Matching extendability of balanced hypercubes*

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Abstract

The balanced hypercube, which is a variant of the hypercube, is proposed as a novel inter-processor network. Among the attractive properties of the balanced hypercube, the most special one is that each processor has a backup processor sharing the same neighborhood. A connected graph G with at least $2m+2$ vertices is said to be *m-extendable* if it possesses a matching of size m and every such matching can be extendeded to a perfect matching of G. In this paper, we prove that the balanced hypercube BH_n is m-extendable for every m with $1 \leq m \leq 2n-2$, and our result is optimal.

Key words: Interconnection networks; The balanced hypercube; Perfect matching; Matching extendability

1 Introduction

Let $G = (V, E)$ be a simple undirected graph, where V is vertex-set of G and E is edge-set of G. A *matching* of G is a set of independent edges of G and a *perfect matching* of G is a matching covers all vertices of G. A connected graph G with at least $2m+2$ vertices is said to be

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m-*extendable* if it possesses a matching of size m and every such matching can be extended to a perfect matching of G. A graph is *bipartite* if its vertex-set can be partitioned into two subsets such that each edge has its ends in different subsets. A *path* P from v_0 to v_n is a sequence of vertices $v_0v_1 \cdots v_n$ from v_0 to v_n such that every pair of consecutive vertices are adjacent and all vertices are distinct except for v_0 and v_n. We also denote the path $P = v_0v_1 \cdots v_n$ by $\langle v_0, P, v_n \rangle$. The *length* of a path P is the number of edges in P, denoted by $l(P)$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. For the graph terminologies and notations not defined here, we refer the reader to [28].

It is useful to design distributed processor architectures that offers high connectivity and reliability. The system topology, which defines the inter-processor network (called interconnection network), is an important part of such a distributed system. The hypercube network is one of most popular instances of interconnection networks, which has many attractive properties such as regularity, strong connectivity and symmetry. With such excellent properties, the hypercube has received much attention of graph theorists and computer scientists. However, the hypercube has its own drawback, such as its large diameter. So many variants of the hypercube have been proposed, see [1, 7–11, 17, 25, 29, 30].

Among these variants, the balanced hypercube is the only one that each vertex has a backup (matching) vertex sharing the same neighborhood. Thus, tasks running on a faulty vertex can be easily transferred to its backup vertex. It is also known [29] that the odd-dimensional balanced hypercube BH_n, which has the same number of vertices as Q_{2n}, is of the smaller diameter than that of the hypercube Q_{2n}. Owing to attractive properties above, the balanced hypercube has been extensively studied in literatures, see [4, 13, 14, 20, 21, 31–33, 35].

Brigham et al. [3] showed that if each vertex has a special partner at any time, especially under the event of edge failure, then the network performs robustness in this sense. In order to measure this property of networks, they introduced the concept of matching preclusion. Recently, matching preclusion of famous interconnection networks was extensively studied in [5, 6, 12, 20, 23, 27]. In this paper we consider the opposite aspect of this property, that is, given a matching M of G, can it be extended to a perfect matching of G? Such a problem was first proposed by Plummer in 1980 [24], and later studied in many kinds of graphs [2, 16, 18, 19, 22, 26, 34]. Especially in [18], the authors obtained that matching extendability of the hypercube Q_n is $n-1$. Recently, Vandenbussche et al. [26] extended the result of [18] by using the concept of k-suitable matching. Since the balanced hypercube possesses some novel properties such that the hypercube does not have, it is of interest to consider such a problem for the balanced hypercube. It is also known that a polynomial algorithm for matching extendability of bipartite graphs has been obtained in [15]. However, it is another flavor to derive a combinatorial formula of matching extendability for the balanced hypercube.

The rest of this paper is organized as follows. In the next section, we present some necessary definitions and properties of balanced hypercubes as preliminaries. Some useful lemmas and the main result (Theorem 3.7) are shown in Section 3. Finally, conclusions are given in Section 4.

2 Preliminaries

Wu and Huang [29] gave two equivalent definitions of BH_n as follows:

Definition 1. An n-dimension balanced hypercube BH_n consists of 4^n vertices $(a_0, a_1, \ldots, a_{n-1})$, where $a_i \in \{0, 1, 2, 3\}$ for each $0 \leq i \leq n-1$. An arbitrary vertex $(a_0, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1}), 1 \leq i \leq n-1$, in BH_n has the following $2n$ neighbors:

(1). $((a_0 + 1) \mod 4, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$,
$((a_0 - 1) \mod 4, a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n-1})$, and

(2). $((a_0 + 1) \mod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) \mod 4, a_{i+1}, \ldots, a_{n-1})$,
$((a_0 - 1) \mod 4, a_1, \ldots, a_{i-1}, (a_i + (-1)^{a_0}) \mod 4, a_{i+1}, \ldots, a_{n-1})$.

As a variant of the hypercube, BH_n also has its hierarchical structure as the definition follows.

Definition 2. BH_n consists of four copies of BH_{n-1} labeled by $BH_{n-1}^{(0)}$, $BH_{n-1}^{(1)}$, $BH_{n-1}^{(2)}$, $BH_{n-1}^{(3)}$ respectively, with a new dimension i $(0 \leq i \leq 3)$ added as the $(n-1)$-dimension index of every vertex in each $BH_{n-1}^{(i)}$. For a given vertex $v = (a_0, a_1, \ldots, a_{n-2}, i)$ in $BH_{n-1}^{(i)}$, besides the neighbors in $BH_{n-1}^{(i)}$, there exist two extra neighbors: $((a_0+1) \mod 4, a_1, \ldots, a_{n-2}, (i+1) \mod 4)$ and $((a_0 - 1) \mod 4, a_1, \ldots, a_{n-2}, (i+1) \mod 4)$ if a_0 is even, or $((a_0 + 1) \mod 4, a_1, \ldots, a_{n-2}, (i-1) \mod 4)$ and $((a_0 - 1) \mod 4, a_1, \ldots, a_{n-2}, (i-1) \mod 4)$ if a_0 is odd.

Since the hierarchical structure of BH_n is naturally in accord with the induction method, it plays an important role in our proof of the main theorem.

BH_1 and BH_2 are illustrated in Figs. 1 and 2, respectively.

In what follows, we show some basic properties of BH_n, which will be used in the following paper.

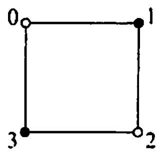

Fig. 1. BH_1.

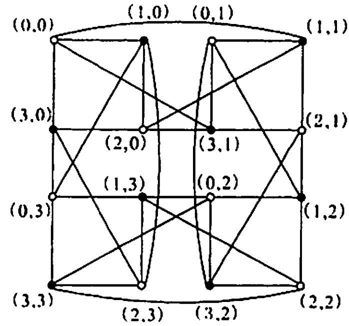

Fig. 2. BH_2.

Proposition 2.1 [29, 35]. The balanced hypercube is bipartite, vertex-transitive and edge-transitive.

Proposition 2.2 [29]. The vertices $(a_0, a_1, \ldots, a_{n-1})$ and $((a_0 + 2) \bmod 4, a_1, \ldots, a_{n-1})$ of BH_n have the same neighborhood.

3 Main result

In BH_n, the first coordinate a_0 of $(a_0, \ldots, a_i, \ldots, a_{n-1})$ is called *inner index*, and the other coordinates a_i $(1 \leq i \leq n-1)$ *i-dimension index*. Let u and v be two adjacent vertices in BH_n. If u and v differ only the inner index, then uv is said to be *0-dimension* edge, additionally, u and v are mutually called 0-dimension neighbor. If u and v differ not only the inner index, but also some i-dimension index $(i \neq 0)$, then uv is called *i-dimension* edge, analogously, u and v are mutually called i-dimension neighbor. Let $BH_{n-1}^{(i)}$ $(0 \leq i \leq 3)$ be the subgraph of BH_n induced by the vertices of BH_n with the $(n-1)$-dimension index i. That is, the $BH_{n-1}^{(i)}$'s can be obtained from BH_n by deleting all $(n-1)$-dimension edges. Let ∂D_i $(0 \leq i \leq n-1)$ be the set of i-dimension edges (also denoted the graph induced by ∂D_i when there is no ambiguity). So $BH_{n-1}^{(i)} \cong BH_{n-1}$ for each $0 \leq i \leq 3$. For simplicity, we denote $BH_{n-1}^{(i)}$s by B_i for $i \in \{0, 1, 2, 3\}$ respectively, and denote the set of edges between B_i and B_j by E_{ij} for $i, j \in \{0, 1, 2, 3\}$ if exists. We also use w_i and u_i (resp. b_i and v_i) to denote white (resp. black) vertices in B_i $(0 \leq i \leq 3)$.

By Proposition 2.1, it can be known that BH_n is bipartite. We can use V_0 and V_1 to denote the two partite sets of BH_n such that V_0 and V_1 consist of vertices with even inner indices and odd inner indices, respectively. For convenience, the vertices of V_0 and V_1 are colored white and black, respectively.

The following statements will be used in our main theorem, we now present it.

Lemma 3.1 [20]. In BH_n, $\partial D_i (0 \le i \le n-1)$ can be divided into 4^{n-1} edge-disjoint 4-cycles.

Theorem 3.2 [31]. The balanced hypercube BH_n is Hamiltonian laceable for $n \ge 1$.

Lemma 3.3 [31]. Let uv be an edge of BH_n. Then uv is contained in a cycle C of length 8 in BH_n such that $|E(C) \cap E(B_i)| = 1$ for each $i = 0, 1, 2, 3$.

Lemma 3.4 [4]. Let V_0 and V_1 be two partite sets of BH_n. Additionally, $u, x \in V_0$ and $v, y \in V_1$. Then there exist two vertex-disjoint paths P and Q such that: (1) P connects u to v, (2) Q connects x to y, (3) $V(P) \cup V(Q) = V(BH_n)$.

Lemma 3.5. Let $P = v_i u_i' v_i' u_i$ be a 3-path in B_i. In addition, $f = u_j v_j$ is an edge in B_j ($j \in \{0,1,2,3\} \setminus \{i\}$). Then P is contained in a cycle C of BH_n with $10 \le |E(C)| \le 14$ and $u_j, v_j \notin V(C)$, such that $|E(C) \cap E_{01})| = 1$, $|E(C) \cap E_{12})| = 1$, $|E(C) \cap E_{23})| = 1$, $|E(C) \cap E_{30})| = 1$ and $|E(B_j) \cap E(C)| = 1$ or 3.

Proof. By symmetry of BH_n, suppose that $i = 0$. Moreover, we may assume that f is contained in B_3, that is $f = u_3 v_3$.

Case 1: $u_0 v_0$ is an edge of B_0. By Lemma 3.3, there exists a cycle C of length 10 containing P. If $u_3, v_3 \notin V(C)$, we are done. Thus, $|\{u_3, v_3\} \cap V(C)| \le 2$. If $|\{u_3, v_3\} \cap V(C)| = 1$. That is, exact one of u_3 and v_3, say u_3, is contained in C. Let w_3 (resp. b_3) be the white (resp. black) vertex differ only the inner index of u_3 (resp. v_3). We only need to replace $u_3 \in V(C)$ by w_3. If $|\{u_3, v_3\} \cap V(C)| = 2$, then $f \in E(C)$. Note that w_3 and u_3 (resp. b_3 and v_3) have the same neighborhood. We delete u_3 and v_3 from C, and add $w_3 b_3$ to C by connecting w_3 (resp. b_3) to the neighbor of u_3 (resp. v_3) on C. Thus, a cycle of length 10 containing P but no endpoints of f follows.

Case 2: u_0 and v_0 are non-adjacent. Then P contains at most two edges of the same dimension, otherwise, u_0 and v_0 are adjacent.

Case 2.1: P contains exact two edges of the same dimension, say k ($0 \le k < n-1$). Then two k-dimension edges must be $v_0 u_0'$ and $v_0' u_0$. Otherwise, u_0 and v_0 are adjacent, a contradiction. Therefore, we may assume that $v_0' u_0'$ is an l-dimension edge ($0 \le l < n-1$, $l \ne k$). Let v_1 be an $(n-1)$-dimension neighbor of u_0. and $v_1 u_1 \in E(B_1)$ be an l-dimension edge. There exists an edge $u_1 v_2$ from B_1 to B_2. Let $P_2 = v_2 u_2' v_2' u_2$ be a 3-path in B_2

such that v_2u_2' and $v_2'u_2$ are both 0-dimension edges, and $u_2'v_2'$ is a k-dimension edge. In addition, there exists an edge u_2v_3 from B_2 to B_3. Similarly, let $P_3 = v_3u_3'v_3'u_3$ be a 3-path in B_3 such that v_3u_3' and $v_3'u_3$ are both 0-dimension edges, and $u_3'v_3'$ is a k-dimension edge. It implies that u_3 and v_0 are adjacent. Thus, $C = \langle v_0, P, u_0, v_1, u_1, v_2, P_2, u_2, v_3, P_3, u_3, v_0\rangle$ is a cycle of length 14 containing P. (Note that P_2 (resp. P_3) will be degenerated to a 0-dimension edge v_2u_2 (resp. v_3u_3) when $k = 0$, then C is a cycle of length 10 containing P.) If $u_3, v_3 \notin E(C)$, we are done. Thus, we may assume that $|\{u_3, v_3\} \cap V(C)| = 2$. Similarly, using the approach of Case 1, we can obtain a cycle of length 14 containing P but no endpoints of f.

Case 2.2: The edges contained in P are of dimension k, l and m ($0 \leq k < l < m < n-1$). If $k \neq 0$, analogously, by Case 2.1, we can obtain a cycle of length 14 containing P but no endpoints of f. Now we study the case $k = 0$. If $u_0'v_0'$ is a 0-dimension edge, we can also obtain a cycle of length 14 containing P but no endpoints of f. So we assume v_0u_0' or $v_0'u_0$, say v_0u_0', is a 0-dimension edge. Thus, we may assume that $u_0'v_0'$ (resp. $v_0'u_0$) is an l-dimension (resp. m-dimension) edge. Let v_1 be an $(n-1)$-dimension neighbor of u_0, and $v_1u_1 \in E(B_1)$ be an l-dimension edge. There exists an edge u_1v_2 from B_1 to B_2. Let $v_2u_2 \in E(B_2)$ be a 0-dimension edge. In addition, there exists an edge u_2v_3 from B_2 to B_3. Again, let $P_3 = v_3u_3'v_3'u_3$ be a 3-path in B_3 such that v_3u_3' and $v_3'u_3$ are both 0-dimension edges, and $u_3'v_3'$ is an m-dimension edge. It implies that u_3 and v_0 are adjacent. Thus, $C = \langle v_0, P, u_0, v_1, u_1, v_2, u_2, v_3, P_3, u_3, v_0\rangle$ is a cycle of length 12 containing P. If $u_3, v_3 \notin E(C)$, we are done. Using the approach of Case 1 again, we can obtain a cycle of length 12 containing P but no endpoints of f. □

Next we present the following lemma as the basis of the main theorem of this paper.

Lemma 3.6. BH_2 is 2-extendable.

Proof. Let e_0 and e_1 be two independent edges of BH_2. By Definition 1, it follows from Lemma 3.1 that, ∂D_i ($i = 0, 1$) can be divided into four vertex-disjoint 4-cycles. If e_0 and e_1 are both in ∂D_0 or ∂D_1, then it can be easily derived a perfect matching of BH_2 containing e_0 and e_1. Thus we only consider the case when $e_0 \in \partial D_0$ and $e_1 \in \partial D_1$. By Proposition 2.1, BH_2 is edge-transitive, without loss of generality (w.l.o.g.), suppose that $e_1 = (0,0)(1,1)$. By symmetry of BH_2, it remains to consider $e_0 = (1,0)(2,0)$ or $(0,3)(1,3)$. Two perfect matchings containing e_0 and e_1 are shown in Table 1. □

Table 1. Two perfect matchings of BH_2 with e_0=(1,0)(2,0),(0,3)(1,3) and e_1=(0,0)(1,1).

e_0	e_1	perfect matchings containing e_0 and e_1
(1,0)(2,0)	(0,0)(1,1)	(1,0)(2,0),(0,0)(1,1),(0,1)(3,1),(3,0)(0,3),(2,1)(1,2),(1,3)(2,3),(0,2)(3,2),(2,2)(3,3)
(0,3)(1,3)	(0,0)(1,1)	(0,3)(1,3),(0,0)(1,1),(2,0)(3,0),(1,0)(2,3),(2,2)(3,3),(0,1)(3,2),(2,1)(3,1),(0,2)(1,2)

Now we state our main theorem of this paper.

Theorem 3.7. BH_n is $(2n-2)$-extendable.

Proof. We use induction on n. It follows from Lemma 3.6 that the theorem is true for $n = 2$, thus, the induction step holds. So we assume that the theorem holds for all integers $3 \leq k < n$. Next we consider BH_n. Let M be a set of independent edges of BH_n such that $|M| = 2n-2$. If for each $i \in \{0,1,2,3\}$, $|E(B_i) \cap M| \leq 2n-4$, by induction hypothesis, it can be derived a perfect matching of each B_i, which obviously yields a perfect matching of BH_n. Thus, suppose that there exists some $i \in \{0,1,2,3\}$ such that $2n-3 \leq |E(B_i) \cap M| \leq 2n-2$. Suppose w.l.o.g. that $i = 0$. Next we consider the following two cases:

Case 1: $|E(B_0) \cap M| = 2n-3$. Let $e \in M \cap E(B_0)$ and $f \in M - E(B_0)$, by induction hypothesis, $M \cap E(B_0) - e$ can be extended to a perfect matching M_0 of B_0.

Case 1.1: $e \in M_0$.

Case 1.1.1: $f \in E_{01}$ or E_{03}. Suppose w.l.o.g. that $f \in E_{01}$. Let $f = u_0v_1$, where u_0 is a white vertex in B_0 and v_1 is a black vertex in B_1. Thus there exists an edge $u_0v_0 \in M_0$. Observe that the edges in M are independent, then $u_0v_0 \notin M$. By Lemma 3.3, there exists an 8-cycle $C = u_0v_0u_3v_3u_2v_2u_1v_1u_0$ containing u_0v_0, where u_3v_3 (resp. u_2v_2, u_1v_1) is an edge in B_3 (resp. B_2, B_1). By Lemma 3.2, there exists a Hamiltonian path P_1 (resp. P_2, P_3) from u_1 to v_1 (resp. u_2 to v_2, u_3 to v_3) in B_1 (resp. B_2, B_3). Thus $C' = \langle u_0, v_0, u_3, P_3, v_3, u_2, P_2, v_2, u_1, P_1, v_1, u_0 \rangle$ is a cycle containing f. Note that $l(P_1) = l(P_2) = l(P_3) = 4^{n-1} - 1$, C' is an even cycle, thus an alternating cycle. So there exists a perfect matching $M_{C'}$ of C' containing f. Hence $M_0 \cup M_{C'} \setminus \{u_0v_0\}$ is a perfect matching of BH_n (see Fig. 3, heavy lines mean edges of M_0, solid lines mean edges of $M_{C'}$).

Case 1.1.2: $f \in E_{12}$ or E_{23}. W.l.o.g, suppose that $f \in E_{12}$, i.e. $f = u_1v_2$. The proof is analogous to that of Case 1.1.1, we omit it.

Case 1.1.3: $f \in E(B_i)$ $(1 \leq i \leq 3)$. Let $u_0v_0 \in M_0 \setminus M$ be an edge in B_0. We first claim that there exists an 8-cycle $C = u_0v_0u_3v_3u_2v_2u_1v_1u_0$ of BH_n such that neither endpoints of f is on C, where u_1v_1 is an edge of B_1, u_2v_2 is an edge of B_2, and u_3v_3 is an edge of B_3. It follows from Lemma 3.3 that there exists an 8-cycle $C = u_0v_0u_3v_3u_2v_2u_1v_1u_0$ of BH_n. If neither endpoints of f is on C, we are done. Otherwise, we may assume that $f \in E(C)$, then $f = u_iv_i$ (i=1,2,3). Suppose, w.l.o.g, that $f = u_3v_3$. Let

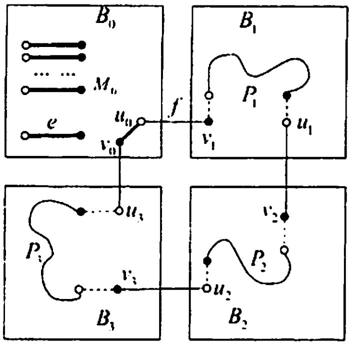

Fig. 3. Illustration for Case 1.1.1.

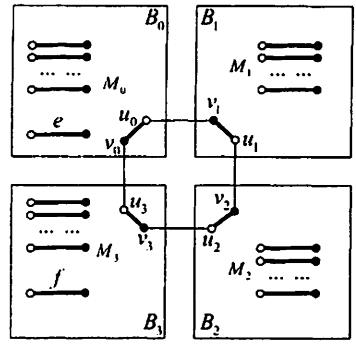

Fig. 4. Illustration for Case 1.1.3.

u_3 and u_3' (resp. v_3 and v_3') be two vertices having the same neighborhood. Then $u_0v_0u_3'v_3'u_2v_2u_1v_1u_0$ is the 8-cycle as required. For convenience, we also denote it by C. Let $M_{u_0v_1}$ be the perfect matching of C containing u_0v_1. By Lemma 3.6, u_1v_1 (resp. u_2v_2) can be extended to a perfect matching M_1 (resp. M_2) of B_1 (resp. B_2). Especially, f and u_3v_3 can be extended to a perfect matching M_3 of B_3. Thus, $M_0 \cup M_1 \cup M_2 \cup M_3 \cup M_{u_0v_1} \setminus \{u_0v_0, u_1v_1, u_2v_2, u_3v_3\}$ is a perfect matching of BH_n containing M (see Fig. 4, heavy lines mean edges of $M_0 \cup M_1 \cup M_2 \cup M_3$).

Case 1.2: $e \notin M_0$. Let $e = u_0'v_0'$. Suppose that u_0' and v_0' are saturated by $u_0'v_0, v_0'u_0 \in M_0$, respectively.

Case 1.2.1: $f \in E_{01} \cup E_{30}$.

Case 1.2.1.1: f is incident to u_0 or v_0. Assume w.l.o.g. that f is incident to u_0. Therefore $f = u_0v_1$. There exist an edge u_3v_0 from B_3 to B_0, an edge u_2v_3 from B_2 to B_3, and an edge u_1v_2 from B_1 to B_2. By Lemma 3.2, there exist a Hamiltonian path P_1 from v_1 to u_1 in B_1, a Hamiltonian path P_2 from v_2 to u_2 in B_2, and a Hamiltonian path P_3 from v_3 to u_3 in B_3. Thus, $C' = \langle u_0', v_0', u_0, v_1, P_1, u_1, v_2, P_2, u_2, v_3, P_3, u_3, v_0, u_0' \rangle$ is a cycle of even length. So C' has two perfect matchings, say M_f and $M_{u_0v_0'}$. Suppose that $M_{u_0v_0'}$ contains v_0u_0' and $v_0'u_0$, and M_f contains e and f. Hence, $M_0 \cup M_f \setminus \{v_0u_0', v_0'u_0\}$ is a perfect matching of BH_n containing M (see Fig. 5, heavy lines mean edges of M_0, dotted lines means edges of M_f).

Case 1.2.1.2: f is not incident to u_0 or v_0. Assume that $f = w_0b_1$ is an edge from B_0 to B_1. Additionally, $b_0w_0 \in M_0$ but not in M. Thus, there exist an edge w_3b_0 (resp. u_3v_0) from B_3 to B_0, and an edge u_0v_1 from B_0 to B_1. Let b_3 and v_3 be two distinct black vertices in B_3, and let w_1 and u_1 be two distinct white vertices in B_1. Thus,

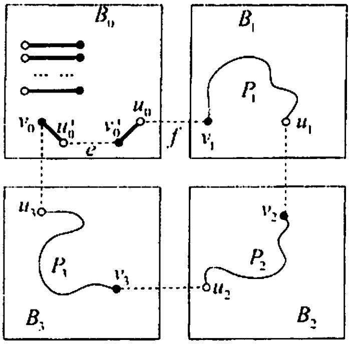

Fig. 5. Illustration for Case 1.2.1.1.

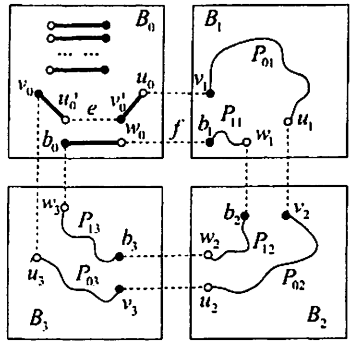

Fig. 6. Illustration for Case 1.2.1.2.

by Lemma 3.4, there exist two vertex-disjoint paths P_{03} and P_{13} such that: (1) P_{03} connects u_3 to v_3, (2) P_{13} connects w_3 to b_3, (3) $V(P_{03}) \cup V(P_{13}) = V(B_3)$. Similarly, there exist two vertex-disjoint paths P_{01} and P_{11} such that: (1) P_{01} connects u_1 to v_1, (2) P_{11} connects w_1 to b_1, (3) $V(P_{01}) \cup V(P_{11}) = V(B_1)$. Thus, there exist an edge w_1b_2 (resp. u_1v_2) from B_1 to B_2, and an edge w_2b_3 (resp. u_2v_3) from B_2 to B_3. Again, there exist two vertex-disjoint paths P_{02} and P_{12} such that: (1) P_{02} connects u_2 to v_2, (2) P_{12} connects w_2 to b_2, (3) $V(P_{02}) \cup V(P_{12}) = V(B_2)$. Then $C_1 = \langle v_0, u_0', v_0', u_0, v_1, P_{01}, u_1, v_2, P_{02}, u_2, v_3, P_{03}, u_3, v_0\rangle$ and $C_2 = \langle b_0, w_0, b_1, P_{11}, w_1, b_2, P_{12}, w_2, b_3, P_{13}, w_3, b_0\rangle$ are two vertex-disjoint cycles of even length. Let M_{C_1} (resp. M_{C_2}) be the perfect matching of C_1 (resp. C_2) containing e (resp. f). Thus $M_{C_1} \cup M_{C_2} \cup M_0 \setminus \{b_0w_0, v_0u_0', v_0'u_0\}$ is a perfect matching of BH_n containing M (see Fig. 6, heavy lines mean edges of M_0, dotted lines mean edges of $M_{C_1} \cup M_{C_2}$).

Case 1.2.2: $f \in E_{12} \cup E_{23}$. We may assume that $f \in E_{12}$. In addition, let $f = u_1v_2$. For convenience, we also make use of the proof of Case 1.2.1.1. Given a black vertex v_1 in B_1, for an arbitrary white vertex u_1 in B_1, there always exists a Hamiltonian path of B_1 from u_1 to v_1. So the choice of f does not affect the existence of C' in BH_n. Therefore, the statement holds.

Case 1.2.3: $f \in E(B_1) \cup E(B_2) \cup E(B_3)$. We may assume that $f \in E(B_3)$. It follows from Lemma 3.5 that, there exists an even cycle C such that $10 \leq |E(C)| \leq 14$, satisfying: (1) $P = u_0v_0'u_0'v_0$ is contained in C, but no endpoints of f is contained in C; (2) $|E(C) \cap E_{01})| = 1$, $|E(C) \cap E_{12})| = 1$, $|E(C) \cap E_{23})| = 1$, $|E(C) \cap E_{30})| = 1$, $|E(B_1) \cap E(C)| = 1$ or 3, $|E(B_2) \cap E(C)| = 1$ or 3, $|E(B_3) \cap E(C)| = 1$. W.l.o.g, suppose that $|E(C)| = 14$, then $|E(B_1) \cap E(C)| = 3$ and $|E(B_2) \cap E(C)| = 3$. Therefore, let $C = \langle u_0v_0'u_0'v_0u_3v_3u_2v_2'u_2'v_2u_1v_1'u_1'v_1u_0\rangle$, where u_3v_3 is an edge in B_3,

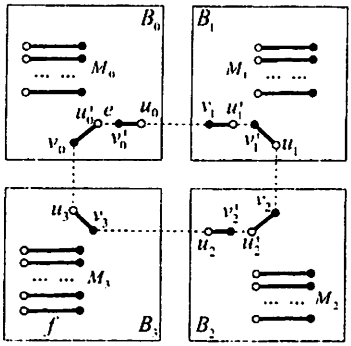

Fig. 7. Illustration for Case 1.2.3.

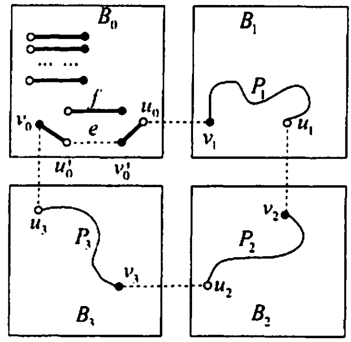

Fig. 8. Illustration for Case 2.2.

and $u_2v_2'u_2'v_2$ (resp. $u_1v_1'u_1'v_1$) is a 3-path in B_2 (resp. B_1). Since C is of even order, it consists of two perfect matchings, say M_e and $M_{v_0'u_0}$, where $e \in M_e$.

By Lemma 3.6, u_1v_1' and $u_1'v_1$ (resp. u_2v_2' and $u_2'v_2$) can be extended to a perfect matching of B_1 (resp. B_2), say M_1 (resp. M_2). Observe that $f \cap u_3v_3 = \emptyset$, then f and u_3v_3 can be extended to a perfect matching of B_3, say M_3. Then $M_e \cup M_0 \cup M_1 \cup M_2 \cup M_3 \setminus \{u_3v_3, v_0u_0', v_0'u_0, v_1u_1', v_1'u_1, v_2u_2', v_2'u_2\}$ is a perfect matching of BH_n containing M (see Fig. 7, heavy lines mean edges of $M_0 \cup M_1 \cup M_2 \cup M_3$, dotted lines mean edges of M_e).

Case 2: $|E(B_0) \cap M| = 2n - 2$. Let e and f be two edges in $M \cap E(B_0)$, then by induction hypothesis, $M - e - f$ can be extended to a perfect matching M_0 of B_0. Next we consider the following cases.

Case 2.1: $e, f \in M_0$. It can be easily obtained a perfect matching of B_1, B_2 and B_3, say M_1, M_2 and M_3, respectively. Thus, $M_0 \cup M_1 \cup M_2 \cup M_3$ is a perfect matching of BH_n containing M.

Case 2.2: Either $e \in M_0$ or $f \in M_0$ but not both. W.l.o.g. suppose that $f \in M_0$ but $e \notin M_0$. Let $e = u_0'v_0'$, and $u_0'v_0, v_0'u_0 \in M_0$. By Definition 1, there exists an edge u_0v_1 (resp. u_1v_2, u_2v_3, u_3v_0) from B_0 to B_1 (resp. B_1 to B_2, B_2 to B_3, B_3 to B_0). By Lemma 3.2, there exists a Hamiltonian path P_1 (resp. P_2, P_3) from v_1 to u_1 (resp. v_2 to u_2, v_3 to u_3) in B_1 (resp. B_2, B_3). Therefore, $C' = \langle u_0, v_1, P_1, u_1, v_2, P_2, u_2, v_3, P_3, u_3, v_0, u_0', v_0', u_0 \rangle$ is a cycle of even length, thus has two perfect matchings, i.e. M_e and $M_{u_0v_0'}$, where $e \in M_e$. Then, $M_0 \cup M_e \setminus \{u_0v_0', u_0'v_0\}$ is a perfect matching of BH_n containing M (see Fig. 8, heavy lines mean edges of M_0, dotted lines mean edges of M_e).

Case 2.3: $e, f \notin M_0$. Let $e = u_0'v_0'$ and $f = w_0'b_0'$. Suppose that $u_0'v_0, v_0'u_0, w_0'b_0, b_0'w_0 \in M_0$. Then there exist $w_0b_1, u_0v_1 \in E_{01}$ (resp.

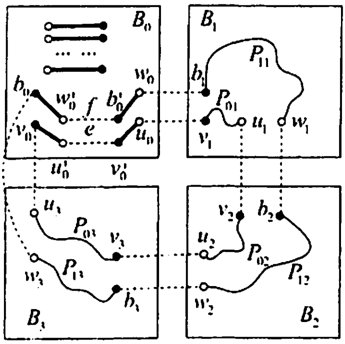

Fig. 9. Illustration for Case 2.3.

$w_3b_0, u_3v_0 \in E_{30}$). Again, there exist $w_1b_2, u_1v_2 \in E_{12}$ (resp. $w_2b_3, u_2v_3 \in E_{23}$). By Lemma 3.4, there exist two vertex-disjoint paths P_{01} and P_{11} in B_1 such that P_{01} connects v_1 to u_1, P_{11} connects b_1 to w_1 and $V(P_{01}) \cup V(P_{11}) = V(B_1)$; there exist two vertex-disjoint paths P_{02} and P_{12} in B_2 such that P_{02} connects v_2 to u_2, P_{12} connects b_2 to w_2 and $V(P_{02}) \cup V(P_{12}) = V(B_2)$; there exist two vertex-disjoint paths P_{03} and P_{13} in B_3 such that P_{03} connects v_3 to u_3, P_{13} connects b_3 to w_3 and $V(P_{03}) \cup V(P_{13}) = V(B_3)$. Therefore, $C_1 = \langle u_0', v_0', u_0, v_1, P_{01}, u_1, v_2, P_{02}, u_2, v_3, P_{03}, u_3, v_0, u_0' \rangle$ and $C_2 = \langle w_0', b_0', w_0, b_1, P_{11}, w_1, b_2, P_{12}, w_2, b_3, P_{13}, w_3, b_0, w_0' \rangle$ are two cycles of even length. Let M_e (resp. M_f) be the perfect matching of C_1 (resp. C_2) containing e (resp. f). Thus $M_e \cup M_f \cup M_0 \setminus \{b_0w_0', b_0'w_0, v_0u_0', v_0'u_0\}$ is a perfect matching of BH_n containing M (see Fig. 9, heavy lines mean edges of M_0, dotted lines mean edges of M_e and M_f). □

4 Conclusions

In this paper, we consider matching extendability of BH_n, and use induction to prove that BH_n is $(2n-2)$-extendable. As there exists a matching M of BH_n with $|M| = 2n-1$, which cover neighbors of the same vertex, thus M can not be extended to a perfect matching of BH_n. For example, let u and w be two distinct vertices differing only the inner index. Additionally, assume that b_i $(1 \le i \le 2n)$ be all the neighbors of u and w. One may assume that M covers all but one neighbors of u and w, say b_{2n}, so M can not cover u and w simultaneously. Obviously, one of u and w can not be saturated by any perfect matching (see Fig. 10, heavy lines mean edges of M). Therefore, our result is optimal. Moreover, it is of interest to consider matching extendability of other famous interconnection networks.

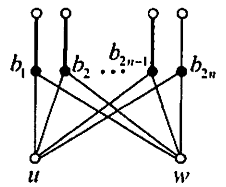

Fig. 10. Illustration for the case that M with $2n-1$ edges can not be extended to a perfect matching of BH_n.

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