Two-parameters generalization of Pell numbers in graphs

Urszula Bednarz, Małgorzata Wołowiec-Musiał

Rzeszów University of Technology
Faculty of Mathematics and Applied Physics
al. Powstańców Warszawy 12, 35-359 Rzeszów, Poland
e-mail: ubednarz@prz.edu.pl, wolowiec@prz.edu.pl

Abstract

In this paper we introduce a new kind of two-parameters generalization of Pell numbers. We give two distinct graph interpretations and prove some identities for these numbers. Moreover we define matrix generators and derive the generalized Cassini formula for the introduced numbers.

Keywords: generalized Pell numbers, edge-colouring of the graph, corona of two graphs, matching, matrix generators, Cassini formula

MSC 11B37, 11C20, 15B36, 05C69

1 Introduction

The sequences of the Fibonacci type are given by the sth order linear equation of the form

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + \ldots + b_s a_{n-s} \tag{1}$$

for $n \geq s$, where $s \geq 2, b_i \geq 0, i = 1, \ldots, s$ are given integers and $a_0, \ldots, a_{s-1}$ are fixed integers. It is clear that for special values of s and b_i, where $i = 1, \ldots, s$ the equation (1) gives well-known recurrences. For example if $s = 2$ and $b_1 = b_2 = 1$, then we obtain classical Fibonacci numbers, usually denoted by F_n and given by the equation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with $F_0 = F_1 = 1$. If $s = 2$ and $b_1 = 2, b_2 = 1$, then we get the well-known Pell numbers, denoted by P_n and defined by the equation $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, with $P_0 = 0$ and $P_1 = 1$. Similarly, if $s = 3$ and $b_1 = b_2 = b_3 = 1$, then we get Tribonacci numbers, denoted by T_n and given by the equation $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$, with the initial conditions $T_0 = 0, T_1 = T_2 = 1$. Other more or less known examples of the sequences of the Fibonacci type can be found in [16]. In this paper we focus on Pell numbers and their generalizations.

It is worth mentioning that in the literature there are many kinds of generalizations of Pell numbers with respect to one or more parameters. Such generalizations have been intensively studied by E. Kiliç and D Tasci, mostly by matrix methods (for details see [6]-[9]). An interesting one-parameter generalization of Pell numbers and their connections with special number partitions, graphs and the usual Fibonacci numbers can be also found in [11]-[14]. In [15] a generalization of Pell numbers in a distance sense is defined. Let us recall this generalization. The generalized Pell numbers $P(k,n)$ are given by the following recurrence relation

$$P(k,n) = P(k,n-1) + P(k,n-k+1) + P(k,n-k) \quad \text{for} \quad n \geq k+1$$

with the initial conditions $P(2,0) = 0$, $P(k,0) = 1$ for $k \geq 3$, $P(k,1) = 1$ for $k \geq 2$ and $P(k,n) = 2n-2$ for $2 \leq n \leq k$.

In this paper we introduce two-parameters generalization of Pell numbers which is closely related to the concept of the generalized Pell numbers $P(k,n)$.

2 Generalized Pell numbers and their graph interpretations

We begin this section with a definition.

Let $k \geq 1$, $t \geq 1$, $n \geq 0$ be integers. The generalized Pell numbers $P_{k,t}(n)$ are given by the following recurrence relation

$$P_{k,t}(n) = P_{k,t}(n-1) + tP_{k,t}(n-k) + P_{k,t}(n-k-1) \tag{2}$$

for $n \geq k$, with the initial conditions $P_{k,t}(0) = 0$ and $P_{k,t}(n) = 1$ for $n = 1, \ldots, k$.

Note that setting $k = 1$ and $t = 1$ in (2) we obtain classical Pell numbers P_n and for $k = 2$ and $t = 1$ we get Tribonacci numbers T_n. Actually, if $t = 1$ then $P_{k,1}(n) = P(k+1, n-k+1)$ for $n \geq k-1$.

The table below presents the first few elements of $P_{k,t}(n)$ numbers for special values of k and n.

Tab.1. The generalized Pell numbers $P_{k,t}(n)$.

n	0	1	2	3	4	5
$P_{1,t}(n)$	0	1	$t+1$	$(t+1)^2+1$	$(t+1)[(t+1)^2+2]$	$(t+1)^2[(t+1)^2+3]+1$
$P_{2,t}(n)$	0	1	1	$t+1$	$2(t+1)$	$(t+1)(t+2)+1$
$P_{3,t}(n)$	0	1	1	1	$t+1$	$2(t+1)$
$P_{4,t}(n)$	0	1	1	1	1	$(t+1)$
$P_{5,t}(n)$	0	1	1	1	1	1

Based on the above table one can easily observe the following relations between introduced numbers

$$P_{k,t}(n) = (n-k)(t+1) \quad \text{for} \quad n = k+1, \dots, 2k, \tag{3}$$

$$P_{k,t}(2k+1) = (t+1)(t+k)+1. \tag{4}$$

Before giving the graph interpretations of the generalized Pell numbers $P_{k,t}(n)$ let us recall some history. Graph interpretations of the Fibonacci type numbers were introduced for the first time by H.Prodinger and R.F.Tichy (see [10]). They showed connections between Fibonacci and Lucas numbers and the number of all independent sets in some graphs. More precisely, they proved that if G is an undirected, connected graph of size m, where size means number of edges of a graph G, and $\mu(G)$ is the number of all independent sets in G, then $\mu(\mathbb{P}(m)) = F_m$ and $\mu(\mathbb{C}(m)) = L_m$, where $\mathbb{P}(m)$ and $\mathbb{C}(m)$ denote a path of size m and a cycle of size m, respectively. Those results gave an impetus for studying connections between numbers of the Fibonacci type and the number of all independent sets in different kinds of graphs and their products. Pell numbers also have such an interpretation. It is commonly known that if $Z(G)$ is the number of all matchings of G, then $Z(\mathbb{P}(m-1) \circ K_1) = P_m$, where $G \circ H$ denotes the corona of two graphs (for details see [2] and [5]). It is worth noting that the index $Z(G)$, called the Hosoya index, has many applications in combinatorial chemistry. The graph interpretation of $P(k,n)$ numbers is closely related to the concept of k-independent sets in graphs too (see [15]). For other results connected with the problem of counting of independent sets in graphs and its relation with the Fibonacci numbers see also [4]. Those various graph interpretations were our motivation for further research.

Now let us turn to the graph interpretations of the generalized Pell numbers $P_{k,t}(n)$. The first graph interpretation of these numbers we will present is connected with a special edge-colouring of the graph. An edge-colouring of the graph is a fundamental topic of the graph theory and it has been intensively studied in the literature. A new concept of an edge-colouring i.e. an edge-colouring by monochromatic paths has been introduced recently (see [1]). Let us now recall some basic definitions.

Let G be an undirected, connected, simple graph, $\mathcal{I} = \{1, \dots, k\}, k \geq 2$ and $\mathcal{I}_i = \{1, \dots, b_i\}, b_i \geq 1$. In particular, $\mathcal{I}_i$ can be empty (then we put $b_i = 0$). Moreover, let $\mathcal{C} = \bigcup_{i \in \mathcal{I}} \mathcal{C}_i$ be a nonempty family of colours, where $\mathcal{C}_i = \{A^i_j; j \in \mathcal{I}_i\}$ for $i = 1, \dots, k$. The set $\mathcal{C}_i$ is called the set of b_i shades of the colour i. Particularly, if $\mathcal{C}_i = \{A^i_1, A^i_2, \dots, A^i_t\}$ then we speak of t shades of the colour i, and if the set $\mathcal{C}_i$ has exactly one element, then we label the colour A^i_1. Naturally, a shade of the colour is also the colour. Therefore, the family $\mathcal{C}$ has exactly $\sum_{i=1}^{k} |\mathcal{C}_i| = \sum_{i=1}^{k} b_i$ colours.

A graph G is said to be *$(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)$-edge coloured by monochromatic paths* if for every maximal A_j^i-monochromatic subgraph H of G, where $A_j^i \in C_i$ and $1 \le i \le k, 1 \le j \le b_i$, there exists a partition of H into edge disjoint paths of length i. Less formally we can say that the edges of G are partitioned into paths, and paths of length i can be coloured with one of b_i possible shades. Obviously, we have to consider all possible partitions into paths of every monochromatic subgraph. Note that if $b_1 \neq 0$ then $(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)$-edge colouring by monochromatic paths always exists.

Assume that a graph G can be $(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)$-edge coloured by monochromatic paths. Let $\mathcal{F}$ be a family of distinct $(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)$-edge coloured graphs obtained by the colouring of a graph G and

$$\mathcal{F} = \{G^{(1)}, G^{(2)}, \dots, G^{(l)}\}, l \ge 1,$$

where $G^{(p)}$, $1 \le p \le l$ denotes a graph obtained by $(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)$-edge colouring by monochromatic paths of the graph G. By $\theta(G^{(p)})$ we denote the number of all partitions of $G^{(p)}$ for $1 \le p \le l$. Let

$$\sigma_{(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)}(G) = \sum_{p=1}^{l} \theta(G^{(p)}).$$

It was proved in [1] that for $G = \mathbb{P}(m)$, where $\mathbb{P}(m)$ is a path of size m the following theorem holds.

Theorem 1 *Let $k \ge 2$ and $m \ge k$ be integers. Then*

$$\sigma_{(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)}(\mathbb{P}(m)) = b_1 \sigma_{(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)}(\mathbb{P}(m-1)) + \dots$$

$$+ b_k \sigma_{(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)}(\mathbb{P}(m-k)).$$

It shows that the graph parameter $\sigma_{(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)}(G)$ is closely related to the recurrence equation (1). Many connections between the parameter $\sigma_{(A_j^i; i \in \mathcal{I}, j \in \mathcal{I}_i)}(\mathbb{P}(m))$ and the numbers of the Fibonacci type have been proved. We list only a few of them: for Pell numbers P_n, Tribonacci numbers T_n and generalized Pell numbers $P(k,n)$. For other results see [1].

Theorem 2 *Let $k \ge 2$ be integers. Then*

1. $\sigma_{(A_1^1, A_2^1, A_1^2)}(\mathbb{P}(m)) = P_{m+1}$ *for* $m \ge 1$.
2. $\sigma_{(A_1^1, A_1^2, A_1^3)}(\mathbb{P}(m)) = T_{m+1}$ *for* $m \ge 1$.
3. $\sigma_{(A_1^1, A_1^{(k-1)}, A_1^k)}(\mathbb{P}(m)) = P(k, m-k+3)$ *for* $m \ge k-3$.

It should be mentioned that such a graph interpretation is not valid for all numbers of the Fibonacci type. For example, it does not work for the Tribonacci numbers of the second type given by the equation $T^*_n = T^*_{n-1} + T^*_{n-2} + T^*_{n-3}$ for $n \geq 3$, with $T^*_0 = T^*_1 = T^*_2 = 1$.

Now we shall show that this interpretation works for numbers $P_{k,t}(n)$.

Theorem 3 *Let $k \geq 1, t \geq 1, m \geq 1$ be integers. Then*

$$\sigma_{(A^1_1, A^k_1, \ldots, A^k_t, A^{(k+1)}_1)}(\mathbb{P}(m)) = P_{k,t}(m+1).$$

Proof. By Theorem 1 it suffices to check only the initial conditions. Let us consider $(A^1_1, A^k_1, \ldots, A^k_t, A^{(k+1)}_1)$-edge colouring by monochromatic paths of a graph $\mathbb{P}(m)$. If $m = 1, 2, \ldots, k-1$ for $k \geqslant 1$ then there is a unique $(A^1_1, A^k_1, \ldots, A^k_t, A^{(k+1)}_1)$-edge colouring of $\mathbb{P}(m)$ using only the colour A^1_1. Then

$$\sigma_{(A^1_1, A^k_1, \ldots, A^k_t, A^{(k+1)}_1)}(\mathbb{P}(m)) = 1 = P_{k,t}(k).$$

If $m = k$ then we can colour edges of the path $\mathbb{P}(k)$ by the colour A^1_1 or we can use one of t shades of the colour k. Therefore

$$\sigma_{(A^1_1, A^k_1, \ldots, A^k_t, A^{(k+1)}_1)}(\mathbb{P}(k)) = t + 1 = P_{k,t}(k+1).$$

Let $m = k + 1$. If $k = 1$ then we consider $(A^1_1, \ldots, A^1_{t+1}, A^2_1)$-edge colouring by monochromatic paths of $\mathbb{P}(2)$. In this case the path is A^2_1-monochromatic or can be coloured by one of $t + 1$ shades of the colour 1 i.e. by $A^1_j, j = 1, \ldots, t+1$ on $t + 1$ ways. Then

$$\sigma_{(A^1_1, \ldots, A^1_{t+1}, A^2_1)}(\mathbb{P}(2)) = (t+1)^2 + 1 = P_{1,t}(3).$$

If $k \geq 2$ then we have the following possibilities of colouring the path $\mathbb{P}(k)$: use only the colour A^1_1 or only the colour $A^{(k+1)}_1$ or to choose simultaneously one of t shades of the colour k and the colour A^1_1 on $2t$ ways. Consequently

$$\sigma_{(A^1_1, A^k_1, \ldots, A^k_t, A^{(k+1)}_1)}(\mathbb{P}(k+1)) = 2t + 2 = P_{k,t}(k+2) \quad \text{for} \quad k \geq 2.$$

Thus by the initial conditions and by Theorem 1 the proof is complete. □

To give the second graph interpretation of the generalized Pell numbers $P_{k,t}(n)$ we recall the concept of distance (H, k)-matchings introduced by A.Włoch (see [13]).

Let G and H be two graphs, $k \geq 1$ be an integer. We say that a subgraph M of a graph G is a distance (H, k)-matching if all connected components of M are isomorphic to H and for each two components H_i, H_j from M such that $i \neq j$ we have $d_G(H_i, H_j) \geq k$, where $d_G(H_i, H_j) = \min\{d_G(x,y) : x \in V(H_i), y \in V(H_j)\}$ and $d_G(x,y)$ denotes distance between vertices x and y in the graph G. Note that the distance $(K_2, 1)$-matching is a matching in the classical sense.

Now let us consider a corona $\mathbb{P}(m) \circ N_t$ of two graphs $\mathbb{P}(m)$ and N_t, where $\mathbb{P}(m)$ is a path of size m and N_t is an edgeless graph of order t (i.e. with t vertices). By $P(0)$ we mean a graph with a unique vertex i.e. $P(0) = N_1$.

Theorem 4 *If $m \geq 0, k \geq 1$ are integers then the number of all distance (K_2, k)- matchings of a graph $\mathbb{P}(m) \circ N_t$ is equal to $P_{k,t}(m+k-1)$.*

Proof (by induction on m). Let us denote by $p\,(\mathbb{P}(m) \circ N_t)$ the number of all distance (K_2, k)-matchings of a graph $\mathbb{P}(m) \circ N_t$. For $m = 0$ a path $\mathbb{P}(m)$ reduces to one vertex and then $p\,(\mathbb{P}(0) \circ N_t) = t + 1 = P_{k,t}(k+1)$.

Assume now that $m \geq 1$ and the statement is true for an arbitrary m. We shall prove that it is true for $m+1$. Consider a graph $\mathbb{P}(m+1) \circ N_t$ (see Fig. 1).

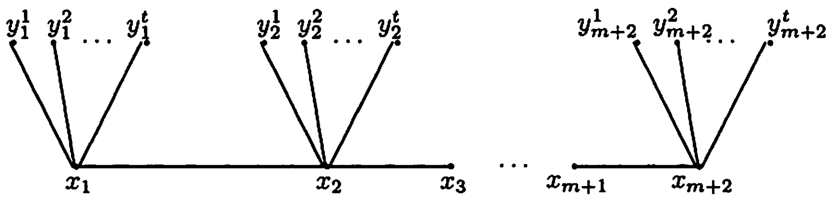

Figure 1.

Let M be any (K_2, k)-matching of the graph $\mathbb{P}(m+1) \circ N_t$. The following cases are possible:

1) the edge $x_{m+2}y_{m+2}^i \in M$ for some $i = 1, \ldots, t$,

2) the edge $x_{m+2}y_{m+2}^i \notin M$ for any $i = 1, \ldots, t$ and the edge $x_{m+1}x_{m+2} \in M$,

3) edges $x_{m+2}y_{m+2}^i$ and $x_{m+1}x_{m+2}$ do not belong to M.

Let $p_1(\mathbb{P}(m+1) \circ N_t)$, $p_2(\mathbb{P}(m+1) \circ N_t)$ and $p_3(\mathbb{P}(m+1) \circ N_t)$ be numbers of all distance (K_2, k)- matchings of the graph $\mathbb{P}(m+1) \circ N_t$ in the case 1), the case 2) and the case 3), respectively.

In the first case we have

$$p_1(\mathbb{P}(m+1) \circ N_t) = tp(\mathbb{P}(m+1-k) \circ N_t),$$

and by the induction hypothesis we get

$$p_1(\mathbb{P}(m+1) \circ N_t) = tP_{k,t}(m).$$

In the second case we have

$$p_2(\mathbb{P}(m+1)\circ N_t) = p(\mathbb{P}(m-k)\circ N_t),$$

and by the induction hypothesis we get

$$p_2(\mathbb{P}(m+1)\circ N_t) = P_{k,t}(m-1).$$

In the third case we have

$$p_3(\mathbb{P}(m+1)\circ N_t) = p(\mathbb{P}(m-1)\circ N_t),$$

and by the induction hypothesis we get

$$p_3(\mathbb{P}(m+1)\circ N_t) = P_{k,t}(m-1+k).$$

It is obvious that

$$p(\mathbb{P}(m+1)\circ N_t) = p_1(\mathbb{P}(m+1)\circ N_t)+p_2(\mathbb{P}(m+1)\circ N_t)+p_3(\mathbb{P}(m+1)\circ N_t).$$

From the above cases and the definition of $P_{k,t}(n)$ we obtain

$$p(\mathbb{P}(m+1)\circ N_t) = tP_{k,t}(m)+P_{k,t}(m-1)+P_{k,t}(m-1+k) = P_{k,t}(m+k).$$

Thus the proof is complete. □

3 Identities

In this section we present some identities for generalized Pell numbers $P_{k,t}(n)$ and extend these numbers to negative integers. We show how to use the graph interpretation given in the previous section for proving identities.

Theorem 5 *Let $k \geq 1, t \geq 1, n \geq 2k$ be integers. Then*

$$P_{k,t}(n) = P_{k,t}(n-2k) + (t+1)\sum_{i=0}^{k-1} P_{k,t}(n-k-i).$$

Proof. Assume that $n \geq 2k$. Then using (2) and some calculations we get
$P_{k,t}(n) = P_{k,t}(n-1) + tP_{k,t}(n-k) + P_{k,t}(n-k-1) =$
$= P_{k,t}(n-2)+tP_{k,t}(n-k)+(t+1)P_{k,t}(n-k-1)+P_{k,t}(n-k-2) =$
$= P_{k,t}(n-3)+tP_{k,t}(n-k)+(t+1)[P_{k,t}(n-k-1)+P_{k,t}(n-k-2)] + P_{k,t}(n-k-3).$
After $k-1$ steps we obtain
$P_{k,t}(n) = P_{k,t}(n-2k)+(t+1)[P_{k,t}(n-k)+P_{k,t}(n-k-1)+\ldots+P_{k,t}(n-2k+1)] =$
$= P_{k,t}(n-2k) + (t+1)\sum_{i=0}^{k-1} P_{k,t}(n-k-i).$ □

Theorem 6 *Let $k \geq 1, t \geq 1, m \geq k+1, 1 \leq n \leq m-k-1$ be integers. Then*

$$P_{k,t}(m+1) = P_{k,t}(n)P_{k,t}(m-n+1)+t\sum_{r=1}^{k} P_{k,t}(n-r+1)P_{k,t}(m-n-k+r+1)+$$

$$\sum_{r=1}^{k+1} P_{k,t}(n-r+1)P_{k,t}(m-n-k+r).$$

Proof. To prove this identity we use the first graph interpretation of the number $P_{k,t}(m)$. Let us consider the path $\mathbb{P}(m)$ of size m. Let $n \in E(\mathbb{P}(m))$ and $1 \leq n \leq m-k-1$. By $\sigma_{(A_1^1, A_1^k, \ldots, A_t^k, A_1^{(k+1)})}(\mathbb{P}(m))$ we denote the number of all $(A_1^1, A_1^k, \ldots, A_t^k, A_1^{(k+1)})$-edge colouring by monochromatic paths of the graph $\mathbb{P}(m)$. For convenience we will write $\sigma(\mathbb{P}(m))$ instead of $\sigma_{(A_1^1, A_1^k, \ldots, A_t^k, A_1^{(k+1)})}(\mathbb{P}(m))$. We have to consider the following possibilities. The edge n could be coloured by the colour A_1^1 or by one of t shades of the colour k, or by $A_1^{(k+1)}$. Then the number of all colouring of the path $\mathbb{P}(m)$ in every case we denote by $\sigma^{A_1^1}(\mathbb{P}(m))$, $\sigma^{A_j^k}(\mathbb{P}(m)), \sigma^{A_1^{(k+1)}}(\mathbb{P}(m))$, respectively. Taking into account that there are t shades of the colour k it is clear that

$$\sigma(\mathbb{P}(m)) = \sigma^{A_1}(\mathbb{P}(m)) + t\sigma^{A_j^k}(\mathbb{P}(m)) + \sigma^{A_1^{(k+1)}}(\mathbb{P}(m)).$$

Let us consider the possibilities mentioned previously.

(*i*) If n is coloured by the colour A_1^1 then

$$\sigma^{A_1^1}(\mathbb{P}(m)) = \sigma(\mathbb{P}(n-1))\sigma(\mathbb{P}(m-n)).$$

(*ii*) If n is coloured by one of shades of the colour k then the edge n could be the $vth, v = 1, \ldots, k$ edge of the subpath of $\mathbb{P}(m)$ of length k. Then

$$\sigma^{A_j^k}(\mathbb{P}(m)) = \sum_{r=1}^{k} \sigma(\mathbb{P}(n-r))\sigma(\mathbb{P}(m-n-k+r)).$$

(*iii*) If n is coloured by the colour $A_1^{(k+1)}$ then, analogously like in the previous case, we have

$$\sigma^{A_1^{(k+1)}}(\mathbb{P}(m)) = \sum_{r=1}^{k+1} \sigma(\mathbb{P}(n-r))\sigma(\mathbb{P}(m-n-k+r-1)).$$

Finally from the above cases we obtain

$$\sigma(\mathbb{P}(m)) = \sigma(\mathbb{P}(n-1))\sigma(\mathbb{P}(m-n)) + t\sum_{r=1}^{k}\sigma(\mathbb{P}(n-r))\sigma(\mathbb{P}(m-n-k+r)) +$$

$$+\sum_{r=1}^{k+1}\sigma(\mathbb{P}(n-r))\sigma(\mathbb{P}(m-n-k+r-1)).$$

Applying Theorem 3 we obtain

$$P_{k,t}(m+1) = P_{k,t}(n)P_{k,t}(m-n+1) + t\sum_{r=1}^{k}P_{k,t}(n-r+1)P_{k,t}(m-n-k+r+1) +$$

$$\sum_{r=1}^{k+1}P_{k,t}(n-r+1)P_{k,t}(m-n-k+r).$$

Thus ends the proof. □

The generalized Pell numbers $P_{k,t}(n)$ can be extended to negative integers n. Let $k \geq 1$, $n \geq 1$ and $t \geq 1$ be integers. Then

$$P_{k,t}(-n) = P_{k,t}(k-n+1) - P_{k,t}(k-n) - tP_{k,t}(1-n) \text{ for } n \geq 1 \tag{5}$$

with initial conditions $P_{k,t}(0) = 0, P_{k,t}(n) = 1$ for $n = 1, \ldots, k$.

Table 2 includes the first few elements of $P_{k,t}(n)$ sequence for special k and negative n.

Tab.2. The generalized Pell numbers $P_{k,t}(n)$ for negative n.

n	-4	-3	-2	-1	0	1
$P_{1,t}(n)$	$-(t+1)[(t+1)^2+2]$	$(t+1)^2+1$	$-t-1$	1	0	1
$P_{2,t}(n)$	$-1+t^2$	$-t$	1	0	0	1
$P_{3,t}(n)$	$-t$	1	0	0	0	1
$P_{4,t}(n)$	1	0	0	0	0	1

Notice that for $k = 1, t = 1$ we get the well-known extension of Pell numbers for negative numbers.

4 Matrix generators

In this section we give matrix generators for generalized Pell numbers $P_{k,t}(n)$ using the same idea as in [14].

Based on the equation (2), which describes generalized Pell numbers, let us define a matrix $P_{k,t} = [p_{ij}]_{(k+1)\times(k+1)}$ as follows. For $j = 1$ and $i = 1, 2 \ldots, k+1$ an element p_{ij} of the matrix $P_{k,t}$ is equal to the coefficient of $P_{k,t}(n-i)$ in the right-hand-side of the equation (2). For $2 \le j \le k+1$ and $i = 1, 2 \ldots k+1$ we put

$$p_{ij} = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{otherwise }. \end{cases}$$

By the above definition for $k = 1, 2, 3, 4, \ldots$ we obtain matrices:

$$P_{1,t} = \begin{bmatrix} t+1 & 1 \\ 1 & 0 \end{bmatrix},\ P_{2,t} = \begin{bmatrix} 1 & 1 & 0 \\ t & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},\ P_{3,t} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$P_{4,t} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ t & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ldots, P_{k,t} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ t & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We will call the matrix $P_{k,t}$ the generalized Pell matrix.

Note that for $k = t = 1$ we get the matrix $P_{1,1} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ having the property $P_{1,1}^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$, and being the classical matrix generator of Pell numbers

defined by J. Ercolano [3].

Let $k \ge 1, t \ge 1$ be integers. For fixed $k \ge 1, t \ge 1$ we define a matrix $A_{k,t}$ of order $k+1$ as the matrix of initial conditions

$$A_{k,t} = \begin{bmatrix} P_{k,t}(2k) & P_{k,t}(2k-1) & \cdots & P_{k,t}(k+1) & P_{k,t}(k) \\ P_{k,t}(2k-1) & P_{k,t}(2k-2) & \cdots & P_{k,t}(k) & P_{k,t}(k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{k,t}(k+1) & P_{k,t}(k) & \cdots & P_{k,t}(2) & P_{k,t}(1) \\ P_{k,t}(k) & P_{k,t}(k-1) & \cdots & P_{k,t}(1) & P_{k,t}(0) \end{bmatrix}.$$

Theorem 7 *Let $k \ge 1$, $t \ge 1$, $n \ge 1$ be integers. Then*

$$A_{k,t}P_{k,t}^n = \begin{bmatrix} P_{k,t}(n+2k) & P_{k,t}(n+2k-1) & \cdots & P_{k,t}(n+k+1) & P_{k,t}(n+k) \\ P_{k,t}(n+2k-1) & P_{k,t}(n+2k-2) & \cdots & P_{k,t}(n+k) & P_{k,t}(n+k-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{k,t}(n+k+1) & P_{k,t}(n+k) & \cdots & P_{k,t}(n+2) & P_{k,t}(n+1) \\ P_{k,t}(n+k) & P_{k,t}(n+k-1) & \cdots & P_{k,t}(n+1) & P_{k,t}(n+0) \end{bmatrix}. \quad (6)$$

Proof (by induction on n). Let $k \geq 1$ and $t \geq 1$ be fixed integers. If $n = 1$, then by simple calculations and recurrence relation (2) we get (6). Assume now that the statement is true for all integers $1, ..., n$. We shall show that it is true for an integer $n+1$. Since $A_{k,t}P_{k,t}^{n+1} = A_{k,t}P_{k,t}^{n}P_{k,t}$, thus by our assumption and the recurrence relation (2) we obtain

$$A_{k,t}P_{k,t}^{n+1} = \begin{bmatrix} P_{k,t}(n+2k) & \cdots & P_{k,t}(n+k+1) & P_{k,t}(n+k) \\ P_{k,t}(n+2k-1) & \cdots & P_{k,t}(n+k) & P_{k,t}(n+k-1) \\ \vdots & \ddots & \vdots & \vdots \\ P_{k,t}(n+k+1) & \cdots & P_{k,t}(n+2) & P_{k,t}(n+1) \\ P_{k,t}(n+k) & \cdots & P_{k,t}(n+1) & P_{k,t}(n+0) \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} P_{k,t}(n+1+2k) & P_{k,t}(n+2k) & \cdots & P_{k,t}(n+2+k) & P_{k,t}(n+1+k) \\ P_{k,t}(n+2k) & P_{k,t}(n-1+2k) & \cdots & P_{k,t}(n+1+k) & P_{k,t}(n+k) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{k,t}(n+2+k) & P_{k,t}(n+1+k) & \cdots & P(k,n+3) & P(k,n+2) \\ P_{k,t}(n+1+k) & P(k,n+k) & \cdots & P_{k,t}(n+2) & P_{k,t}(n+1) \end{bmatrix}.$$

□

Theorem 8 *Let $k \geq 1$, $t \geq 1$ be integers. Then*

$$det(P_{k,t}) = (-1)^{k}, \tag{7}$$

$$det(A_{k,t}) = (-1)^{\frac{k(k+1)}{2}}. \tag{8}$$

Proof. Equality (7) follows from the definition of $P_{k,t}$ and basic properties of determinants. To prove equality (8) it is enough to notice that by definition of $A_{k,t}$, recurrence relation (2) and observation (3) we have

$$A_{k,t} = \begin{bmatrix} k(t+1) & (k-1)(t+1) & \cdots & t+1 & 1 \\ (k-1)(t+1) & (k-2)(t+1) & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t+1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

Now we use elementary operations. Subtracting the $(k+1-s)$th row from the $(k-s)$th row, for $s = 0, \ldots, k-1$, and then by Laplace expansion of determinant with respect to the last column we get

$$\det A_{k,t} = \det \begin{bmatrix} t+1 & t+1 & \cdots & t & 0 \\ t+1 & t+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} =$$

$$= -\det \begin{bmatrix} t+1 & t+1 & \cdots & t+1 & t \\ t+1 & t+1 & \cdots & t & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t+1 & t & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Analogously, subtracting the $(k-s)$th column from the $(k-1-s)$th column, for $s = 0, \ldots k-2$, and then expanding with respect to the last row we get

$$\det A_{k,t} = -\det \begin{bmatrix} 0 & 0 & \cdots & 1 & t \\ 0 & 0 & \cdots & t & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & t & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} =$$

$$= -\det \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & t & \cdots & 0 & 0 \end{bmatrix} = (-1)^{\frac{k(k+1)}{2}}.$$

□

As an immediate consequence of Theorem (8) and Cauchy's Theorem for determinants we obtain two generalizations of the Cassini formula.

Theorem 9 *Let $k \geq 1, t \geq 1$ be integers. Then*

$$det(P_{k,t}^n) = (-1)^{nk} \tag{9}$$

$$det(A_{k,t}P_{k,t}^n) = (-1)^{\frac{k(k+1+2n)}{2}} \tag{10}$$

Putting $k = t = 1$ to the equality (9) we get the well-known Cassini formula $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$.

Acknowledgment

The authors wish to thank the referee for all suggestions which have improved this paper.

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