The Number of 1-Factors and Edge-Colorings of Möbius Ladder Graphs and Triangular Embeddings of K_{12m+7}[1]

Cao Ni, Ren Han

Mathematics Department of East China Normal University.

Shanghai 200062, P.R.China

Abstract — In this paper we study the number of 1-factors and edge-colorings of the Möbius ladder graphs. We find exact formulae for such numbers and show that there are exponentially many 1-factors and edge-colorings in such graphs. As applications, we show that every "men-made" triangular embedding for K_{12m+7} by the current graphs by these of Youngs and Ringel permits exponentially many "*Grünbaum colorings*" (i.e.,3-edge-colored triangulations in such a way that each triangle receives three distinct colors.)

Key words — Möbius ladder graph, 1-factor, edge-coloring.
Mathematics Subject Classifications: 05C15

[1]Supported by the NNSF of China (Grant No.:10671073,11171114)

§1. Introduction

The graphs throughout this paper are finite simple without loops or multi-edges. All terms are standard and follow from Bondy's book[BM].

Tutte's 1-factor theorem shows that a cubic graph without a cut-edge must contain a 1-factor (or perfect matching as some scholars named). But what about the number of 1-factors of such graphs? In this field, Lovasz and Plummer conjectured that there are exponentially many 1-factors in such graphs[LP]. Since then, this conjecture has challenged every one in graph theory and some important progresses are made.

(1) Edmonds, Lovasz and Pulleyblank showed that there are at least $1+\beta(G)$ 1-factors in a cubic bridge-less graph G [ELP], where $\beta(G)$ is the Betti number of G.

(2) Kral, Sereni and Stiebitz improved this bound to $\frac{n}{2}+2$ [KSS].

(3) Voovhoeve solved this conjecture in the case of bipartite graphs [V].

(4) Shrijver extended the above result to k-regular bipartite graphs [S].

(5) Very recently M. Chudnovsky and P. Seymour proved this conjecture for planar cubic graphs [CS].

Here in this article we consider the number of 1-factors and edge-colorings in the Möbius ladder graphs $C(2m,m)$ which is formed by introducing a chord $(i,i+m)$ on a $2m$-cycle $C=(1,2,3,\cdots,2m)$ for $1\le i\le 2m$ where the sum is under modular $2m$.. What surprises us most is that the number of 1-factors in $C(2m,m)$ can be expressed by the famous Fibonacci series, i.e.,

Theorem 1. *Let $h(m)$ denote the number of 1-factors in the Möbius ladder graphs $C(2m,m)$ $(m\ge 2)$. Then*

$$h(m)=\begin{cases} f(m)+2f(m-1)+2, & m\equiv 1(mod2), \quad (1)\\ f(m)+2f(m-1), & m\equiv 0(mod2). \quad (1')\end{cases}$$

where $\{f(m)\}$ is a Fibonacci series with the condition

$$\begin{cases} f(m)=f(m-1)+f(m-2), & m\ge 3 \quad (2)\\ f(1)=f(2)=1 & \quad (2')\end{cases}$$

Corollary 1. *The number of 1-factors in the Möbius ladder graphs $C(2m,m)$ $(m\ge 2)$ is*

$$h(m)=\begin{cases} f(m)+2f(m-1)+2, & m\equiv 1(mod2), \quad (3)\\ f(m)+2f(m-1), & m\equiv 0(mod2). \quad (3')\end{cases}$$

where $f(m) = \frac{1}{\sqrt{5}}\{(\frac{1+\sqrt{5}}{2})^m - (\frac{1-\sqrt{5}}{2})^m\}(m \geq 2)$

This result differs from that of (1)-(5) since the Möbius ladder graphs $C(2m, m)$ may be nonbipartite and nonplanar. Further more, we study the number of 3-edge-colorings of Möbius ladder graphs $C(2m, m)$ (i.e., the number of 1-factorizations in such graphs) and show that the number of (proper) edge-colorings of the Möbius ladder graphs $C(2m, m)$ is also exponentially large. More precisely, we obtain the following

Theorem 2. *The number of (proper) edge-colorings of the Möbius ladder graphs* $C(2n, n)(n \geq 2)$ *is*

$$\sigma(n) = \begin{cases} 2(2^{n-1} - (-1)^{n-1}) + 6, & n \equiv 1(mod2), \quad (4) \\ 2(2^{n-1} - (-1)^{n-1}), & n \equiv 0(mod2). \quad (4') \end{cases}$$

In the history of solving of the famous *Heawood Conjecture* in coloring graphs on general surfaces, all the triangular embeddings for the complete graph K_n are "men-made"(i.e., induced from current graphs with few indexes). One may asks that what does these triangulations look like? One may readily see that a Möbius ladder graph is a *underline graph* of a type of current graph by Youngs. Therefore, current graphs will certainly have some actions on their triangular embeddings. There should exists some connection between current graphs and triangular embeddings of complete graph K_{12m+7}. We show that a current graph has great influence on the combinatorial structures of the corresponding triangular embeddings of K_{12m+7}. As applications, we have the following

Theorem 3. *A current graph (by Youngs or Ringel) is bipartite if and only if the dual of the corresponding triangular embedding for* K_{12m+7} *is bipartite. Further more, a 3-edge-coloring of a such current graph will imply a Grunbaum coloring of the corresponding triangular embedding of* K_{12m+7}.

As we had shown[RG] that a simple 3-regular graph G of order n may have at least $2^{n-\gamma_M(G)}(\geq \frac{\sqrt{2}}{2}2^{\frac{3n}{4}})$ distinct maximum genus embeddings, where $\gamma_M(G)$ is the maximum genus of G, and each index 1 current graph of Youngs will induce at least one oreintable cyclic triangular embeddings of K_{12m+7},we have

Corollary K_{12m+7} *has at least* $\frac{2^{9m+5}}{\sqrt[4]{2}}$ *triangular embeddings such that (1)The dual of such embeddings are of 3-regular nonbipartite;(2)Each of such embeddings permits* $\sigma(4m+2) = 2(2^{4m+1} - 1)$ *Grumbaum colorings.*

§2. Proof of Main Results

In this section, we shall prove Theorems 1 and 2. We first consider the validity of Theorem 1.

Proof of Theorem 1. Let the vertices of the Möbius ladder graphs $C(2m, m)$ be listed as

$$x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_{2m}$$

It is easy to see that the result stands for smaller natural number.So assume that the integer $m \geq 4$ in the following discussions. Now every 1-factor f of $C(2m, m)$ contains one of the three edges $x_1x_{m+1}, x_1x_2, x_1x_{2m}$. Let $f_1(m)$ and $f_2(m)$ denote, respectively, the number of 1-factors of $C(2m, m)$ containing the edge x_1x_{m+1} and the edge x_1x_2. By symmetry, the number of 1-factors of $C(2m, m)$ containing the edge x_1x_{2m} is also $f_2(m)$. Therefore, the number of 1-factors of $C(2m, m)$ is

$$h(m) = f_1(m) + 2f_2(m).$$

Case 1 1-factors containing the edge x_1x_{m+1}.

It is clear that the 1-factors containing x_1x_{m+1} are determined by those of $C(2m, m) - \{x_1, x_{m+1}\}$(as shown in Fig.1).

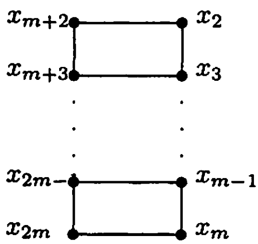

Fig.1

One may readily see that there are $m-1$ horizontal edges in $G(m-1)$ $= C(2m, m) - \{x_1, x_{m+1}\}$. Let g(m-1) denote the number of 1-factors of $G(m-1)$. Then its 1-factors contain either the edge x_2x_3 or x_2x_{m+2}. If a 1-factor contains the edge x_2x_{m+2}, then $G(m-1) - \{x_2, x_{m+2}\} = G(m-2)$ which has g(m-2) 1-factors; if a 1-factors contains the x_2x_3, then it must contains the edge $x_{m+2}x_{m+3}$, and $G(m-2) - \{x_3, x_{m+3}\} = G(m-3)$ which has g(m-3) 1-factors. Hence, we obtain a recursive relation for $\{g(m)\}$,

$$\begin{cases} g(m-1) = g(m-2) + g(m-3) & (5) \\ g(1) = 1 & (5') \\ g(2) = 2 & (5'') \end{cases}$$

which implies that $\{g(m)\}$ is a Fibonacci series and

$$f_1(m) = g(m-1) = f(m)$$

Case 2 1-factors containing the edge x_1x_2.

It is clear that any 1-factor of $C(2m,m)$ containing x_1x_2 is determined by those of $C(2m,m) - \{x_1, x_2\}$(as shown in Fig.2).

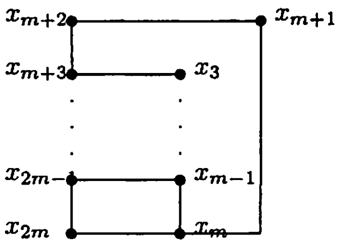

Fig.2

If a 1-factor contains x_1x_2 and $x_{m+1}x_m$, then there is exactly one of such a 1-factor provided $m \equiv 1(mod2)$; if $m \equiv 0(mod2)$, there is no such 1-factor contains both of the edges x_1x_2 and x_mx_{m+1}. If a 1-factor contains both of the edges x_1x_2 and $x_{m+1}x_{m+2}$, then

$$G(m-2) = C(2m,m) - \{x_1, x_2, x_{m+1}, x_{m+2}\}$$

will have $g(m-2)$ many 1-factors. By our reasonings in Case 1, $g(m-2) = f(m-1)$. Hence

$$f_2(m) = \begin{cases} f(m-1)+1, & m \equiv 1(mod2) \qquad (6) \\ f(m-1) & m \equiv 0(mod2) \qquad (6') \end{cases}$$

Now substitute $f_1(m)$ and $f_2(m)$ into $h(m) = f_1(m) + f_2(m)$, we have the following formula:

$$h(m) = \begin{cases} f(m)+2f(m-1)+2, & m \equiv 1(mod2), \qquad (7) \\ f(m)+2f(m-1), & m \equiv 0(mod2). \qquad (7') \end{cases}$$

This ends the proof of Theorem 1.

In the following, we shall prove Theorem 2.

Proof of Theorem 2. Let $\sigma(n)$ denote the number of (proper) edge-colorings of the Möbius ladder graphs $C(2n,n)$, where the color-set is $\{1,2,3\}$. We still draw $C(2n,n)$ in the projective plane as show in Fig.3

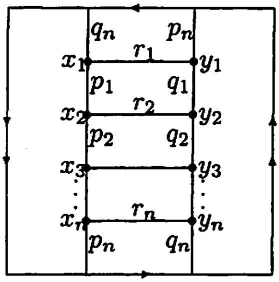

Fig.3

Let the edge $x_ix_{i+1} = p_i, y_iy_{i+1} = q_i, x_iy_i = r_i, 1 \leq i \leq n-1, x_ny_1 = p_n, y_nx_1 = q_n, x_ny_n = r_n$. We use $\tau(n)$ to count the number of edge-colorings of $C(2n, n)$ such that the color of the edge r_1 is $c(r_1) = 1$, here $c(e)$ indicates the color received by the edge e. Then we have

Claim 1. $\sigma(n) = 3\tau(n)$.

Next, we shall determine the value of $\tau(n)$ (i.e. determining of the number of edge-colorings such that $c(r_1) = 1$). There are two situations to be considered (i.e. $c(p_1) = c(q_1)$ or not).

Case 1 $c(p_1) = c(q_1)$.

We use $\tau_1(n)$ to denote the number of edge-coloring of $C(2n, n)$ with $c(r_1) = 1$ and $c(p_1) = c(q_1)$. (Obviously $\tau_2(n)$ counts those of $c(r_1) = 1$ and $c(p_1) \neq c(q_1)$). Then $c(p_1) = c(q_1) \in \{2, 3\}$. Let $\alpha(n)$ to count the number of edge-colorings of $C(2n, n)$ such that $c(r_1) = 1, c(p_1) = c(q_1) = 2$).Then we have the following

Claim 2. $\tau_1(n) = 2\alpha(n)$.

Now $c(p_2) = a \in \{1, 2, 3\} - \{2\}$ and $c(r_2) = a \in \{1, 3\} - \{a\}$, and so $c(q_2) = a$. In general, we obtain the following

Claim 3. $c(p_i) = c(q_i), 1 \leq i \leq n$.

Let A_n be the set of integer series $\{a_1, a_2, \cdots, a_n\}$ with $a_1 = 2, a_{i+1} \in \{1, 2, 3\} - \{a_i\}, 1 \leq i \leq n-1$.
Let B_n and C_n be the subsets of A_n such that

$$B_n = \{\{a_1, a_2, \cdots, a_n\} \in A_n | a_n = 3\}.$$

$$C_n = \{\{a_1, a_2, \cdots, a_n\} \in A_n | a_n = 1 \text{ or } 2\}.$$

The following property says that B_n almost determines the edge-colorings of $C(2n, n)$.

Claim 4. The number of edge-colorings of $C(2n, n)$ satisfying $c(r_1) = 1$ and $c(p_1) = c(q_1) = 2$ is $|B_n| = \alpha(n)$.

Proof. Let c be an edge-coloring of $C(2n, n)$ as defined above, Then

$$s_1 = c(p_1) = 2, s_n = c(p_n) = 3, s_i = c(p_i), s_i \neq s_{i+1}, 1 \leq i \leq n-1$$

Therefore, $\{s_1, s_2, \cdots, s_n\} \in B_n$.

Conversely, let $\{s_1, s_2, \cdots, s_n\} \in B_n$. We may define an edge-coloring c of $C(2n, n)$ as follows:

$$c(p_i) = c(q_i) = a_i, c(r_i) = z \in \{1, 2, 3\} - \{a_{i-1}, a_i\}, (a_{n+1} = a_1).$$

It is easy to see that c is a proper edge-coloring of $C(2n, n)$ such that

$$a_1 = c(p_1) = c(q_1) = 2, a_n = 3 = c(p_n) = c(q_n), \; c(r1) = 1$$

Claim 5. $|C_n| = |C_{n-1}| + 2|C_{n-2}|, |B_n| = |B_{n-1}| + 2|B_{n-2}|$

Proof. Let $\{a_1, a_2, \cdots, a_{k-1}, a_k\} \in B_k$. Then $a_k = 3$ and $a_{k-1} \in \{1, 2\}$. Hence, $|B_k| = |C_{k-1}|$. Let $\{a_1, a_2, \cdots, a_{k-1}, a_k\} \in C_k$. Then $a_k \in \{1, 2\}$ and $a_{k-1} \in \{1, 2, 3\} - \{a_k\}$. If $a_{k-1} = 3$, then there are $2|B_{k-1}|$ subsequences in C_k; if $a_{k-1} \in \{1, 2\} - \{a_k\}$, then there are $|C_{k-1}|$ subsequences in C_k, Hence, $|C_k| = 2|B_{k-1}| + |C_{k-1}|$. Substitute $|B_k| = |C_{k-1}|$ into $|C_n| = 2|B_{n-1}| + |C_{n-1}|$, the results follows.

Solving the recursions in Claim 5, we obtain the following

$$|B_n| = \frac{1}{3}(2^{n-1} - (-1)^{n-1}) \; (n \geq 2)$$

$$\tau_1(n) = 2|\alpha(n)| = 2|B_n| = \frac{2}{3}(2^{n-1} - (-1)^{n-1})$$

Case 2 $c(p_1) \neq c(q_1)$.

Under this assumption, there are exactly 2 edge-coloring of $C(2n, n)$ provided $n \equiv 1(mod 2)$ and no proper edge-coloring of $C(2n, n)$ whenever $n \equiv 0(mod 2)$.

Based on the above analysis, we arrive at

$$\tau(n) = \begin{cases} \frac{2}{3}(2^{n-1} - (-1)^{n-1}) + 2, & n \equiv 1(mod 2); \quad (8) \\ \frac{2}{3}(2^{n-1} - (-1)^{n-1}), & n \equiv 0(mod 2), \quad (8') \end{cases}$$

which implies that the number of proper edge-colorings of $C(2n, n)$ is

$$\sigma(n) = \begin{cases} 2(2^{n-1} - (-1)^{n-1}) + 6, & n \equiv 1(mod2); \quad (9) \\ 2(2^{n-1} - (-1)^{n-1}), & n \equiv 0(mod2). \quad (9') \end{cases}$$

§3. Applications in graph embedings

In this section we shall discuss the relationship between current graphs and triangular embeddings of K_{12m+7}. Readers may refer Ringel's monograph[GR] for the concepts and terminologies.

Claim 6 *There are two types of current graphs designed by Youngs and Rigngel, respectively, (as depicted below).*

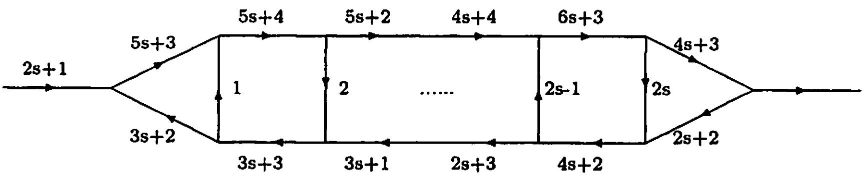

Fig.4

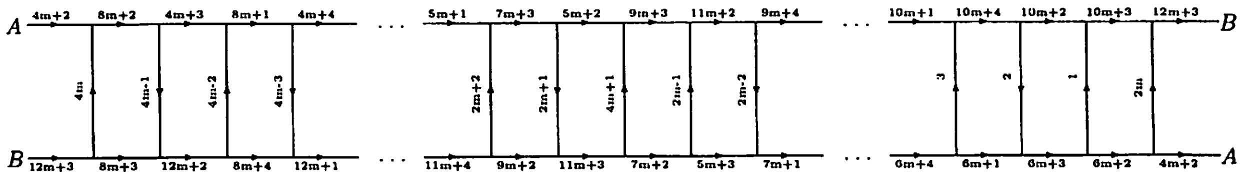

Fig.5: Orientable current graph for $n = 24m + 7, m \geq 2$

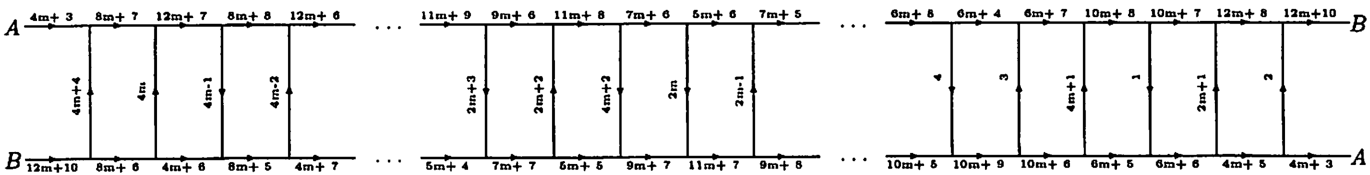

Fig.6: Orientable current graph for $n = 24m + 19, m \geq 3$

It is clear that the underline graph of the current graph by Youngs is a type of nonbipartite and nonplanar Möbius ladder graph. Although these two types current graphs induce triangular embeddings for the same graph K_{12m+7}, such embeddings may have different combinatorial structures.

Claim 7. *A current graph(as depicted before) is bipartite if and only if the corresponding triangular embeddings is 2-face colorable. Further more, if a current graph has a 3-edge-coloring, then so does the dual of the corresponding triangular embeddings (i.e., such triangular embedding permit a Grunbaum coloring).*

Proof. Let (K, Z_{12m+7}, λ) be an index 1 current graph satisfying the following conditions
(i) there is exactly one 2-cell(called *index 1*);
(ii)this 2-cell is an $(n-1)$-gon;
(iii) each element of $Z_{12m+7} - \{0\}$ appears exactly once as a current on some (clockwise,say) oriented edge of the $(n-1)$–gon;
(iv) K is 3-regular;
(v) the Kirchoff current low (KCL)holds at each vertex.

Let $\Delta(ijk)$ be a triangular face of K_{12m+7}. Then its local situation is shown below

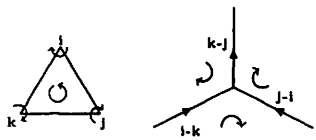

Fig.7

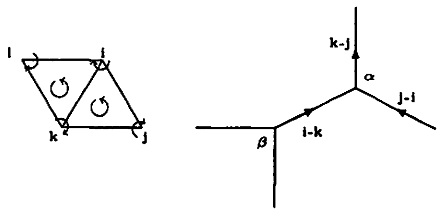

Fig.8

When one walks around Δ in counter-clock way, by KCL one may imagine that the weight of the oriented edge ji is $\lambda(ji) = j - i$. Similarly we have $\lambda(ik) = i - k, \lambda(kj) = k - j$, each of which are distributed on three edges sharing the same vertex in the current graph. Let $\Delta' = (ikl)$ be another triangular face sharing a common oriented edge ik with $\Delta(ijk)$. Since ik is on the common boundary of Δ and Δ', these two triangles determines the two ends of the with current $\lambda = i - k$ in (K, Z_{12m+7}, λ). Thus, when one moves from a triangle $\Delta(ijk)$ to the next one,say $\Delta' =$

(ikl), the corresponding walk on (K, Z_{12m+7}, λ) is from one end of the oriented edge having current $\lambda = i - k$ to another.

Claim 8. *A closed face-chain* $(\Delta_1\Delta_2...\Delta_k)$ *in a triangular embedding of* K_{12m+7} *with* $E(\Delta_i)\bigcap E(\Delta_{i+1}) \neq \phi, (1 \leq i \leq k)$ *corresponds to a closed eulerian walk in its current graph and a simple cycle* C_k *of length* k *in* (K, Z_{12m+7}, λ) *determines a simple closed face-chain in the corresponding triangular embedding of* K_{12m+7}.

This ends the proof of the first part of Theorem 3.

Now we begin to prove the second part of Theorem 3. Let C be a 3-edge-coloring of the underline graph of (K, Z_{12m+7}, λ) and $\Delta = (ijk)$ be a triangular face of K_{12m+7}. As we have indicated before, the oriented edges are ji, ik and kj such that $j-i, i-k$ and $k-i$ are, respectively, the current of three oriented edges (also denoted as $j-i, i-k$ and $k-i$ in convention) incident to the same vertex in the current graph preserving KCL. Now we define a coloring C^* such that

$$C^*(ji) = C(j-i); \quad C^*(ik) = C(i-k); \quad C^*(kj) = C(k-i).$$

It is easy to check that C^* is a Grunbaum coloring of the corresponding triangular embedding of K_{12m+7} and distinct edge-coloring of the underline graph of a current graph will induce different Grunbaum coloring of the corresponding triangular embeddings.

References

[BM] J.A.Bondy and U.S.R.Murty, Graph theory with applications, Macmillan, London, (1978).

[T] Tutte W.T. The factorization of linear graphs. J.Lond.Math. Soc. 22:107-111, 1947.

[CS] M.Chudnovsky and P.Seymour. Prefect matching in planar cubic graph, Preprint.

[KSS] D.Kral, J.S.Sereni and M.Stiebitz, A new lower bound on the number of perfect matching in cubic graph.

[P] J.Peterson, Die Theorie der regularen graph. Acta Math. 15(1):193-220, 1891.

[ELP] J.Edmonds, L.Lovasz and W.R.Pulleyblank. Brick decompositions and the matching of graphs. Combinatorica, 2(3):247-274, 1982.

[L] L.Lovasz Matching structure and the matching lattice. J.Combin. Theory Ser. B, 43(2):187-222, 1987.

[N] D.Naddef. Rank of maximum matching in a graph. Math. Programming, 22(1):52-70, 1982.

[S] A. Schrijver. Counting 1-factors in regular bipartite graphs. J.Combin.Theory Ser.B, 72(1):122-135, 1998.
[V] M. Voorhoeve. A lower bound for the permanents of certain (0,1)-matrices. Nederl. Akad. Wetensch. Indag. Math, 41(1):83-86, 1979.
[LP] L.lovasz and M.D.Plummer, Some recent results on graph matching, Graph Theory and Its Applications East and West: Proceedings of the First China-USA International Graph Theory Conference, Annals of the New York Academy of Sciences, 576(1989) 389-395.
[RG] Ren Han and Gao Yanbo, Lower Bound of the Number of Maximum Genus Embeddings and Genus Embeddings of K_{12s+7}, Graphs & Combinatorics,2011.
[GR] G.Ringel, *Map Color Theorem*, Springer-Verlag,Berlin,1974.