

A note on Erdős-Faber-Lovász Conjecture and edge coloring of complete graphs.

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Abstract

A hypergraph is intersecting if any two different edges have exactly one common vertex and an n -quasicluster is an intersecting hypergraph with n edges each one containing at most n vertices and every vertex is contained in at least two edges. The Erdős-Faber-Lovász Conjecture states that the chromatic number of any n -quasicluster is at most n . In the present note we prove the correctness of the conjecture for a new infinite class of n -quasiclusters using a specific edge coloring of the complete graph.

Introduction

A *hypergraph* $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ consists of a finite no empty set \mathcal{V} , the vertices of \mathcal{H} , and a finite collection \mathcal{E} of subsets of \mathcal{V} , the edges of \mathcal{H} . It is assumed that each vertex belongs to at least one edge. A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is *linear* if $|E \cap F| \leq 1$, for all $E, F \in \mathcal{E}$, and \mathcal{H} is *intersecting* if $|E \cap F| = 1$,

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for all $E, F \in \mathcal{E}$. In the remainder of this note each edge contains at least two vertices.

Let $\{1, 2, \dots, k\}$ be a set of k colors. A k -vertex-coloring of \mathcal{H} is a surjective map $\varphi : \mathcal{V} \rightarrow \{1, 2, \dots, k\}$ such that if $u, v \in \mathcal{V}$ are adjacent, then $\varphi(u) \neq \varphi(v)$. In other words, no two vertices with the same color belong in the same edge. The chromatic number of \mathcal{H} , denoted by $\chi(\mathcal{H})$, is the minimum $k \in \mathbb{N}$ for which there is a k -vertex-coloring of \mathcal{H} .

Thus, in the hypergraph setting the original Erdős-Faber-Lovász Conjecture states:

Conjecture 0.1. *If \mathcal{H} is a linear hypergraph consisting of n edges, each one containing n vertices, then $\chi(\mathcal{H}) = n$.*

An n -quasicluster is an intersecting hypergraph consisting of n edges, each one with at most n vertices and each vertex is contained in at least two edges.

Conjecture 0.2. *If \mathcal{H} is an n -quasicluster, then $\chi(\mathcal{H}) \leq n$.*

It is not difficult to prove that the conjecture 0.1 and the conjecture 0.2 are equivalent. For this, consider an n -quasicluster and add new vertices in each edge such that the edges of this new intersecting hypergraph \mathcal{H} has size n . By assumption Conjecture 0.1 is true for intersecting hypergraphs, the hypergraph \mathcal{H} has an n -vertex-coloring, which in turn induces an n -vertex-coloring to n -quasicluster. On the other hand, consider a linear hypergraph \mathcal{H} which satisfies the conjecture 0.1. By the theorem 3 of [18] there exists an intersecting hypergraph $\tilde{\mathcal{H}}$ consisting of n edges, each of size n , such that $\chi(\mathcal{H}) = \chi(\tilde{\mathcal{H}})$. Now, deleting from $\tilde{\mathcal{H}}$ vertices of degree one we get an n -quasicluster \mathcal{H}' . Assuming that Conjecture 0.2 is true we have $\chi(\mathcal{H}') \leq n$ and this coloring can be easily extended to n -vertex-coloring to $\tilde{\mathcal{H}}$ (using non-used colors to each vertex of degree one), thus $\chi(\mathcal{H}) = n$.

In the present note we say that \mathcal{H} is an *instance* of the conjecture or theorem, if \mathcal{H} satisfies the hypothesis of its statements.

There exist works related with some equivalences of Conjecture 0.1 and also many advances, but it is clear that its proof is, in this moment, far

from being attained. There are some results about upper bounds on the number of colors required. Specifically, Mitchem [16], and independently Chang and Lawler [6] had shown that if \mathcal{H} is an instance of the conjecture 0.1, then the chromatic number of \mathcal{H} is at most $\lceil \frac{3n}{2} - 2 \rceil$. Kahn [14] had proved, as an asymptotic result, that if \mathcal{H} is an instance of the conjecture 0.1, then the chromatic number of \mathcal{H} is at most $n + o(n)$. There are some works about specific classes of linear hypergraphs that satisfy the conjecture 0.1; see for example [2], [4], [13], [17], [18] and [20]. Also there are interesting equivalences of this conjecture; see for example [11], [12], [15] and [19]. Recently, Faber [10] proved that for regular and uniform hypergraphs of fixed degree there can only be a finite number of counterexamples for conjecture 0.1.

In this work we expose a new method to approach it, using a specific edge coloring of the complete graph, giving a new infinite class of n -quasiclusters that satisfy the conjecture 0.2.

1 The result

Let G be a simple graph. A *decomposition* of G is a collection $\mathcal{D} = \{G_1, \dots, G_k\}$ of subgraphs of G such that every edge of G belongs to exactly one subgraph in \mathcal{D} , denoted by (G, \mathcal{D}) a decomposition of G .

Let $\{1, \dots, k\}$ be a set of k colors. A k - \mathcal{D} -coloring of (G, \mathcal{D}) is a surjective map $\varphi' : \mathcal{D} \rightarrow \{1, \dots, k\}$ such that for every $G, H \in \mathcal{D}$ if $V(G) \cap V(H) \neq \emptyset$ then $\varphi'(G) \neq \varphi'(H)$. Here φ' means that every edge of the subgraph G is colored with the color $\varphi'(G)$. The *chromatic index* of a decomposition (G, \mathcal{D}) , denoted by $\chi'((G, \mathcal{D}))$, is the minimum $k \in \mathbb{N}$ for which there is a k - \mathcal{D} -coloring of (G, \mathcal{D}) . Now the (K_n, \mathcal{D}) denotes a decomposition where the elements of \mathcal{D} are complete subgraphs of K_n .

It is not difficult to see that there exists a bijection between the decompositions (K_n, \mathcal{D}) and the n -quasiclusters ¹. Because, if we have a decom-

¹The elements of the decomposition (K_n, \mathcal{D}) can be thought as the cliques of the intersection graph of the corresponding n -quasicluster (see [12], [15], [17], [22]).

position (K_n, \mathcal{D}) , each vertex of K_n is associated with an edge of the n -quasiclusters and each element $G \in \mathcal{D}$ with a vertex of the n -quasiclusters; the intersection vertex of the edges associated with the vertices of G . Then a k -vertex-coloring of an n -quasicluster \mathcal{H} is a k - \mathcal{D} -coloring of (K_n, \mathcal{D}) and vice-versa. Therefore, the Conjecture 0.2 is equivalent to:

Conjecture 1.1. *If (K_n, \mathcal{D}) is a decomposition, then $\chi'((K_n, \mathcal{D})) \leq n$.*

As example of our interpretation consider an n -quasicluster where each vertex is a member of exactly two edges, that is, each vertex has degree two, then the elements of corresponding decomposition (K_n, \mathcal{D}) are subgraphs of order two, namely $\mathcal{D} = E(K_n)$. Then a $\chi'((K_n, \mathcal{D}))$ -edge-coloring of (K_n, \mathcal{D}) is equivalent to $\chi'(K_n)$ -edge-coloring of K_n and by Vizing's Theorem we have that $\chi'((K_n, \mathcal{D})) \leq n$.

In this note we will work with a specific n -edge-coloring of the complete graph K_n given by the following: suppose that $K_n = (\mathbb{Z}_n, E)$ and that $\{c_1, \dots, c_n\}$ is a set of n different colors. If $ab \in E$ is an edge then the associated color for this edge is c_{a+b} (with $a + b \in \mathbb{Z}_n$). This assignment is an n -edge-coloring of K_n (for every $n \geq 2$). Let G_0, \dots, G_{n-1} be the n chromatic classes of K_n with respect to this coloring. If we think that $G_i = (\mathbb{Z}_n, E_i)$, where $E_i = \{ab \in E : a + b \equiv i \pmod{n}\}$ for $i = 0, \dots, n-1$, then these subgraphs satisfy:

1. For $i = 0, \dots, n-1$, the degree of the vertices of G_i is at most one.
2. If n is odd then the subgraph G_i has an isolated vertex, say u_i and $G_i - u_i$ is a perfect matching. If n is even and i is even then the subgraph G_i has two isolated vertices, say u_i and v_i and $G_i - u_i - v_i$ is a perfect matching. When i is odd the subgraph G_i is a perfect matching.

For example, Figures 1 and 2 show the subgraphs corresponding to K_5 and K_6 , respectively.

Our results are related with a previous result given by Romero and Sánchez-Arroyo in [18]. We define some similar concepts like them, but

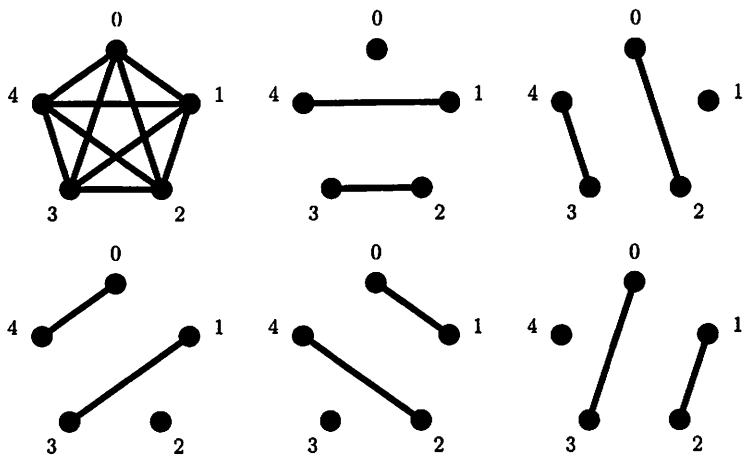


Figure 1: Subgraphs corresponding to the 5-edge-coloring of K_5 .

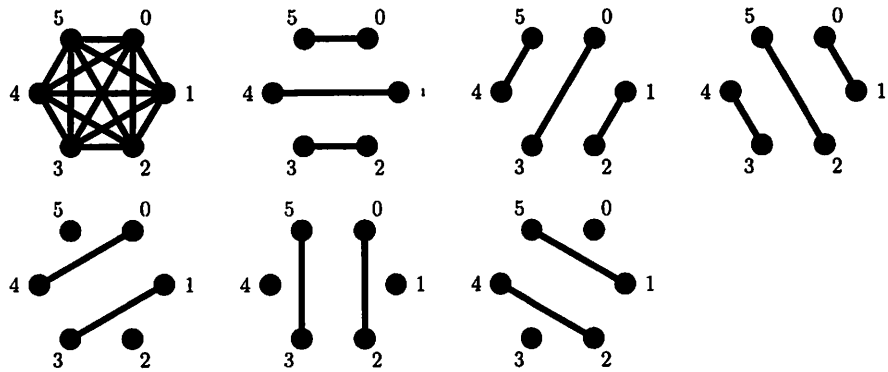


Figure 2: Subgraphs corresponding to the 6-edge-coloring of K_6 .

their approach to solve this is totally algorithmic. On the other hand, our method, as we previously exposed, is related with edge colorings in the complete graph.

In [18] the following concepts were defined: A nonempty set W of nonnegative integers is *compact*, if either $|W| = 1$, or there is an order $(a_1, \dots, a_{|W|})$ on W , such that $a_{i+1} = a_i + 1$, for $i = 1, \dots, |W| - 1$. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with n edges. \mathcal{H} is *edge conformable* if there is a bijection $\psi : \mathcal{E} \rightarrow \{0, \dots, n-1\}$, called *conformal labeling*, such that for each vertex $v \in \mathcal{V}$, the set $F(v) = \{\psi(E) : v \in E \in \mathcal{E}\}$ can be partitioned into two compact sets. The main result of [18] is that any intersecting hypergraph consisting of n edges, each of size n , and edge conformable has an n -vertex-coloring.

Now we introduce the following definitions: Let $W = \{w_1, \dots, w_r\}$ be a subset of \mathbb{Z}_n ; W is *k-arithmetic* if $w_{i+1} - w_i \equiv k \pmod n$, for $i = 1, \dots, r-1$ and some $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. A decomposition (K_n, \mathcal{D}) is called *arithmetic decomposition*, if there exists a bijection $\varphi : V(K_n) \rightarrow \mathbb{Z}_n$, called *arithmetic labeling*, such that for every $G \in \mathcal{D}$ either $V(G)$ is *k-arithmetic* or $V(G)$ can be partitioned into two *k-arithmetic* sets of same cardinality, for some $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Let (K_n, \mathcal{D}) be an arithmetic decomposition and G be an element of \mathcal{D} , where $V(G) = \{v_1, \dots, v_l\}$ has odd cardinality, then $v_{\frac{l+1}{2}}$ is called the *central vertex* of G . Also, we say that a decomposition (K_n, \mathcal{D}) has different central vertices if any pair of central vertices (corresponding to elements of \mathcal{D} of odd order) are different.

Theorem 1.2. *Let (K_n, \mathcal{D}) be an arithmetic decomposition with different central vertices, then $\chi'((K_n, \mathcal{D})) \leq n$.*

Proof

Let (K_n, \mathcal{D}) be an arithmetic decomposition, G be an element of \mathcal{D} and $\{c_1, \dots, c_n\}$ be a set of n different colors.

Case (i) If $V(G) = \{v_1, \dots, v_l\}$ is *k-arithmetic*, for some $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ then by hypothesis $v_{i+1} - v_i \equiv k \pmod n$, for $i = 1, \dots, l-1$. The edges of G will be colored as follows:

- (i.a) If $V(G)$ has even cardinality, then $v_1 + v_l \equiv v_2 + v_{l-1} \equiv \dots \equiv v_{\frac{l}{2}} + v_{\frac{l}{2}+1} \equiv j$, for some $j \in \mathbb{Z}_n$. Assign the color c_j to the edges $M = \{v_1v_l, v_2v_{l-1}, \dots, v_{\frac{l}{2}}v_{\frac{l}{2}+1}\}$. As M is a perfect matching then there are no incident edges to M of color c_j different from M , so that we can assign the color c_j to all edges of G .
- (i.b) If $V(G)$ has odd cardinality, let $v_G = v_{\frac{l+1}{2}} \in V(G)$ be the central vertex of G . Then $v_1 + v_l \equiv v_2 + v_{l-1} \equiv \dots \equiv v_{\frac{l-1}{2}} + v_{\frac{l+1}{2}+1} \equiv j$, for some $j \in \mathbb{Z}_n$, so that we can assign the color c_j to all edges of $G - v_G$ as Case (i.a).

Note that the color c_j is not incident to v_G : otherwise there exists $u_G \in V(K_n)$ such that $v_G + u_G \equiv j \pmod n$. As $V(G)$ is k -arithmetic and v_G is the central vertex of G then $(v_G - rk) + (v_G + rk) \equiv j \pmod n$, for $r = 1, \dots, \frac{|V(G)|-1}{2}$, that is $2v_G \equiv j \pmod n$. Since $2v_G \equiv j \pmod n$ and by hypothesis $v_G + u_G \equiv j \pmod n$ then $2v_G \equiv v_G + u_G \pmod n$; this implies that $u_G \equiv v_G \pmod n$, which is a contradiction. Therefore there are no incident edges to v_G of color c_j , and so we can assign the color c_j to all edges of G .

Case (ii) Now, if $V(G)$ can be partitioned into two k -arithmetic sets of same cardinality, for some $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ then the subgraph G will be colored as follows: suppose that $V(G) = \{v_1, \dots, v_l\} \cup \{u_1, \dots, u_l\}$. By hypothesis $\{v_1, \dots, v_l\}$ and $\{u_1, \dots, u_l\}$ are k -arithmetics, for some $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Then $v_{i+1} - v_i \equiv u_{i+1} - u_i \equiv k \pmod n$, for $i = 1, \dots, l-1$; so that $v_1 + u_l \equiv v_2 + u_{l-1} \equiv \dots \equiv v_l + u_1 \equiv j$, for some $j \in \mathbb{Z}_n$. Assign the color c_j to the edges $M = \{v_1u_l, v_2u_{l-1}, \dots, v_lu_1\}$. As M is a perfect matching, there are no incident edges to M of color c_j different from M , so that we can assign the color c_j to all edges of G .

Remains prove that for all $G, H \in \mathcal{D}$, with $V(G) \cap V(H) = \{v\}$ and different central vertex (in case of having it) have different colors. Let $G, H \in \mathcal{D}$, then

Case (i) If $V(G)$ and $V(H)$ have even cardinality then the corresponding perfect matching of G and H does not share edges (by linearity), therefore they have different colors.

Case (ii) Suppose that $V(G)$ and $V(H)$ have odd cardinality and the edges of G have the same color that the edges of H . Let v_G and v_H be the central vertices of G and H respectively.

(ii.a) If $\{v\} = V(G) \cap V(H) \not\subset \{v_G, v_H\}$ then there exists $u_G \in V(G)$ and $u_H \in V(H)$ such that $u_G + v \equiv u_H + v \equiv j \pmod{n}$, for some $j \in \mathbb{Z}_n$, implying that $u_G \equiv u_H \pmod{n}$, which is a contradiction.

(ii.b) If $v_H = V(G) \cap V(H)$ then there exists $u_G \in V(G)$ such that $u_G + v_H \equiv (v_H - rk) + (v_H + rk) \pmod{n}$, for $r = 1, \dots, \frac{|V(G)|-1}{2}$. Since $2v_H = (v_H - rk) + (v_H + rk)$, for $r = 1, \dots, \frac{|V(G)|-1}{2}$ then $u_G \equiv v_H \pmod{n}$, which is a contradiction.

Therefore $E(G)$ and $E(H)$ have different colors.

Case (iii) Finally, if $V(G)$ and $V(H)$ have different cardinality it is not difficult to see that the edges of G and H have different colors because if $u_G \in V(G)$ is the central vertex of G then the perfect matching of $G - u_G$ and the perfect matching of H does not share edges (by linearity), and so they have different colors. \square

To continue, in Figure 3, we exhibit the theorem giving an example. Let $V(G_0) = \{0, 3, 6\}$, $V(G_1) = \{1, 4, 7\}$, $V(G_2) = \{5, 8, 2\}$, $V(H_0) = \{0, 2, 4\}$, $V(H_1) = \{4, 6, 8\}$, $V(H_2) = \{8, 1, 3\}$ and $V(H_3) = \{3, 5, 7\}$ be the vertices of the complete graphs of \mathcal{D} of cardinality larger than two and the rest of the elements of \mathcal{D} are edges.

Note that:

1. $V(G_i)$ is 3-arithmetic and $V(H_j)$ is 2-arithmetic, for $i = 0, 1, 2$ and $j = 0, 1, 2, 3$.
2. The central vertices of $V(G_0)$, $V(G_1)$, $V(G_2)$, $V(H_0)$, $V(H_1)$, $V(H_2)$, $V(H_3)$ are 3, 4, 8, 2, 6, 1 and 5 respectively.

Hence, this decomposition is a (K_9, \mathcal{D}) arithmetic decomposition. By the Theorem 1.2 this decomposition satisfies the conjecture 0.2.

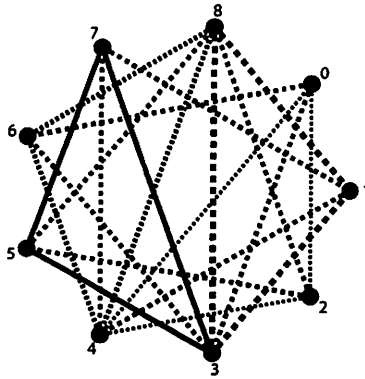


Figure 3: Elements of \mathcal{D} with order larger than two.

To finish this note, it is important to establish which is the correspondence, regarding the previous definitions, given in arithmetic decompositions to n -quasiclusters. We state the Theorem 1.2 in these terms. To do this we give the definitions in terms of hypergraphs (or n -quasiclusters).

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be an n -quasicluster. \mathcal{H} is *edge arithmetic* if there is a bijection $\varphi : \mathcal{E} \rightarrow \mathbb{Z}_n$, that we call arithmetic labeling, such that for each vertex $u \in \mathcal{V}$, the set $F(u) = \{\varphi(E) : u \in E \in \mathcal{E}\}$ is k -arithmetic or can be partitioned in two k -arithmetic sets of same cardinality. Let \mathcal{H} be an n -quasicluster edge arithmetic, u be a vertex of \mathcal{H} of odd degree and $F(u) = \{E_1, \dots, E_l\}$, then $E_{\lfloor \frac{l+1}{2} \rfloor}$ is called *central edge* of u . We say that an n -quasicluster edge arithmetic has different central edges if any pair of central edges (corresponding to vertices of odd degree) is different. Then, the main result (Theorem 1.2) in terms of hypergraphs states:

Theorem 1.3. *Let \mathcal{H} be an n -quasicluster. If \mathcal{H} is edge arithmetic and has different central edges, then $\chi(\mathcal{H}) \leq n$.*

Finally we can note that any edge arithmetic n -quasicluster \mathcal{H} which has at most one vertex of odd degree in each edge immediately has different central edges and then we have the following:

Corollary 1.1. *Let \mathcal{H} be an n -quasicluster edge arithmetic with all the edges with at most one vertex of odd degree, then $\chi(\mathcal{H}) \leq n$.*

Acknowledgment

The authors want to thank the anonymous referees for their kind help and valuable suggestions which led to an improvement of this paper.

Also, the authors want to thank to CINNMA (Centro de Innovación Matemática) and their people by their academic support and friendship.

Research supported by PAPPIT-México under project IN101912

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