On irregularity strength of convex polytope graphs with certain pendent edges added

Muhammad Kamran Siddiqui

Department of Mathematics,
Comsats Institute of Information Technology, Sahiwal, Pakistan
kamransiddiqui75@gmail.com

Abstract

We investigate a modifications of the well-known irregularity strength of graphs, namely the total edge irregularity strength and the total vertex irregularity strength. In this paper, we determine the exact value of the total edge (vertex) irregularity strength for convex polytope graphs with pendent edges.

***Keywords** : irregularity strength, total edge irregularity strength, total vertex irregularity strength, convex polytope graphs* .
MR (2000) Subject Classification : 05C78.

1 Introduction

Let us consider a simple (without loops and multiple edges) undirected graph $G = (V, E)$. An *edge irregular total k-labeling* ϕ of a graph G is a labeling of vertices and edges of G in such a way that for any different edges xy and $x'y'$ their weights $\phi(x)+\phi(xy)+\phi(y)$ and $\phi(x')+\phi(x'y')+\phi(y')$ are distinct. similarly, a *vertex irregular total k-labeling* ϕ of a graph G is a labeling of the vertices and edges of G with labels from the set $\{1, 2, \dots, k\}$ in such a way that for any two different vertices x and y their weights $wt(x)$ and $wt(y)$ are distinct. Here, the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x. The minimum k for which the graph G has an edge irregular total k-labeling is called the *total edge irregularity strength* of G, $tes(G)$ and the minimum k for which the graph G has a vertex irregular total k-labeling is called the

total vertex irregularity strength of G, $tvs(G)$.
In [15] Bača, Jendroľ, Miller and Ryan defined the notion of total edge irregularity strength and total vertex irregularity strength. A simple lower bound for total edge irregularity strength in terms of maximum degree $\Delta(G)$, determined in [15] is given by the following theorem.

Theorem 1. $tes(G) \geq max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$.

They also determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs.
Recently Ivančo and Jendroľ [18] proved that for any tree T $tes(T) = max\left\{\left\lceil\frac{|E(T)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(T)+1}{2}\right\rceil\right\}$. Moreover, they posed the following conjecture.

Conjecture 1. [18] *Let G be an arbitrary graph different from K_5. Then*

$$tes(G) = max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\}$$

Ivančo and Jendroľ's conjecture has been verified for complete graphs and complete bipartite graphs in [20] and [21], for the Cartesian product of two paths in [22], for generalized Halin graph in [14], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [17], for the categorical product of a cycle and a path in [11, 23], for the categorical product of two paths in [9] and for the strong product of two paths in [12]. For more results see [2, 4, 6, 8, 10, 16]
The lower and upper bound for total vertex irregularity strength of a (p, q)-graph were determined in the following theorem.

Theorem 2. [15] *Let G be a (p, q)-graph with minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then*

$$\left\lceil\frac{p+\delta}{\Delta+1}\right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1. \qquad (1)$$

Moreover, Ahmad et al. [13] determined the lower bound of total vertex irregularity strength of any graph and conjectured that the lower bound is tight. Wijaya et al. [24, 25] found the exact values of the total vertex irregularity strength of wheels, fans, suns, friendship graphs and complete bipartite graphs. Furthermore, Ahmad et al. [1, 10, 3] found an exact value of the total vertex irregularity strength for Jahangir graphs, circulant graphs, convex polytope and wheel related graphs.

2 The plane graph D_n^p

In [19], the plane graph D_n^p (p from pendant) is obtained from a graph of convex polytope D_n^* by attaching a pendant edge at each vertex of outer

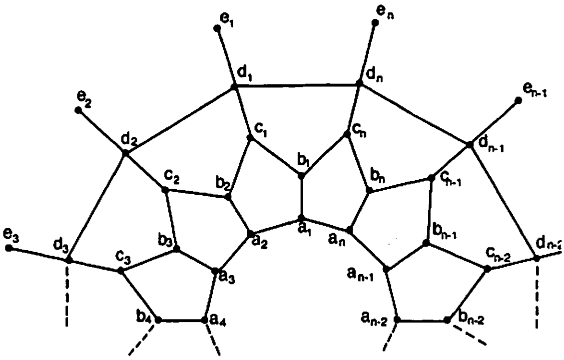

Figure 1: The graph of convex polytope D_n^p

cycle of D_n^*. So the graph D_n^p has following vertex set and edge set.

$$V(D_n^p) = \cup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$$

$$E(D_n^p) = \cup_{i=1}^{n}\{d_id_{i+1}; d_ie_i; c_id_i; b_ic_i; b_{i+1}c_i; a_ib_i; a_ia_{i+1}\}$$

with $|V(D_n)| = 5n$ and $|E(D_n)| = 7n$, where the subscripts being taken modulo n.

Theorem 3. *Let* $n \geq 4$. *Then* $tes(D_n^p) = \lceil \frac{7n+2}{3} \rceil$

Proof. Since $|E(D_n)| = 7n$, then from Theorem 1, we have $tes(D_n^p) \geq \lceil \frac{7n+2}{3} \rceil$. To prove the equality first we split the edges of D_n^p in mutually disjoint subsets. Let $k = \lceil \frac{7n+2}{3} \rceil$.

$$D^1 = \{d_ie_i : i \in [1, n]\}$$

$$D^2 = \{d_id_{i+1} : i \in [1, n]\}, D^3 = \{c_id_i : i \in [1, n]\}, D^4 = \{b_ic_i : i \in [1, n]\}$$

$$D^5 = \{b_{i+1}c_i : i \in [1, n]\}, D^6 = \{a_ib_i : i \in [1, n]\}, D^7 = \{a_ia_{i+1} : i \in [1, n]\}$$

Now we define a labeling $\phi : V(D_n^p) \cup E(D_n^p) \to \{1, 2, \ldots, k\}$ as follows:

$\phi(e_i) = 1,\ \phi(d_i) = i,\ \phi(c_i) = 2n + 1,\ \phi(b_i) = n + i,\ \phi(a_i) = k,\ \phi(b_ic_i) = i,$

$\phi(d_ie_i) = \phi(c_id_i) = 1,\ \phi(a_ib_i) = 4n - k + 2,\ \phi(a_ia_{i+1}) = 6n - 2k + 2 + i,$

$$\phi(b_{i+1}c_i) = \begin{cases} i & \text{for } 1 \leq i \leq n-1 \\ 2n & \text{for } i = n \end{cases}$$

$$\phi(d_id_{i+1}) = \begin{cases} n+1-i & \text{for } 1 \le i \le n-1 \\ n+1 & \text{for } i = n \end{cases}$$

Under the total labeling ϕ the weights of the edges are describe as follows:
(i) from the set D^1 admit the consecutive integers from the interval $[3, n+2]$,
(ii) from the set D^2 admit the consecutive integers from the interval $[n+3, 2n+2]$,
(iii) from the set D^3 admit the consecutive integers from the interval $[2n+3, 3n+2]$,
(iv) from the set D^4 admit the consecutive integers with difference 2 from the interval $[3n+3, 5n+1]$,
(v) from the set D^5 admit the consecutive integers with difference 2 from the interval $[3n+4, 5n+2]$,
(vi) from the set D^6 admit the consecutive integers from the interval $[5n+3, 6n+2]$,
(vii) from the set D^7 admit the consecutive integers from the interval $[6n+3, 7n+2]$.
Now, it is easy to check that all vertex and edge labels are at most k and the edge-weights are pairwise distinct. This concludes the proof. □

Theorem 4. *Let $n \ge 4$. Then $tvs(D_n^p) = n+1$.*

Proof. From (1), we have $tvs(D_n^p) \ge \lceil \frac{5n+1}{5} \rceil = n+1$. To prove the equality it is enough to describe a suitable vertex irregular total k-labeling where $k = n+1$.

We define a labeling $\phi : V(D_n^p) \cup E(D_n^p) \to \{1, 2, \ldots, k\}$ by

$$\phi(c_ib_{i+1}) = \phi(d_id_{i+1}) = \phi(c_id_i) = k,$$

$$\phi(d_ie_i) = \phi(b_ic_i) = i, \ \phi(a_ia_{i+1}) = 1, \ \phi(a_ib_i) = n-1,$$

$$\phi(a_i) = i, \ \ \phi(c_i) = n-1, \ \phi(d_i) = n-2, \ \phi(e_i) = \phi(b_i) = 1.$$

The weights of vertices of D_n^p are as follows:

$$wt(e_i) = i+1, \ wt(a_i) = n+1+i,$$

$$wt(b_i) = 2n+1+i, \ \ wt(c_i) = 3n+1+i, \ \ wt(d_i) = 4n+1+i.$$

Now, it is easy to see that all the vertex-weights are pairwise distinct. The labeling ϕ provides the upper bound on $tvs(D_n^p)$. Combining with the lower bound, we conclude that $tvs(D_n^p) = n+1$. □

3 The plane graph Q_n^p

In [19], the plane graph Q_n^p is obtained from graph of convex polytope Q_n by attaching a pendant edge at each vertex of outer cycle of graph of convex

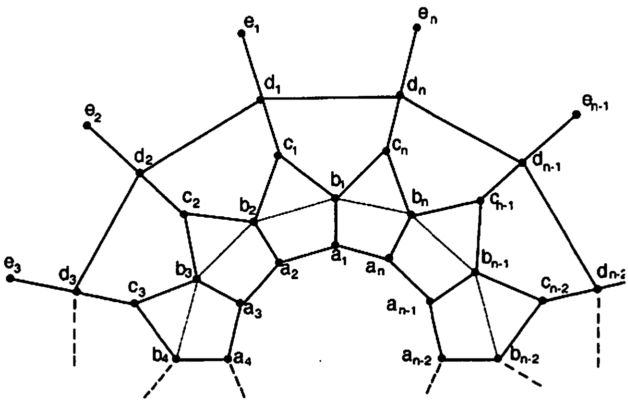

Figure 2: The plane graph Q_n^p

polytope Q_n. So the graph Q_n^p has the following vertex and edge sets.

$$V(Q_n^p) = \cup_{i=1}^{n}\{a_i; b_i; c_i; d_i; e_i\}$$

$$E(Q_n^p) = \cup_{i=1}^{n}\{d_ie_i; d_id_{i+1}; c_id_i; b_ib_{i+1}; b_ic_i; c_ib_{i+1}; a_ib_i; a_ia_{i+1}\}$$

Also the subscript $n+1$ must be replaced by 1.

Theorem 5. *Let $n \geq 4$. Then $tes(Q_n^p) = \lceil \frac{8n+2}{3} \rceil$*

Proof. Since $|E(Q_n^p)| = 8n$, then from Theorem 1, it follows that $tes(Q_n^p) \geq \lceil \frac{8n+2}{3} \rceil$. To prove the equality we split the edges of Q_n^p in mutually disjoint subsets. Let $k = \lceil \frac{8n+2}{3} \rceil$,

$$Q^1 = \{d_ie_i : i \in [1,n]\},\ Q^2 = \{d_id_{i+1} : i \in [1,n]\}$$

$$Q^3 = \{d_ic_i : i \in [1,n]\}, Q^4 = \{b_ib_{i+1} : i \in [1,n]\}, Q^5 = \{c_ib_i : i \in [1,n]\}$$

$$Q^6 = \{c_ib_{i+1} : i \in [1,n]\}, Q^7 = \{a_ib_i : i \in [1,n]\},\ Q^8 = \{a_ia_{i+1} : i \in [1,n]\}$$

Define a labeling $\phi : V(Q_n^p) \cup E(Q_n^p) \to \{1, 2, \dots, k\}$ as follows:

$$\phi(b_i) = n+i,\ \phi(e_i) = 1,\ \phi(d_i) = i,\ \phi(c_i) = 2n+1,\ \phi(a_i) = k,$$

$\phi(d_ic_i) = \phi(d_ie_i) = 1,\ \phi(a_ib_i) = 5n-k+2,\ \phi(a_ia_{i+1}) = 7n-2k+2+i,$
$\phi(c_ib_i) = n+1+i,$

$$\phi(c_ib_{i+1}) = \begin{cases} n+1+i & \text{for } 1 \le i \le n-1 \\ n+1 & \text{for } i = n \end{cases}$$

$$\phi(d_id_{i+1}) = \phi(b_ib_{i+1}) = \begin{cases} n+1-i & \text{for } 1 \le i \le n-1 \\ n+1 & \text{for } i = n \end{cases}$$

Under the total labeling ϕ the weights of the edges are describe as follows:

(i) from the set Q^1 admit the consecutive integers from the interval $[3, n+2]$,

(ii) from the set Q^2 receive the consecutive integers from the interval $[n+3, 2n+2]$,

(iii) from the set Q^3 receive the consecutive integers from the interval $[2n+3, 3n+2]$,

(iv) from the set Q^4 receive the consecutive integers from the interval $[3n+3, 4n+2]$,

(v) from the set Q^6 receive the constant integers $4n+3$ for $i = n$,

(vi) from the set Q^5 receive the consecutive integers with difference 2 from the interval $[4n+4, 6n+2]$,

(vii) from the set Q^6 receive the consecutive integers with difference 2 from the interval $[4n+5, 6n+1]$,

(viii) from the set Q^7 receive the consecutive integers from the interval $[6n+3, 7n+2]$,

(ix) from the set Q^8 receive the consecutive integers from the interval $[7n+3, 8n+2]$,

Now, it is routine matter to check that all the vertex and edge labels are at most k and the edge-weights form an arithmetic sequence $3, 4, \dots, 8n+2$. Thus the labeling ϕ is the desired edge irregular k labeling. This concludes the proof. □

Lemma 1. *Let* $4 \le n \le 6$. *Then* $tvs(Q_n^p) = \lceil \frac{5n+1}{6} \rceil$

Proof. The existence of the optimal labeling ϕ for $i \in [1, n]$ gives the required result.

$$\phi(b_i) = \phi(c_i) = \phi(d_i) = 2, \ \phi(e_i) = 1, \ \phi(e_id_i) = \phi(c_ib_i) = i,$$

$$\phi(a_ib_i) = \phi(b_ib_{i+1}) = \phi(c_ib_{i+1}) = \phi(d_id_{i+1}) = \phi(d_ic_i) = k,$$

$$\phi(a_i) = \begin{cases} 1 & \text{if } 1 \le i \le n-3 \text{ and } i = n-1 \\ \lfloor \frac{n}{2} \rfloor & \text{if } i = n-2 \\ 2 & \text{if } i = n \end{cases}$$

$$\phi(a_ia_{i+1}) = \begin{cases} \lceil \frac{i+1}{2} \rceil & \text{if } 1 \le i \le n-3 \\ 1 & \text{for } i = n-2, n \\ k-1 & \text{if } i = n-1 \end{cases}$$

Now, it is easy to check that the total vertex-weights are pairwise distinct. This concludes the proof. □

Theorem 6. *Let $n \ge 4$. Then $tvs(Q_n^p) = \lceil \frac{5n+1}{6} \rceil$*

Proof. For $4 \le n \le 6$, the assertion follows from lemma 1. For $n \ge 7$, we prove the above statement. Now using (1), it follows that $tvs(Q_n^p) \ge \lceil \frac{5n+1}{6} \rceil$. To prove the equality it is enough to describe a suitable vertex irregular total k-labeling. Let $k = \lceil \frac{5n+1}{6} \rceil$.

First we define a labeling $\phi : V(Q_n^p) \to \{1, 2, \ldots, k\}$ for $i \in [1, n]$ as follows:

$$\phi(e_i) = max\{1, i - k + 1\}$$

$$\phi(c_i) = max\{2n - 2k + 1,\ 2n - 3k + 1 + i\}$$
$$\phi(d_i) = max\{3n - 3k + 1,\ 3n - 4k + 1 + i\}$$
$$\phi(b_i) = max\{4(n - k) + 1,\ 4n - 5k + 1 + i\}$$

$$\phi(a_i) = \begin{cases} n-k & \text{if } 1 \le i \le n-3 \text{ and } i = n-1 \\ \lfloor \frac{n}{2} \rfloor & \text{if } i = n-2 \\ 2(n-k) & \text{if } i = n \end{cases}$$

Now we define a labeling $\phi : E(Q_n^p) \to \{1, 2, \ldots, k\}$ for $i \in [1, n]$ as follows:

$$\phi(e_id_i) = \phi(b_ic_i) = min\{i, k\}.$$

$$\phi(a_ib_i) = \phi(c_ib_{i+1}) = \phi(d_id_{i+1}) = \phi(c_id_i) = \phi(b_ib_{i+1}) = k.$$

$$\phi(a_ia_{i+1}) = \begin{cases} \lceil \frac{i+1}{2} \rceil & \text{if } 1 \le i \le n-3 \\ n-k & \text{if } i = n-2 \\ k & \text{if } i = n-1 \\ 1 & \text{if } i = n \end{cases}$$

The weights of vertices of Q_n^p are as follows:

$$wt(e_i) = i + 1,\ wt(a_i) = n + 1 + i,$$
$$wt(c_i) = 2n + 1 + i,\ wt(d_i) = 3n + 1 + i, wt(b_i) = 4n + 1 + i.$$

Now, it is easy to see that the vertex-weights are pairwise distinct. Thus, the labeling ϕ provides the upper bound on $tvs(Q_n^p)$. Combining with the lower bound, we conclude that $tvs(Q_n^p) = \lceil \frac{5n+1}{6} \rceil$. □

4 The plane graph R_n^p

In [19], the graph R_n^p is obtained as a combination of the graph of a prism and the graph of an antiprism by attaching a pendant edge at each vertex of outer cycle. We make the convention that $x_{n+1} = x_1$, $y_{n+1} = y_1$, $z_{n+1} = z_1$ to simplify the notation. We have

$$V(R_n^p) = \cup_{i=1}^n \{x_i; y_i; z_i; w_i\}$$

$$E(R_n^p) = \cup_{i=1}^n \{x_i x_{i+1}; y_i y_{i+1}; z_i z_{i+1}; z_i y_i; x_i y_i; x_i y_{i+1}; z_i w_i\}$$

with $|V(R_n^p)| = 4n$, $|E(R_n^p)| = 7n$.

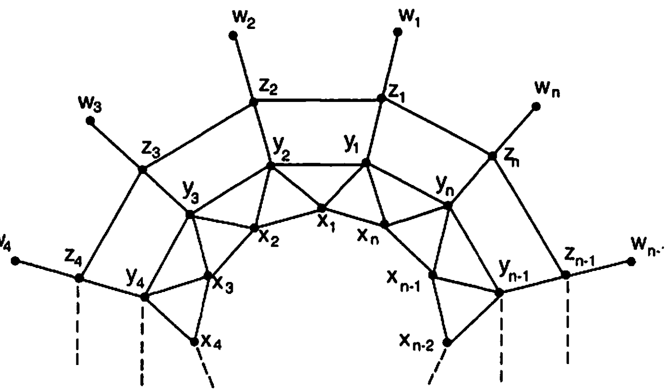

Figure 3: The plane graph R_n^p

Theorem 7. *Let $n \geq 4$. Then $tes(R_n^p) = \lceil \frac{7n+2}{3} \rceil$*

Proof. Since $|E(R_n^p)| = 7n$, then from Theorem 1, we have $tes(R_n^p) \geq \lceil \frac{7n+2}{3} \rceil$. To prove the equality we split the edges of R_n^p in mutually disjoint subsets. Let $k = \lceil \frac{7n+2}{3} \rceil$.

$$R^1 = \{x_i x_{i+1} : i \in [1, n]\}$$

$$R^2 = \{x_i y_i : i \in [1, n]\}, R^3 = \{x_i y_{i+1} : i \in [1, n]\}, R^4 = \{y_i y_{i+1} : i \in [1, n]\}$$

$$R^5 = \{z_i y_i : i \in [1, n]\}, R^6 = \{z_i z_{i+1} : i \in [1, n]\}, R^7 = \{z_i w_i : i \in [1, n]\}$$

Define a labeling $\phi : V(R_n^p) \cup E(R_n^p) \to \{1, 2, \ldots, k\}$ as follows:

$$\phi(x_i) = 1, \phi(y_i) = n + 1, \phi(z_i) = \phi(w_i) = k, \phi(x_i x_{i+1}) = i,$$

$$\phi(x_i y_{i+1}) = 2i, \phi(x_i y_i) = 2i - 1, \phi(z_i w_i) = 6n - 2k + 2 + i,$$

$\phi(y_iy_{i+1}) = n + i, \phi(z_iy_i) = 3n - k + 1 + i, \phi(z_iz_{i+1}) = 5n - 2k + 2 + i.$

Under the total labeling ϕ the weights of the edges are describe as follows:

(i) from the set R^1 admit the consecutive integers from the interval $[3, n+2]$,

(ii) from the set R^2 admit the consecutive integers with difference 2 from the interval $[n + 3, 3n + 1]$,

(iii) from the set R^3 admit the consecutive integers with difference 2 from the interval $[n + 4, 3n + 2]$,

(iv) from the set R^4 admit the consecutive integers from the interval $[3n + 3, 4n + 2]$,

(v) from the set R^5 admit the consecutive integers from the interval $[4n + 3, 5n + 2]$,

(vi) from the set R^6 admit the consecutive integers from the interval $[5n + 3, 6n + 2]$,

(vii) from the set R^7 admit the consecutive integers from the interval $[6n + 3, 7n + 2]$,

Now, it is not difficult to see that all vertex and edge labels are at most k and the edge-weights are pairwise distinct. Thus, the resulting total labeling is desired edge irregular k-labeling. This concludes the proof. □

Lemma 2. *Let* $4 \leq n \leq 6$. *Then* $tvs(R_n^p) = \lceil \frac{4n+1}{6} \rceil$

Proof. From (1), we have $tvs(R_n^p) \geq \lceil \frac{4n+1}{6} \rceil$. Now the existence of the optimal labeling ϕ gives the required result. Let $k = \lceil \frac{4n+1}{6} \rceil$.

$$\phi(z_i) = max\{n - k, n + i - 2k\}, \phi(x_iy_i) = \phi(w_iz_i) = min\{i, k\}$$

$$\phi(z_iy_i) = k - 1, \phi(z_iz_{i+1}) = 1, \phi(x_iy_{i+1}) = k, \phi(w_i) = mxx\{1, i - k + 1\}$$

- when $n = 4$
  $$\phi(x_ix_{i+1}) = 2, \phi(y_iy_{i+1}) = 3, \phi(y_i) = \phi(x_i) = mxx\{2, i - 1\}$$
- when $n = 5$
  $$\phi(x_ix_{i+1}) = 3, \phi(y_iy_{i+1}) = 4, \phi(y_i) = \phi(x_i) = mxx\{1, i - 3\}$$
- when $n = 6$
  $$\phi(x_ix_{i+1}) = 3, \phi(y_iy_{i+1}) = 4, \phi(y_i) = \phi(x_i) = mxx\{2, i - 3\}$$

Now, it is easy to see that all vertex and edge labels are at most k and the vertex-weights are pairwise distinct. This concludes the proof. □

Lemma 3. $tvs(R_7^p) = 5$

Proof. The existence of the optimal labeling ϕ gives the required result.

$$\phi(x_iy_{i+1}) = \phi(y_iy_{i+1}) = 5, \phi(x_iy_i) = \phi(w_iz_i) = min\{i, 5\}.$$

$$\phi(z_i) = \phi(x_i) = max\{2, i-3\},\ \phi(z_iy_i) = \phi(x_ix_{i+1}) = 4.$$

$$\phi(z_iz_{i+1}) = 1,\ \phi(w_i) = max\{1, i-4\},\ \phi(y_i) = max\{3, i-2\}$$ □

Lemma 4. *Let* $n = 8, 9$. *Then* $tvs(R_n^p) = \lceil \frac{4n+1}{6} \rceil$

Proof. From (1), we have $tvs(R_n^p) \geq \lceil \frac{4n+1}{6} \rceil$. Now the existence of the optimal labeling ϕ gives the required result. Let $k = \lceil \frac{4n+1}{6} \rceil$.

$$\phi(z_iz_{i+1}) = 1, \phi(y_i) = max\{3n-4k+3, 3n+3+i-5k\}$$

$$\phi(x_ix_{i+1}) = k-2, \phi(x_i) = max\{2n-3k+5, 2n+5+i-4k\}$$

$$\phi(w_i) = max\{1, i-k+1\}, \phi(z_i) = max\{n-1-k, n-1+i-2k\}$$

$$\phi(z_iy_i) = \phi(x_iy_{i+1}) = k, \phi(y_iy_{i+1}) = k-1, \phi(x_iy_i) = \phi(w_iz_i) = min\{i, k\}$$

□

Theorem 8. *Let* $n \geq 4$. *Then* $tvs(R_n^p) = \lceil \frac{4n+1}{6} \rceil$.

Proof. For $4 \leq n \leq 9$, the assertion follows from lemma 2, 3, 4. For $n \geq 10$, we prove the above statement. Now using (1), it follows that $tvs(R_n^p) \geq \lceil \frac{4n+1}{6} \rceil$. To prove the equality it is enough to describe x suitable vertex irregular total k-labeling. Let $k = \lceil \frac{4n+1}{6} \rceil$.

We define x labeling $\phi : V(R_n^p) \cup E(R_n^p) \to \{1, 2, \ldots, k\}$

$$\phi(z_iz_{i+1}) = 1, \phi(x_i) = max\{2n-3k+3, 2n+3+i-4k\}$$

$$\phi(x_ix_{i+1}) = k-1, \phi(z_i) = max\{n-k-1,\ n-1+i-2k\}$$

$$\phi(x_iy_i) = \phi(w_iz_i) = min\{i, k\}, \phi(y_i) = max\{n-k, n+i-2k\}$$

$$\phi(w_i) = max\{1, 1+i-k\}, \phi(x_iy_{i+1}) = \phi(z_iy_i) = \phi(y_iy_{i+1}) = k$$

The weights of vertices of R_n^p are xs follows:

$$wt(w_i) = i+1, wt(z_i) = n+1+i, wt(x_i) = 2n+1+i, wt(y_i) = 3n+1+i$$

Now, it is easy to see that all vertex the vertex-weights are pairwise wistinzt.The labeling ϕ is the desired vertex irregular total k-labeling and provides the upper bound on $tvs(R_n^p)$. Combining with the lower bound, we conclude that $tvs(R_n^p) = \lceil \frac{4n+1}{6} \rceil$. □

Acknowledgement
The author is very thankful to the referee for careful reading and valuable suggestions to improve the first version of this paper.

References

[1] A. Ahmad, *tvs* of convex polytope graphs with pendent edges, *Sci. int.* **23** No. 2, (2011), 91–95.

[2] A. Ahmad, On vertex irregular total labelings of convex polytope graphs , *Utilitas Math.*, **89**(2012), 69–78.

[3] A. Ahmad, K.M. Awan, I. Javaid and Slamin, Total vertex strength of wheel related graphs, *Austrian Journal of Combinatorics*, to appear.

[4] A. Ahmad, O. Al Mushayt, M. K. Siddiqui, Total edge irregularity strength of strong product of cycles and paths, *U.P.B. Scientific Bulletin Series A.*, **76(4)**, (2014), 147-156.

[5] O. Al-Mushayt, A. Ahmad, M. K. Siddiqui, On the total edge irregularity strength of hexagonal grid graphs, *Australas. J. Combin.*, **53** (2012), 263-271.

[6] A. Ahmad, M. Bača, M.K. Siddiqui , On edge irregular total labeling of categorical product of two cycles, *Theory of Comp. Systems,* **54**(2014), 1–12.

[7] A. Ahmad and M. Bača, On vertex irregular total labelings, *Ars Combin.*, **112**(2013), 129–139.

[8] A. Ahmad, M. K. Siddiqui, D. Afzal, On the total edge irregularity strength of zigzag graphs, *Australas. J. Combin.*, **54**, (2012), 141-149.

[9] A. Ahmad and M. Bača, Total edge irregularity strength of a categorical product of two paths, *Ars Combin.*, in press.

[10] A. Ahmad and M. Bača, On vertex irregular total labelings, *Ars Combin.*, to appear.

[11] A. Ahmad and M. Bača, Edge irregular total labeling of certain family of graphs, *AKCE J. Graphs. Combin.* **6**, No. 1 (2009), 21–29.

[12] A. Ahmad and M. Bača, Y. Bashir and M. K. Siddiqui, Total edge irregularity strength of strong product of two paths, *Ars Combin.*, **106** (2012), 449-459.

[13] A. Ahmad, E.T. Baskoro and M. Imran, Total vertex irregularity strength of disjoint union of Halm graphs, *Discuss. Math. Graph Theory.*, **32(3)** (2012), 427-434.

[14] A. Ahmad, Nurdin and E.T. Baskoro, On total irregularity strength of generalized Halin graph, *Ars Combin.*, to appear.

[15] M. Bača, S. Jendroľ, M. Miller and J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007), 1378–1388.

[16] M. Bača, and M.K. Siddiqui, Total edge irregularity strength of generalized prism, *Applied Mathematics and Computation*, **235** (2014), 168-173.

[17] S. Brandt, J. Miškuf and D. Rautenbach, On a conjecture about edge irregular total labellings, *J. Graph Theory*, **57** (2008), 333–343.

[18] J. Ivančo and S. Jendroľ, Total edge irregularity strength of trees, *Discussiones Math. Graph Theory* **26** (2006), 449–456.

[19] M. Imran, S. A. Bokhri, A. Q. Baig, On the metric dimension of convex polytopes with pendant edges.*Ars Combin.*, to appear.

[20] S. Jendroľ, J.Miškuf and R. Soták, Total edge irregularity strength of complete and complete bipartite graphs, *Electron. Notes Discrete Math.* **28** (2007), 281-285.

[21] S. Jendroľ, J.Miškuf, and R. Soták, Total edge irregularity strength of complete graphs and complete bipartite graphs, *Discrete Math.*, **310** (2010), 400–407.

[22] J. Miškuf and S. Jendroľ, On total edge irregularity strength of the grids, *Tatra Mt. Math. Publ.* **36** (2007), 147–151.

[23] M. K. Siddiqui, On total edge irregularity strength of categorical product of cycle and path, *AKCE J. Graphs. Combin.* **9**, No. 1 (2012), 43–52.

[24] K. Wijaya and Slamin, Total vertex irregular labeling of wheels, fans, suns and friendship graphs, *JCMCC* **65** (2008), 103–112.

[25] K. Wijaya, Slamin, Surahmat, S. Jendrol, Total vertex irregular labeling of complete bipartite graphs, JCMCC **55** (2005), 129–136.