-subset of X is either contained in exactly one block or in exactly one group, but not in both. If $\mathcal{G}$ contains u_i groups of size g_i for $1 \le i \le r$, then we call $g_1^{u_1} g_2^{u_2} \dots g_r^{u_r}$ the group type of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as k-GDD.

Let $\mathcal{H} = \{H_1, H_2, \dots, H_m\}$ be a partition of a finite set X into subsets (called *holes*), where $|H_i| = n_i$ for $1 \le i \le m$. Let $K_{n_1,n_2,\dots,n_m}$ be the complete multipartite graph on X with the i-th part on H_i, and let G be a graph. A G-GDD is a triple $(X, \mathcal{H}, \mathcal{B})$ such that $(X, \mathcal{B})$ is a G-design of $K_{n_1,n_2,\dots,n_m}$. The hole type of the G-GDD is $\{n_1, n_2, \dots, n_m\}$. We also use an exponential notation $g_1^{u_1} g_2^{u_2} \dots g_r^{u_r}$ to describe hole type if there are u_i occurrences of g_i for $1 \le i \le r$, in the hole type. We say a T_i-GDD if G is a T_i graph.

The following construction is a variation of Wilson's Fundamental Construction [13].

Construction 3.1 *(Wilson's Fundamental Construction). Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a K-GDD, and let $\omega : X \to Z^+$ be a weight function. Suppose that for each block $B \in \mathcal{B}$ there is a G-GDD of type $\{\omega(x) : x \in B\}$. Then there exists a G-GDD of type $\{\sum_{x \in G} \omega(x) : G \in \mathcal{G}\}$.*

Theorem 3.2 *For $i = 1,2,3,4$, $I_{T_i}(n) = J_{T_i}(n)$, where $n \equiv 0, 1, 5, 6, 10, 11, 15, 16 \pmod{30}$.*

Proof: The cases when $n = 5, 6, 10, 11, 15$ and 16 follow from the small cases in Section 2. Assume that $n \geq 30$. Let $n = 10u + a$ with $u \equiv 0, 1 \pmod 3$, $u \geq 3$ and $a \in \{0, 1, 5, 6\}$. Start from a 3-GDD of type 2^u from Lemma 1.1. Give each point of the GDD weight 5. By Lemma 2.3 and 2.16, there is a pair of T_i-GDDs of type 5^3 with α common blocks, $\alpha \in \{0, 15\} \subset I_{T_i}(K_{5,5,5})$. Then, apply Construction 3.1 to obtain a pair of T_i-GDDs of type 10^u with $\sum_{i=1}^{x} \alpha$ common blocks, where $x = 2u(u-1)/3$ is the number of blocks of the 3-GDD of type 2^u.

By Lemmas 2.7, 2.9-2.13, 2.18-2.20, we take a pair of T_i-designs of K_{10+a} with β common blocks, $\beta \in J_{T_i}(K_{10+a})$, and $u-1$ pairs of T_i-designs of $K_{10+a} \backslash K_a$ with γ common blocks, $\gamma \in \{0, 9+2a\} \subset I_{T_i}(K_{10+a} \backslash K_a)$. There are a pair of T_i-designs of order $10u+a$ with $\sum_{i=1}^{x} \alpha + \beta + \sum_{i=1}^{u-1} \gamma$ common blocks. Thus, $I_{T_i}(n) \supseteq x \cdot I_{T_i}(K_{5,5,5}) + (u-1) \cdot I_{T_i}(K_{10+a} \backslash K_a) + I_{T_i}(10+a) \supset x \cdot \{0, 15\} + (u-1) \cdot \{0, 9+2a\} + J_{T_i}(K_{10+a}) = J_{T_i}(n)$. This completes the proof. ■

Theorem 3.3 *For $i = 1, 2, 3, 4$, $I_{T_i}(n) = J_{T_i}(n)$, where $n \equiv 20, 21, 25, 26$* (mod 30).

Proof: The cases when n =20, 21, 25 and 26 follow from the small cases in Section 2. Assume that $n \geq 50$. Let $n = 10u + a$ with $u \equiv 2 \pmod 3$, $u \geq 5$ and $a \in \{0, 1, 5, 6\}$. Start from a 3-GDD of type $2^{u-2}4$ from Lemma 1.1. Give each point of the GDD weight 5. By Lemma 2.3 and 2.16, there is a pair of T_i-GDDs of type 5^3 with α common blocks, $\alpha \in \{0, 15\} \subset I_{T_i}(K_{5,5,5})$. Then, apply Construction 3.1 to obtain a pair of T_i-GDDs of type $10^{u-2}20$ with $\sum_{i=1}^{x} \alpha$ common blocks, where $x = 2(u+1)(u-2)/3$ is the number of blocks of the 3-GDD of type $2^{u-2}4$.

By Lemmas 2.7, 2.9, 2.10, 2.12, 2.14, 2.15, 2.18-2.20, 2.22, 2.23, we take a pair of T_i-designs of K_{20+a} with β common blocks, $\beta \in J_{T_i}(20+a)$, and $u-2$ pairs of T_i-designs of $K_{10+a} \setminus K_a$ with γ common blocks, $\gamma \in \{0, 9+2a\} \subset I_{T_i}(K_{10+a} \backslash K_a)$. There are a pair of T_i-designs of order $10u+a$ with $\sum_{i=1}^{x} \alpha + \beta + \sum_{i=1}^{u-2} \gamma$ common blocks. Thus, $I_{T_i}(n) \supseteq x \cdot I_{T_i}(K_{5,5,5}) + (u-2) \cdot I_{T_i}(K_{10+a} \setminus K_a) + I_{T_i}(20+a) \supset x \cdot \{0, 15\} + (u-2) \cdot \{0, 9+2a\} + J_{T_i}(20+a) = J_{T_i}(n)$. This completes the proof. ■

Combining the Theorems 3.2 and 3.3, the Main Theorem can be obtained as follows.

Main Theorem $I_{T_i}(n) = J_{T_i}(n)$ for $n \equiv 0, 1 \pmod 5$ and $n \geq 10$; $I_{T_i}(n) = J_{T_i}(n)$ for $(i, n) = (1, 5), (1, 6), (2, 6), (3, 6)$.

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