Resistance Distance of Three Classes of Join Graphs*

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Abstract

The subdivision graph $\mathcal{S}(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. The set of inserted vertices of $\mathcal{S}(G)$ is denoted by $I(G)$. Let G_1 and G_2 be two vertex disjoint graphs. The *subdivision-edge-vertex join* of G_1 and G_2, denoted by $G_1 \circledcirc G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex in $I(G_1)$ to every vertex in $V(G_2)$. The *subdivision-edge-edge join* of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex in $I(G_1)$ to every vertex in $I(G_2)$. The *subdivision-vertex-edge join* of G_1 and G_2, denoted by $G_1 \circledast G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex in $V(G_1)$ to every vertex in $I(G_2)$. In this paper, we obtain the formulas for resistance distance of $G_1 \circledcirc G_2$, $G_1 \ominus G_2$ and $G_1 \circledast G_2$.

Keywords: Subdivision-edge-vertex join, Subdivision-edge-edge join, Subdivision-vertex-edge join, Resistance distance

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1 Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$.

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Let A_G denote the adjacency matrix of a graph G, and D_G denote the diagonal matrix of vertex degrees of G. The Laplacian matrix of G is defined as $L_G = D_G - A_G$. Let G be a graph with n vertices and m edges. We use $l(G)$ to denote the line graph of G.

Let G be a connected graph. For two vertices u, v in a connected G, the *resistance distance* between u and v is defined to be the effective resistance between them when unit resistors are placed on every edge of G. It is a distance function on graphs introduced by Klein and Randić [7]. As usual, we use $r_{uv}(G)$ to denote the resistance distance between u and v in G. Recently, many results on resistance distance were obtained. The reader is referred to [1,4,7,9–11] to know more.

Example 1.1. Let P_n denote a path of order n. Figure 1 depicts the *subdivision-edge-vertex join* $P_4 \circledcirc P_3$, *subdivision-edge-edge join* $P_4 \ominus P_3$ and *subdivision-vertex-edge join* $P_4 \circledast P_3$.

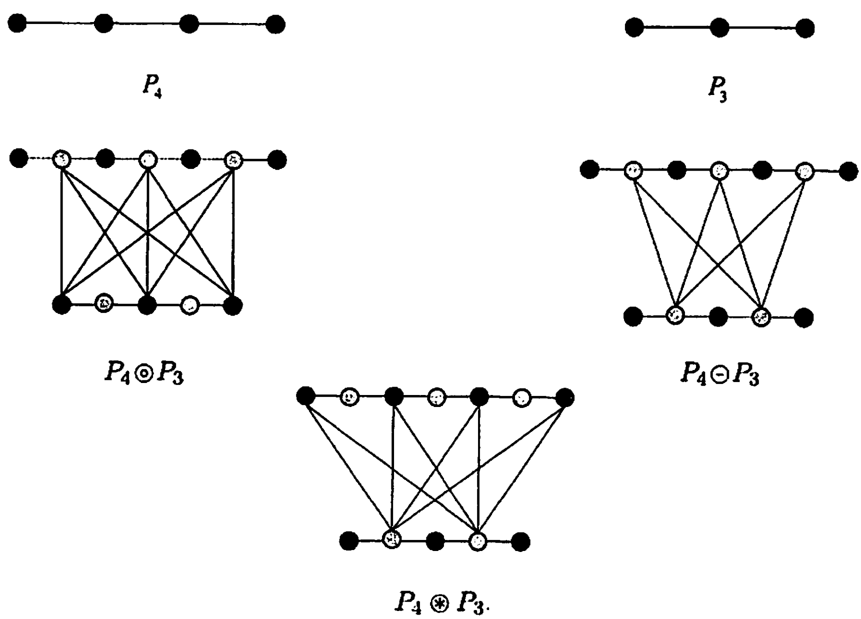

Fig. 1: An example of *subdivision-edge-vertex join*, *subdivision-edge-edge join* and *subdivision-vertex-edge join*.

The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. The set of inserted vertices of

$\mathcal{S}(G)$ is denoted by $I(G)$. Let G_1 and G_2 be two vertex disjoint graphs. The *subdivision-edge-vertex join* of G_1 and G_2, denoted by $G_1 \circledcirc G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex in $I(G_1)$ to every vertex in $V(G_2)$. The *subdivision-edge-edge join* of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex in $I(G_1)$ to every vertex in $I(G_2)$. The *subdivision-vertex-edge join* of G_1 and G_2, denoted by $G_1 \circledast G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $\mathcal{S}(G_2)$ by joining every vertex in $V(G_1)$ to every vertex in $I(G_2)$.

In this paper, we obtain the formulae for resistance distance of $G_1 \circledcirc G_2$, $G_1 \ominus G_2$ and $G_1 \circledast G_2$, respectively.

2 Preliminaries

Let M be a square matrix. The $\{1\}$*-inverse* of M is a matrix X such that $MXM = M$. The *group inverse* of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known that $M^{\#}$ exists if and only if $\text{rank}(M) = \text{rank}(M^2)$. If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$-inverse of M (see [2]).

We use $M^{(1)}$ to denote any $\{1\}$-inverse of matrix M. Let $(M)_{uv}$ denote the (u, v)-entry of M.

Lemma 2.1. *(See [2] [1])Let G be a connected graph. Then*

$$\begin{aligned} r_{uv}(G) &= \left(L_G^{(1)}\right)_{uu} + \left(L_G^{(1)}\right)_{vv} - \left(L_G^{(1)}\right)_{uv} - \left(L_G^{(1)}\right)_{vu} \\ &= \left(L_G^{\#}\right)_{uu} + \left(L_G^{\#}\right)_{vv} - 2\left(L_G^{\#}\right)_{uv}. \end{aligned}$$

Let $\mathbf{1}$ denote the all-ones column vector.

Lemma 2.2. *(See [3]) Let S be a real symmetric matrix such that $S\mathbf{1} = 0$. Then $S^{\#}\mathbf{1} = 0$, $\mathbf{1}^T S^{\#} = 0$.*

Lemma 2.3. *(See [12]) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a nonsingular matrix. If A and D are nonsingular, then*

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \end{aligned}$$

where $S = D - CA^{-1}B$.

Lemma 2.4. *(See [3]) Let* $L = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ *be a Laplacian matrix of a connected graph. If each column vector of* L_2^T *is* $-\mathbf{1}$ *or a zero vector, then* $N = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{pmatrix}$ *is a symmetric* $\{1\}$*-inverse of* L*, where* $S = L_3 - L_2^T L_1^{-1} L_2$.

Lemma 2.5. *(See [3]) Let* L *be the Laplacian matrix of a graph of order* n*. For any* $a>0$*, we have* $\left(L + aI - \frac{a}{n}J_{n\times n}\right)^{\#} = (L + aI)^{-1} - \frac{1}{an}J_{n\times n}$.

3 Resistance distances of $G_1 \circledcirc G_2$

In this section, we obtain the formulae for the resistance distances in $G_1 \circledcirc G_2$, for G_1 a regular graph and G_2 an arbitrary graph.

Theorem 3.1. *Let* G_1 *be an* d_1*-regular(*$d_1>0$*) graph of order* n_1*, and* G_2 *be a graph with* n_2 *vertices and* m_2 *edges. Let* $X = L_{l(G_1)} + d_1 n_2 I_{m_1}$ *and* $Y = \frac{1}{2}L_{G_2} + m_1 I_{n_2}$*. Then the following hold:*

(1) *For any* $i, j \in V(G_1)$*, we have*

$$r_{ij}(G_1 \circledcirc G_2) = 2d_1^{-1} + d_1^{-2}\sum_{u,v=1}^{d_1} r_{e_u f_v}(G_1 \circledcirc G_2) - d_1^{-2}\sum_{u<v} r_{e_u e_v}(G_1 \circledcirc G_2) - d_1^{-2}\sum_{u<v} r_{f_u f_v}(G_1 \circledcirc G_2),$$

where $e_1, \ldots, e_{d_1}$ *are* d_1 *edges incident to* i *in* G_1*,* $f_1, \ldots, f_{d_1}$ *are* d_1 *edges incident to* j *in* G_1*;*

(2) *For any* $i, j \in V(G_2)$*, we have* $r_{ij}(G_1 \circledcirc G_2) = Y_{ii}^{-1} + Y_{jj}^{-1} - 2Y_{ij}^{-1}$*;*

(3) *For any* $i, j \in I(G_1)$*, we have* $r_{ij}(G_1 \circledcirc G_2) = d_1\left(X_{ii}^{-1} + X_{jj}^{-1} - 2X_{ij}^{-1}\right)$*;*

(4) *For any* $i = u_1 v_1$*,* $j = u_2 v_2 \in I(G_2)$*, we have*

$$r_{ij}(G_1 \circledcirc G_2) = 1 + \frac{1}{4}\left[r_{u_1 u_2}(G_1 \circledcirc G_2) + r_{v_1 v_2}(G_1 \circledcirc G_2) + r_{u_1 v_2}(G_1 \circledcirc G_2) + r_{v_1 u_2}(G_1 \circledcirc G_2) - r_{u_1 v_1}(G_1 \circledcirc G_2) - r_{u_2 v_2}(G_1 \circledcirc G_2)\right];$$

(5) *For any* $i \in V(G_1)$*,* $j \in I(G_1)$*, we have*

$r_{ij}(G_1 \circledcirc G_2) = d_1^{-1} + d_1^{-1}\sum_{u=1}^{d_1} r_{e_u j}(G_1 \circledcirc G_2) - d_1^{-2}\sum_{u<v} r_{e_u e_v}(G_1 \circledcirc G_2)$*, where* $e_1, \ldots, e_{d_1}$ *are* d_1 *edges incident to* i *in* G_1*;*

(6) *For any* $i \in V(G_2)$, $j = uv \in I(G_2)$, *we have*
$r_{ij}(G_1 \circledcirc G_2) = \frac{1}{2} + \frac{1}{4}\left[2r_{iu}(G_1 \circledcirc G_2) + 2r_{iv}(G_1 \circledcirc G_2) - r_{uv}(G_1 \circledcirc G_2)\right];$

(7) *For any* $i \in V(G_1)$, $j = uv \in I(G_2)$, *we have* $r_{ij}(G_1 \circledcirc G_2) = d_1^{-1} + r_{kj}(G_1 \circledcirc G_2)$, *where* $k \in I(G_1)$;

(8) *For any* $i \in V(G_2)$, $j \in I(G_1)$, *we have* $r_{ij}(G_1 \circledcirc G_2) = Y_{ii}^{-1} + d_1 X_{jj}^{-1} - \frac{1}{m_1 n_2}$;

(9) *For any* $i \in V(G_1)$, $j \in V(G_2)$, *we have* $r_{ij}(G_1 \circledcirc G_2) = d_1^{-1} + r_{jk}(G_1 \circledcirc G_2)$, *where* $k \in I(G_1)$;

(10) *For any* $i \in I(G_1)$, $j = uv \in I(G_2)$, *we have* $r_{ij}(G_1 \circledcirc G_2) = \frac{1}{2} + \frac{1}{4}\left(Y_{uu}^{-1} + Y_{vv}^{-1} + 2Y_{uv}^{-1}\right) + d_1 X_{ii}^{-1} - \frac{1}{m_1 n_2}$;

Proof. Let R be the vertex-edge incidence matrix of G. Then the Laplacian matrix of $G_1 \circledcirc G_2$ can be written as

$$L_{G_1 \circledcirc G_2} = \begin{pmatrix} d_1 I_{n_1} & -R_1 & 0_{n_1 \times m_2} & 0_{n_1 \times n_2} \\ -R_1^T & (n_2+2)I_{m_1} & 0_{m_1 \times m_2} & -J_{m_1 \times n_2} \\ 0_{m_2 \times n_1} & 0_{m_2 \times m_1} & 2I_{m_2} & -R_2^T \\ 0_{n_2 \times n_1} & -J_{n_2 \times m_1} & -R_2 & D_{G_2} + m_1 I_{n_2} \end{pmatrix}.$$

Let

$$L_1 = \begin{pmatrix} d_1 I_{n_1} & -R_1 \\ -R_1^T & (n_2+2)I_{m_1} \end{pmatrix}.$$

By Lemma 2.3, we have

$$L_1^{-1} = \begin{pmatrix} d_1^{-1} I_{n_1} + d_1^{-1} R_1 \left(L_{l(G_1)} + d_1 n_2 I_{m_1}\right)^{-1} R_1^T & R_1 \left(L_{l(G_1)} + d_1 n_2 I_{m_1}\right)^{-1} \\ \left(L_{l(G_1)} + d_1 n_2 I_{m_1}\right)^{-1} R_1^T & d_1 \left(L_{l(G_1)} + d_1 n_2 I_{m_1}\right)^{-1} \end{pmatrix}.$$

Then, let

$$L_2 = \begin{pmatrix} L_1 & 0_{(m_1+n_1) \times m_2} \\ 0_{m_2 \times (m_1+n_1)} & 2I_{m_2} \end{pmatrix},$$

By Lemma 2.4, we get

$$L_2^{-1} = \begin{pmatrix} L_1^{-1} & 0_{(m_1+n_1) \times m_2} \\ 0_{m_2 \times (m_1+n_1)} & \frac{1}{2} I_{m_2} \end{pmatrix}.$$

Let

$$S = D_{G_2} + m_1 I_{n_2} - \begin{pmatrix} 0_{n_2\times n_1} & -J_{n_2\times m_1} & -R_2 \end{pmatrix} L_2^{-1} \begin{pmatrix} 0_{n_1\times n_2} \\ -J_{m_1\times n_2} \\ -R_2^T \end{pmatrix}$$

$$= D_{G_2} + m_1 I_{n_2} - \left(J_{n_2\times m_1} d_1 \left(d_1 n_2 I_{m_1} + L_{l(G_1)}\right)^{-1} J_{m_1\times n_2} + \frac{1}{2} R_2 R_2^T \right)$$

$$= D_{G_2} + m_1 I_{n_2} - \frac{m_1}{n_2} J_{n_2\times n_2} - \frac{1}{2} A_{G_2} - \frac{1}{2} D_{G_2}$$

$$= \frac{1}{2} L_{G_2} + m_1 I_{n_2} - \frac{m_1}{n_2} J_{n_2\times n_2}.$$

Lemma 2.5 implies that

$$S^\# = \left(\frac{1}{2} L_{G_2} + m_1 I_{n_2}\right)^{-1} - \frac{1}{m_1 n_2} J_{n_2\times n_2}.$$

Let $X = d_1 n_2 I_{m_1} + L_{l(G_1)}$ and $Y = \frac{1}{2} L_{G_2} + m_1 I_{n_2}$. By Lemma 2.4, the following matrix

$$\begin{pmatrix} d_1^{-1} I_{n_1} + d_1^{-1} R_1 X^{-1} R_1^T & R_1 X^{-1} & 0_{n_1\times m_2} & 0_{n_1\times n_2} \\ X^{-1} R_1^T & d_1 X^{-1} & 0_{m_1\times m_2} & 0_{m_1\times n_2} \\ 0_{m_2\times n_1} & 0_{m_2\times m_1} & \frac{1}{2} I_{m_2} + \frac{1}{4} R_2^T S^\# R_2 & \frac{1}{2} R_2^T S^\# \\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & \frac{1}{2} S^\# R_2 & S^\# \end{pmatrix} \quad (3.1)$$

is a symmetric $\{1\}$-inverse of $L_{G_1 \circledcirc G_2}$.

For any $i, j \in V(G_1)$, let $e_1, \ldots, e_{d_1}$ are d_1 edges incident to i in G_1 and $f_1, \ldots, f_{d_1}$ are d_1 edges incident to j in G_1. by Lemma 2.1 and Matrix (3.1), we have

$r_{ij}(G_1 \circledcirc G_2)$

$$= \frac{2}{d_1} + \frac{1}{d_1}\left(R_1 X^{-1} R_1^T\right)_{ii} + \frac{1}{d_1}\left(R_1 X^{-1} R_1^T\right)_{jj} - \frac{2}{d_1}\left(R_1 X^{-1} R_1^T\right)_{ij}$$

$$= 2d_1^{-1} + d_1^{-2} \sum_{u,v=1}^{d_1} r_{e_u f_v}(G_1 \circledcirc G_2) - d_1^{-2} \sum_{u<v} r_{e_u e_v}(G_1 \circledcirc G_2)$$

$$- d_1^{-2} \sum_{u<v} r_{f_u f_v}(G_1 \circledcirc G_2).$$

For any $i,j\in V(G_2)$, by Lemma 2.1 and Matrix (3.1), we have
$r_{ij}(G_1\circledcirc G_2)=S^{\#}_{ii}+S^{\#}_{jj}-2S^{\#}_{ij}=Y^{-1}_{ii}+Y^{-1}_{jj}-2Y^{-1}_{ij}$.
For any $i,j\in I(G_1)$, by Lemma 2.1 and Matrix (3.1), we have

$$\begin{aligned} r_{ij}(G_1\circledcirc G_2)&=\left(d_1X^{-1}\right)_{ii}+\left(d_1X^{-1}\right)_{jj}-2\left(d_1X^{-1}\right)_{ij}\\ &=d_1\left(X^{-1}_{ii}+X^{-1}_{jj}-2X^{-1}_{ij}\right). \end{aligned}$$

For any $i=u_1v_1$, $j=u_2v_2\in I(G_2)$, by Lemma 2.1 and Matrix (3.1), we have

$$\begin{aligned} r_{ij}(G_1\circledcirc G_2)&=1+\frac{1}{4}\left(R_2^TS^{\#}R_2\right)_{ii}+\frac{1}{4}\left(R_2^TS^{\#}R_2\right)_{jj}-\frac{1}{2}\left(R_2^TS^{\#}R_2\right)_{ij}\\ &=1+\frac{1}{4}\left[r_{u_1u_2}(G_1\circledcirc G_2)+r_{v_1v_2}(G_1\circledcirc G_2)+r_{u_1v_2}(G_1\circledcirc G_2)\right.\\ &\quad\left.+r_{u_2v_1}(G_1\circledcirc G_2)-r_{u_1v_1}(G_1\circledcirc G_2)-r_{u_2v_2}(G_1\circledcirc G_2)\right]. \end{aligned}$$

For any $i\in V(G_1)$, $j\in I(G_1)$, let $e_1,\dots,e_{d_1}$ be d_1 edges incident to i in G_1, by Lemma 2.1 and Matrix (3.1), we have

$$\begin{aligned} r_{ij}(G_1\circledcirc G_2)&=d_1^{-1}+d_1^{-1}\left(R_1X^{-1}R_1^T\right)_{ii}+d_1X^{-1}_{jj}-2\left(R_1X^{-1}\right)_{ij}\\ &=d_1^{-1}+d_1^{-1}\sum_{u=1}^{d_1}r_{e_uj}(G_1\circledcirc G_2)-d_1^{-2}\sum_{u<v}r_{e_ue_v}(G_1\circledcirc G_2). \end{aligned}$$

For any $i\in V(G_2)$, $j=uv\in I(G_2)$, by Lemma 2.1 and Matrix (3.1), we have

$$\begin{aligned} r_{ij}(G_1\circledcirc G_2)&=S^{\#}_{ii}+\frac{1}{2}+\frac{1}{4}\left(R_2^TS^{\#}R_2\right)_{jj}-\left(S^{\#}R_2\right)_{ij}\\ &=\frac{1}{2}+\frac{1}{4}\left[2r_{iu}(G_1\circledcirc G_2)+2r_{iv}(G_1\circledcirc G_2)-r_{uv}(G_1\circledcirc G_2)\right]. \end{aligned}$$

For any $i\in V(G_1)$, $j=uv\in I(G_2)$, $k\in I(G_1)$, by Lemma 2.1 and Matrix (3.1), we have

$$\begin{aligned} r_{ij}(G_1\circledcirc G_2)&=d_1^{-1}+d_1^{-1}\left(R_1X^{-1}R_1^T\right)_{ii}+\frac{1}{2}+\frac{1}{4}\left(R_2^TS^{\#}R_2\right)_{jj}\\ &=\frac{1}{2}+d_1^{-1}+\frac{1}{4}\left(R_2^TY^{-1}R_2\right)_{jj}+d_1^{-1}\left(R_1X^{-1}R_1^T\right)_{ii}-\frac{1}{m_1n_2}\\ &=d_1^{-1}+r_{kj}(G_1\circledcirc G_2) \end{aligned}$$

For any $i\in V(G_2)$, $j\in I(G_1)$, by Lemma 2.1 and Matrix (3.1), we have
$$r_{ij}(G_1\circledcirc G_2)=S^{\#}_{ii}+d_1X^{-1}_{jj}=Y^{-1}_{ii}+d_1X^{-1}_{jj}-\frac{1}{m_1n_2}.$$
For any $i\in V(G_1)$, $j\in V(G_2)$, $k\in I(G_1)$, by Lemma 2.1 and Matrix (3.1), we have
$$r_{ij}(G_1\circledcirc G_2)=d_1^{-1}+d_1^{-1}\left(R_1X^{-1}R_1^T\right)_{ii}+S^{\#}_{jj}=d_1^{-1}+r_{jk}(G_1\circledcirc G_2).$$
For any $i\in I(G_1)$, $j=uv\in I(G_2)$, by Lemma 2.1 and Matrix (3.1),

$$r_{ij}(G_1 \circledcirc G_2) = d_1 X_{ii}^{-1} + \frac{1}{2} + \frac{1}{4}\left(R_2^T S^{\#} R_2\right)_{jj}$$
$$= \frac{1}{2} + \frac{1}{4}\left(Y_{uu}^{-1} + Y_{vv}^{-1} + 2Y_{uv}^{-1}\right) + d_1 X_{ii}^{-1} - \frac{1}{m_1 n_2}.$$

This completes the proof. □

4 Resistance distances of $G_1 \ominus G_2$

In this section, we obtain the formulae for the resistance distances in $G_1 \ominus G_2$ for G_1 a regular graph and G_2 a regular graph.

Theorem 4.1. *Let G_1 be an d_1-regular($d_1 > 0$) graph of order n_1, and G_2 be an d_2-regular($d_2 > 0$) graph of order n_2. Let $X = L_{l(G_1)} + d_1 m_2 I_{m_1}$ and $Y = L_{l(G_2)} + d_2 m_1 I_{m_2}$. Then the following hold:*

(1) *For any $i, j \in V(G_1)$ $(i \neq j)$, we have*

$$r_{ij}(G_1 \ominus G_2) = 2d_1^{-1} + d_1^{-2} \sum_{u,v=1}^{d_1} r_{e_u f_v}(G_1 \ominus G_2) - d_1^{-2} \sum_{u<v} r_{e_u e_v}(G_1 \ominus G_2) - d_1^{-2} \sum_{u<v} r_{f_u f_v}(G_1 \ominus G_2),$$

where $e_1, \ldots, e_{d_1}$ are d_1 edges incident to i in G_1, $f_1, \ldots, f_{d_1}$ are d_1 edges incident to j in G_1;

(2) *For any $i, j \in V(G_2)$ $(i \neq j)$, we have*

$$r_{ij}(G_1 \ominus G_2) = 2d_2^{-1} + d_2^{-2} \sum_{u,v=1}^{d_2} r_{e_u f_v}(G_1 \ominus G_2) - d_2^{-2} \sum_{u<v} r_{e_u e_v}(G_1 \ominus G_2) - d_2^{-2} \sum_{u<v} r_{f_u f_v}(G_1 \ominus G_2),$$

where $e_1, \ldots, e_{d_2}$ are d_2 edges incident to i in G_2, $f_1, \ldots, f_{d_2}$ are d_2 edges incident to j in G_2;

(3) *For any $i, j \in I(G_1)$, we have $r_{ij}(G_1 \ominus G_2) = d_1\left(X_{ii}^{-1} + X_{jj}^{-1} - 2X_{ij}^{-1}\right)$;*

(4) *For any $i, j \in I(G_2)$, we have $r_{ij}(G_1 \ominus G_2) = d_2\left(Y_{ii}^{-1} + Y_{jj}^{-1} - 2Y_{ij}^{-1}\right)$;*

(5) *For any $i \in V(G_1)$, $j \in I(G_1)$, we have*

$r_{ij}(G_1 \ominus G_2) = d_1^{-1} + d_1^{-1} \sum_{u=1}^{d_1} r_{e_u j}(G_1 \ominus G_2) - d_1^{-2} \sum_{u<v} r_{e_u e_v}(G_1 \ominus G_2)$, where $e_1, \ldots, e_{d_1}$ are d_1 edges incident to i in G_1;

(6) *For any* $i \in V(G_2)$, $j \in I(G_2)$, *we have*
$r_{ij}(G_1 \ominus G_2) = d_2^{-1} + d_2^{-1} \sum_{u=1}^{d_2} r_{e_u j}(G_1 \ominus G_2) - d_2^{-2} \sum_{u<v} r_{e_u e_v}(G_1 \ominus G_2)$, *where* $e_1, \ldots, e_{d_2}$ *are* d_2 *edges incident to* i *in* G_2;

(7) *For any* $i \in V(G_2)$, $j \in I(G_1)$, *we have* $r_{ij}(G_1 \ominus G_2) = d_2^{-1} + r_{jk}(G_1 \ominus G_2)$, *where* $k \in I(G_2)$;

(8) *For any* $i \in V(G_1)$, $j \in I(G_2)$, *we have* $r_{ij}(G_1 \ominus G_2) = d_1^{-1} + r_{kj}(G_1 \ominus G_2)$, *where* $k \in I(G_1)$;

(9) *For any* $i \in V(G_1)$, $j \in V(G_2)$, *we have* $r_{ij}(G_1 \ominus G_2) = d_1^{-1} + d_2^{-1} + r_{hk}(G_1 \ominus G_2)$, *where* $h \in I(G_1)$ *and* $k \in I(G_2)$;

(10) *For any* $i \in I(G_1)$, $j \in I(G_2)$, *we have* $r_{ij}(G_1 \ominus G_2) = d_1 X_{ii}^{-1} + d_2 Y_{jj}^{-1} - \dfrac{1}{m_1 m_2}$.

Proof. Let R be the vertex-edge incidence matrix of G. The Laplacian matrix of $G_1 \ominus G_2$ has the following form

$$L_{G_1 \ominus G_2} = \begin{pmatrix} d_1 I_{n_1} & -R_1 & 0_{n_1 \times n_2} & 0_{n_1 \times m_2} \\ -R_1^T & (m_2+2) I_{m_1} & 0_{m_1 \times n_2} & -J_{m_1 \times m_2} \\ 0_{n_2 \times n_1} & 0_{n_2 \times m_1} & d_2 I_{n_2} & -R_2 \\ 0_{m_2 \times n_1} & -J_{m_2 \times m_1} & -R_2^T & (m_1+2) I_{m_2} \end{pmatrix}.$$

Let $L_1 = \begin{pmatrix} d_1 I_{n_1} & -R_1 \\ -R_1^T & (m_2+2) I_{m_1} \end{pmatrix}$.
By Lemma 2.3, we have

$$L_1^{-1} = \begin{pmatrix} d_1^{-1} I_{n_1} + d_1^{-1} R_1 \left(L_{l(G_1)} + d_1 m_2 I_{m_1}\right)^{-1} R_1^T & R_1 \left(L_{l(G_1)} + d_1 m_2 I_{m_1}\right)^{-1} \\ \left(L_{l(G_1)} + d_1 m_2 I_{m_1}\right)^{-1} R_1^T & d_1 \left(L_{l(G_1)} + d_1 m_2 I_{m_1}\right)^{-1} \end{pmatrix}.$$

Then, let $L_2 = \begin{pmatrix} L_1 & 0_{(n_1+m_1) \times n_2} \\ 0_{n_2 \times (n_1+m_1)} & d_2 I_{n_2} \end{pmatrix}$.
By Lemma 2.4, we have $L_2^{-1} = \begin{pmatrix} L_1^{-1} & 0_{(n_1+m_1) \times n_2} \\ 0_{n_2 \times (n_1+m_1)} & d_2^{-1} I_{n_2} \end{pmatrix}$.

Let

$$S = (m_1+2)I_{m_2} - \begin{pmatrix} 0_{m_2\times n_1} & -J_{m_2\times m_1} & -R_2^T \end{pmatrix} L_2^{-1} \begin{pmatrix} 0_{n_1\times m_2} \\ -J_{m_1\times m_2} \\ -R_2 \end{pmatrix}$$
$$= (m_1+2)I_{m_2} - d_1 J_{m_2\times m_1}\left(L_{l(G_1)} + d_1 m_2 I_{m_1}\right)^{-1} J_{m_1\times m_2} - d_2^{-1} R_2^T R_2$$
$$= d_2^{-1}\left(L_{l(G_2)} + d_2 m_1 I_{m_2} - \frac{d_2 m_1}{m_2} J_{m_2\times m_2}\right).$$

Lemma 2.5 implies that

$$S^{\#} = d_2\left(L_{l(G_2)} + d_2 m_1 I_{m_2}\right)^{-1} - \frac{1}{m_1 m_2} J_{m_2\times m_2}.$$

Let $X = L_{l(G_1)} + d_1 m_2 I_{m_1}$ and $Y = L_{l(G_2)} + d_2 m_1 I_{m_2}$. By Lemma 2.4, the following matrix

$$\begin{pmatrix} d_1^{-1}I_{n_1} + d_1^{-1}R_1X^{-1}R_1^T & R_1X^{-1} & 0_{n_1\times n_2} & 0_{n_1\times m_2} \\ X^{-1}R_1^T & d_1X^{-1} & 0_{m_1\times n_2} & 0_{m_1\times m_2} \\ 0_{n_2\times n_1} & 0_{n_2\times m_1} & d_2^{-1}I_{n_2} + d_2^{-2}R_2S^{\#}R_2^T & d_2^{-1}R_2S^{\#} \\ 0_{m_2\times n_1} & 0_{m_2\times m_1} & d_2^{-1}S^{\#}R_2^T & S^{\#} \end{pmatrix} \quad (4.1)$$

is a symmetric $\{1\}$-inverse of $L_{G_1\ominus G_2}$.

For any $i,j \in V(G_1)$ $(i \neq j)$, let $e_1,\ldots,e_{d_1}$ are d_1 edges incident to i in G_1 and $f_1,\ldots,f_{d_1}$ are d_1 edges incident to j in G_1. By Lemma 2.1 and Matrix (4.1), we have

$$r_{ij}(G_1\ominus G_2) = 2d_1^{-1} + d_1^{-1}(R_1X^{-1}R_1^T)_{ii} + d_1^{-1}(R_1X^{-1}R_1^T)_{jj} - 2d_1^{-1}(R_1X^{-1}R_1^T)_{ij}$$
$$= 2d_1^{-1} + d_1^{-1}\sum_{u,v=1}^{d_1} X^{-1}_{e_ue_v} + d_1^{-1}\sum_{u,v=1}^{d_1} X^{-1}_{f_uf_v} - 2d_1^{-1}\sum_{u,v=1}^{d_1} X^{-1}_{e_uf_v}$$
$$= 2d_1^{-1} + d_1^{-2}\sum_{u,v=1}^{d_1} r_{e_uf_v}(G_1\ominus G_2) - d_1^{-2}\sum_{u<v} r_{e_ue_v}(G_1\ominus G_2) - d_1^{-2}\sum_{u<v} r_{f_uf_v}(G_1\ominus G_2).$$

For any $i,j \in V(G_2)$ $(i \neq j)$, let $e_1,\ldots,e_{d_2}$ are d_2 edges incident to i in G_2 and $f_1,\ldots,f_{d_2}$ are d_2 edges incident to j in G_2. By Lemma 2.1 and Matrix (4.1), we have

$r_{ij}(G_1 \ominus G_2)$
$= 2d_2^{-1} + d_2^{-2}(R_2S^{\#}R_2^T)_{ii} + d_2^{-2}(R_2S^{\#}R_2^T)_{jj} - 2d_2^{-2}(R_2S^{\#}R_2^T)_{ij}$
$= 2d_2^{-1} + d_2^{-2}\sum_{u,v=1}^{d_2} r_{e_uf_v}(G_1 \ominus G_2) - d_2^{-2}\sum_{u<v} r_{e_ue_v}(G_1 \ominus G_2)$
$\quad - d_2^{-2}\sum_{u<v} r_{f_uf_v}(G_1 \ominus G_2).$

For any $i, j \in I(G_1)$, by Lemma 2.1 and Matrix (4.1), we have
$r_{ij}(G_1 \ominus G_2) = d_1\left(X_{ii}^{-1} + X_{jj}^{-1} - 2X_{ij}^{-1}\right).$
For any $i, j \in I(G_2)$, by Lemma 2.1 and Matrix (4.1), we have
$r_{ij}(G_1 \ominus G_2) = d_2\left(Y_{ii}^{-1} + Y_{jj}^{-1} - 2Y_{ij}^{-1}\right).$
For any $i \in V(G_1)$, $j \in I(G_1)$, let $e_1, \ldots, e_{d_1}$ be d_1 edges incident to i in G_1. By Lemma 2.1 and Matrix (4.1), we have

$$\begin{aligned} r_{ij}(G_1 \ominus G_2) &= d_1^{-1} + d_1^{-1}(R_1X^{-1}R_1^T)_{ii} + d_1X_{jj}^{-1} - 2(R_1X^{-1})_{ij} \\ &= d_1^{-1} + d_1X_{jj}^{-1} + d_1^{-1}\sum_{u,v=1}^{d_1} X_{e_ue_v}^{-1} - 2\sum_{u=1}^{d_1} X_{e_uj}^{-1} \\ &= d_1^{-1} + d_1^{-1}\sum_{u=1}^{d_1} r_{e_uj}(G_1 \ominus G_2) - d_1^{-2}\sum_{u<v} r_{e_ue_v}(G_1 \ominus G_2). \end{aligned}$$

For any $i \in V(G_2)$, $j \in I(G_2)$, let $e_1, \ldots, e_{d_2}$ be d_2 edges incident to i in G_2. By Lemma 2.1 and Matrix (4.1), we have

$$\begin{aligned} r_{ij}(G_1 \ominus G_2) &= d_2^{-1} + d_2^{-1}(R_2Y^{-1}R_2^T)_{ii} + d_2Y_{jj}^{-1} - 2(R_2Y^{-1})_{ij} \\ &= d_2^{-1} + d_2^{-1}\sum_{u=1}^{d_2} r_{e_uj}(G_1 \ominus G_2) - d_2^{-2}\sum_{u<v} r_{e_ue_v}(G_1 \ominus G_2). \end{aligned}$$

For $i \in V(G_2)$, $j \in I(G_1)$, $k \in I(G_2)$, by Lemma 2.1 and Matrix (4.1), we have

$$\begin{aligned} r_{ij}(G_1 \ominus G_2) &= d_2^{-1} + d_2^{-1}(R_2Y^{-1}R_2^T)_{ii} + d_1X_{jj}^{-1} - \frac{1}{m_1m_2} \\ &= d_2^{-1} + r_{jk}(G_1 \ominus G_2). \end{aligned}$$

For $i \in V(G_1)$, $j \in I(G_2)$, $k \in I(G_1)$, by Lemma 2.1 and Matrix (4.1), we have

$$\begin{aligned} r_{ij}(G_1 \ominus G_2) &= d_1^{-1} + d_1^{-1}(R_2X^{-1}R_2^T)_{ii} + d_2Y_{jj}^{-1} - \frac{1}{m_1m_2} \\ &= d_1^{-1} + r_{kj}(G_1 \ominus G_2). \end{aligned}$$

For $i \in V(G_1)$, $j \in V(G_2)$, $h \in I(G_1)$, $k \in I(G_2)$, by Lemma 2.1 and Matrix (4.1), we have

$r_{ij}(G_1 \ominus G_2)$

$= d_1^{-1} + d_2^{-1}(R_1X^{-1}R_1^T)_{ii} + d_2^{-1} + d_2^{-1}(R_2Y^{-1}R_2)_{jj} - \frac{1}{m_1m_2}$

$= d_1^{-1} + d_2^{-1} + r_{hk}(G_1 \ominus G_2).$

For $i \in I(G_1)$, $j \in I(G_2)$, by Lemma 2.1 and Matrix (4.1), we have $r_{ij}(G_1 \ominus G_2) = d_1Y_{ii}^{-1} + d_2Y_{jj}^{-1} - \frac{1}{m_1m_2}$.

This completes the proof. □

5 Resistance distances of $G_1 \circledast G_2$

In this section, we obtain the formulae for the resistance distances in $G_1 \circledast G_2$ for G_1 an arbitrary graph and G_2 a regular graph.

Theorem 5.1. *Let G_1 be a graph with n_1 vertices and m_1 edges, and G_2 be an d_2-regular($d_2 > 0$) graph of order n_2. Let $X = \frac{1}{2}L_{G_1} + m_2I_{n_1}$ and $Y = L_{l(G_2)} + d_2n_1I_{m_2}$. Then the following hold:*

(1) *For any $i, j \in V(G_1)$, we have $r_{ij}(G_1 \circledast G_2) = X_{ii}^{-1} + X_{jj}^{-1} - 2X_{ij}^{-1}$;*

(2) *For any $i, j \in V(G_2)$, we have*

$$r_{ij}(G_1 \circledast G_2) = 2d_2^{-1} + d_2^{-2}\sum_{u,v=1}^{d_2} r_{e_uf_v}(G_1 \circledast G_2) - d_2^{-2}\sum_{u<v} r_{e_ue_v}(G_1 \circledast G_2) - d_2^{-2}\sum_{u<v} r_{f_uf_v}(G_1 \circledast G_2),$$

where $e_1, \ldots, e_{d_2}$ are d_2 edges incident to i in G_2, $f_1, \ldots, f_{d_2}$ are d_2 edges incident to j in G_2;

(3) *For any $i = u_1v_1$, $j = u_2v_2 \in I(G_1)$, we have*

$$r_{ij}(G_1 \circledast G_2) = 1 + \frac{1}{4}\left[r_{u_1u_2}(G_1 \circledast G_2) + r_{v_1v_2}(G_1 \circledast G_2) + r_{u_1v_2}(G_1 \circledast G_2) + r_{v_1u_2}(G_1 \circledast G_2) - r_{u_1v_1}(G_1 \circledast G_2) - r_{u_2v_2}(G_1 \circledast G_2)\right];$$

(4) *For any $i, j \in I(G_2)$, we have $r_{ij}(G_1 \circledast G_2) = d_2\left(Y_{ii}^{-1} + Y_{jj}^{-1} - 2Y_{ij}^{-1}\right)$;*

(5) *For any $i = uv \in I(G_1)$, $j \in V(G_1)$, we have*

$$r_{ij}(G_1 \circledast G_2) = \frac{1}{2} + \frac{1}{4}\left[2r_{uj}(G_1 \circledast G_2) + 2r_{vj}(G_1 \circledast G_2) - r_{uv}(G_1 \circledast G_2)\right];$$

(6) *For any* $i \in V(G_2)$, $j \in I(G_2)$, *we have*
$r_{ij}(G_1 \circledast G_2) = d_2^{-1} + d_2^{-1} \sum_{u=1}^{d_2} r_{e_u j}(G_1 \circledast G_2) - d_2^{-2} \sum_{u<v} r_{e_u e_v}(G_1 \circledast G_2)$, *where* $e_1, \ldots, e_{d_2}$ *are* d_2 *edges incident to* i *in* G_2;

(7) *For any* $i = uv \in I(G_1)$, $j \in V(G_2)$, *we have* $r_{ij}(G_1 \circledast G_2) = d_2^{-1} + r_{ik}(G_1 \circledast G_2)$, *where* $k \in I(G_2)$;

(8) *For any* $i \in V(G_1)$, $j \in I(G_2)$, *we have* $r_{ij}(G_1 \circledast G_2) = X_{ii}^{-1} + d_2 Y_{jj}^{-1} - \dfrac{1}{n_1 m_2}$;

(9) *For any* $i \in V(G_1)$, $j \in V(G_2)$, *we have* $r_{ij}(G_1 \circledast G_2) = d_2^{-1} + r_{ik}(G_1 \circledast G_2)$, *where* $k \in I(G_2)$;

(10) *For any* $i = uv \in I(G_1)$, $j \in I(G_2)$, *we have* $r_{ij}(G_1 \circledast G_2) = \dfrac{1}{2} + \dfrac{1}{4}(X_{uu}^{-1} + X_{vv}^{-1} + 2X_{uv}^{-1}) + d_2 Y_{jj}^{-1} - \dfrac{1}{n_1 m_2}$.

Proof. Let R be the vertex-edge incidence matrix of G. The Laplacian matrix of $G_1 \circledast G_2$ has the following form

$$L_{G_1 \circledast G_2} = \begin{pmatrix} 2I_{m_1} & -R_1^T & 0_{m_1 \times n_2} & 0_{m_1 \times m_2} \\ -R_1 & D_1 + m_2 I_{n_1} & 0_{n_1 \times n_2} & -J_{n_1 \times m_2} \\ 0_{n_2 \times m_1} & 0_{n_2 \times n_1} & d_2 I_{n_2} & -R_2 \\ 0_{m_2 \times m_1} & -J_{m_2 \times n_1} & -R_2^T & (n_1+2) I_{m_2} \end{pmatrix}.$$

Let

$$L_1 = \begin{pmatrix} 2I_{m_1} & -R_1^T \\ -R_1 & D_{G_1} + m_2 I_{n_1} \end{pmatrix}.$$

By Lemma 2.3, we have

$$L_1^{-1} = \begin{pmatrix} \frac{1}{2} I_{m_1} + \frac{1}{4} R_1^T \left(\frac{1}{2} L_{G_1} + m_2 I_{n_1}\right)^{-1} R_1 & \frac{1}{2} R_1^T \left(\frac{1}{2} L_{G_1} + m_2 I_{n_1}\right)^{-1} \\ \frac{1}{2}\left(\frac{1}{2} L_{G_1} + m_2 I_{n_1}\right)^{-1} R_1 & \left(\frac{1}{2} L_{G_1} + m_2 I_{n_1}\right)^{-1} \end{pmatrix}.$$

Then, let $L_2 = \begin{pmatrix} L_1 & 0_{(n_1+m_1) \times n_2} \\ 0_{n_2 \times (n_1+m_1)} & d_2 I_{n_2} \end{pmatrix}$.

By Lemma 2.4, we have $L_2^{-1} = \begin{pmatrix} L_1^{-1} & 0_{(n_1+m_1) \times n_2} \\ 0_{n_2 \times (n_1+m_1)} & d_2^{-1} I_{n_2} \end{pmatrix}$.

Let

$$S=(n_1+2)I_{m_2}-\begin{pmatrix}0_{m_2\times m_1} & -J_{m_2\times n_1} & -R_2^T\end{pmatrix}L_2^{-1}\begin{pmatrix}0_{m_1\times m_2}\\ -J_{n_1\times m_2}\\ -R_2\end{pmatrix}$$

$$=(n_1+2)I_{m_2}-J_{m_2\times n_1}\left(m_2I_{n_1}+\frac{1}{2}L_{G_1}\right)^{-1}J_{n_1\times m_2}-\frac{1}{d_2}R_2^TR_2$$

$$=d_2^{-1}\left(L_{l(G_2)}+d_2n_1I_{m_2}-\frac{d_2n_1}{m_2}J_{m_2\times m_2}\right).$$

Lemma 2.5 implies that $S^\#=d_2\left(L_{l(G_2)}+d_2n_1I_{n_2}\right)^{-1}-\frac{1}{n_1m_2}J_{m_2\times m_2}$.
Let $X=\frac{1}{2}L_{G_1}+m_2I_{n_1}$ and $Y=L_{l(G_2)}+d_2n_1I_{m_2}$. By Lemma 2.4, the following matrix

$$\begin{pmatrix}\frac{1}{2}I_{m_1}+\frac{1}{4}R_1^TX^{-1}R_1 & \frac{1}{2}R_1^TX^{-1} & 0_{m_1\times n_2} & 0_{m_1\times m_2}\\ \frac{1}{2}X^{-1}R_1 & X^{-1} & 0_{n_1\times n_2} & 0_{n_1\times m_2}\\ 0_{n_2\times m_1} & 0_{n_2\times n_1} & \frac{1}{d_2}I_{n_2}+\frac{1}{d_2^2}R_2S^\#R_2^T & \frac{1}{d_2}R_2S^\#\\ 0_{m_2\times m_1} & 0_{m_2\times n_1} & \frac{1}{d_2}S^\#R_2^T & S^\#\end{pmatrix}\tag{5.1}$$

is a symmetric $\{1\}$-inverse of $L_{G_1\circledast G_2}$.
For any $i,j\in V(G_1)$, by Lemma 2.1 and Matrix (5.1), we have $r_{ij}(G_1\circledast G_2)=X_{ii}^{-1}+X_{jj}^{-1}-2X_{ij}^{-1}$.
For any $i,j\in V(G_2)$, let $e_1,\dots,e_{d_2}$ are d_2 edges incident to i in G_2 and $f_1,\dots,f_{d_2}$ are d_2 edges incident to j in G_2. By Lemma 2.1, Matrix (5.1), we have
$r_{ij}(G_1\circledast G_2)$

$$=\frac{2}{d_2}+\frac{1}{d_2}(R_2Y^{-1}R_2^T)_{ii}+\frac{1}{d_2}(R_2Y^{-1}R_2^T)_{jj}-\frac{2}{d_2}(R_2Y^{-1}R_2^T)_{ij}$$

$$=2d_2^{-1}+d_2^{-2}\sum_{u,v=1}^{d_2}r_{e_uf_v}(G_1\circledast G_2)-d_2^{-2}\sum_{u<v}r_{e_ue_v}(G_1\circledast G_2)$$

$$-d_2^{-2}\sum_{u<v}r_{f_uf_v}(G_1\circledast G_2).$$

For any $i=u_1v_1$, $j=u_2v_2\in I(G_1)$, by Lemma 2.1 and Matrix (5.1), we have

$$r_{ij}(G_1 \circledast G_2) = 1 + \frac{1}{4}\left[\left(R_1^T X^{-1} R_1\right)_{ii} + \left(R_1^T X^{-1} R_1\right)_{jj} - 2\left(R_1^T X^{-1} R_1\right)_{ij}\right]$$
$$= 1 + \frac{1}{4}\left[r_{u_1u_2}(G_1 \circledast G_2) + r_{v_1v_2}(G_1 \circledast G_2) + r_{u_1v_2}(G_1 \circledast G_2)\right.$$
$$\left. + r_{v_1u_2}(G_1 \circledast G_2) - r_{u_1v_1}(G_1 \circledast G_2) - r_{u_2v_2}(G_1 \circledast G_2)\right].$$

For any $i, j \in I(G_2)$, by Lemma 2.1 and Matrix (5.1), we have $r_{ij}(G_1 \circledast G_2) = d_2\left(Y_{ii}^{-1} + Y_{jj}^{-1} - 2Y_{ij}^{-1}\right)$

For any $i = uv \in I(G_1)$, $j \in V(G_1)$, by Lemma 2.1 and Matrix (5.1), we have

$$r_{ij}(G_1 \circledast G_2) = \frac{1}{2} + \frac{1}{4}(R_1^T X^{-1} R_1)_{ii} + X_{jj}^{-1} - (R_1^T X^{-1})_{ij}$$
$$= \frac{1}{2} + \frac{1}{4}\left[2r_{uj}(G_1 \circledast G_2) + 2r_{vj}(G_1 \circledast G_2) - r_{uv}(G_1 \circledast G_2)\right]$$

For any $i \in V(G_2)$, $j \in I(G_2)$, let $e_1, \ldots, e_{d_2}$ be d_2 edges incident to j in G_2. By Lemma 2.1 and Matrix (5.1), we have

$$r_{ij}(G_1 \circledast G_2) = \left(d_2^{-1} I_{n_2} + d_2^{-2} R_2 S^{\#} R_2^T\right)_{ii} + S_{jj}^{\#} - 2d_2^{-1}\left(R_2 S^{\#}\right)_{ij}$$
$$= d_2^{-1} + d_2^{-1}\sum_{u=1}^{d_2} r_{e_uj}(G_1 \circledast G_2) - d_2^{-2}\sum_{u<v} r_{e_ue_v}(G_1 \circledast G_2)$$

For any $i = uv \in I(G_1)$, $j \in V(G_2)$, $k \in I(G_2)$, by Lemma 2.1 and Matrix (5.1), we have

$$r_{ij}(G_1 \circledast G_2) = \frac{1}{2} + \frac{1}{4}(R_1^T X^{-1} R_1)_{ii} + d_2^{-1} + d_2^{-1}(R_2 Y^{-1} R_2^T)_{jj} - \frac{1}{n_1 m_2}$$
$$= d_2^{-1} + r_{ik}(G_1 \circledast G_2)$$

For any $i \in V(G_1)$, $j \in I(G_2)$, by Lemma 2.1 and Matrix (5.1), we have $r_{ij}(G_1 \circledast G_2) = X_{ii}^{-1} + d_2 Y_{jj}^{-1} - \frac{1}{n_1 m_2}$.

For any $i \in V(G_1)$, $j \in V(G_2)$, $k \in I(G_2)$, by Lemma 2.1 and Matrix (5.1), we have

$$r_{ij}(G_1 \circledast G_2) = X_{ii}^{-1} + \left(d_2^{-1} I_{n_2} + d_2^{-1} R_2 Y^{-1} R_2^T\right)_{jj} - \frac{1}{n_1 m_2}$$
$$= d_2^{-1} + r_{ik}(G_1 \circledast G_2)$$

For any $i = uv \in I(G_1)$, $j \in I(G_2)$, by Lemma 2.1 and Matrix (5.1), we have $r_{ij}(G_1 \circledast G_2) = \frac{1}{2} + \frac{1}{4}(X_{uu}^{-1} + X_{vv}^{-1} + 2X_{uv}^{-1}) + d_2 Y_{jj}^{-1} - \frac{1}{n_1 m_2}$.

This completes the proof. □

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