The 2-color Rado Number of $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_k$

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Abstract

In 1982, Beutelspacher and Brestovansky determined the 2-color Rado number of the equation

$$x_1 + x_2 + \cdots + x_{m-1} = x_m$$

for all $m \geq 3$. Here we extend their result by determining the 2-color Rado number of the equation

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_k$$

for all $n \geq 2$ and $k \geq 2$. As a consequence, we determine the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_n = a_1 y_1 + \cdots + a_\ell y_\ell$$

in all cases where $n \geq 2$ and $n \geq a_1 + \cdots + a_\ell$, and in most cases where $n \geq 2$ and $2n \geq a_1 + \cdots + a_\ell$.

1. Introduction

A special case of the work of Richard Rado [5] is that for all positive integers n and k such that $n + k \geq 3$, and all positive integers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$, there exists a smallest positive integer r with the following property: for every coloring of the elements of the set $[r] = \{1, \ldots, r\}$ with two colors, there exists a solution of the equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b_1 y_1 + b_2 y_2 + \cdots + b_k y_k$$

using elements of $[r]$ that are all colored the same. (Such a solution is called *monochromatic.*) The integer r is called the *2-color Rado number* of the equation.

In recent years there has been a considerable amount of work aimed at determining the Rado numbers of specific equations. One of the earliest results in this direction appeared in a 1982 paper of Beutelspacher and Brestovansky [1], where it was proved that for every $m \geq 3$, the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_{m-1} = x_m$$

is $m^2 - m - 1$. In 2008 Guo and Sun [2] generalized this result by proving that, for all positive integers $a_1, \ldots, a_{m-1}$, the 2-color Rado number of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} = x_m$$

is $aw^2 + w - a$, where $a = \min\{a_1, \ldots, a_{m-1}\}$ and $w = a_1 + \cdots + a_{m-1}$. In the same year, Schaal and Vestal [8] dealt with the equation

$$x_1 + x_2 + \cdots + x_{m-1} = 2x_m.$$

They proved, in particular, that for every $m \geq 6$, the 2-color Rado number is $\lceil \frac{m-1}{2} \lceil \frac{m-1}{2} \rceil \rceil$. Building on the work of Schaal and Vestal, we investigated the equation

$$x_1 + x_2 + \cdots + x_{m-1} = ax_m, \tag{1}$$

for $a \geq 3$, in [6] and [7].

Notation. We will denote $\lceil \frac{m-1}{a} \lceil \frac{m-1}{a} \rceil \rceil$ by $C(m,a)$, and we will denote the 2-color Rado number of equation (1) by $R_2(m,a)$.

Fact 1 ([7], Theorem 3). Suppose $a \geq 3$. If $3|a$ then $R_2(m,a) = C(m,a)$ when $m \geq 2a+1$ but $R_2(2a,a) = 5$ and $C(2a,a) = 4$. If $3 \nmid a$ then $R_2(m,a) = C(m,a)$ when $m \geq 2a+2$ but $R_2(2a+1,a) = 5$ and $C(2a+1,a) = 4$.

We note that, by the results of [8], the statements in Fact 1 remain valid when $a = 2$.

Results have been obtained for a number of other variations of the equation $x_1 + \cdots + x_{m-1} = x_m$, most of which have had the property that one side of the equation involves only one variable. Our first purpose here is to determine the 2-color Rado number of the equation

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_k, \tag{2}$$

for all $n \geq 2$ and $k \geq 2$.

Notation. We denote the 2-color Rado number of equation (2) by $r_2(n,k)$.

To determine $r_2(n,k)$ for all n and k it clearly suffices to consider the case $n \geq k$. We will deal with this case by relating equation (2) to the equation

$$x_1 + \cdots + x_n = ky \tag{3}$$

and using results from [7] (and [1] and [8], for the cases $k = 1, 2$).

Theorem 1. If $n \geq 2$ and $n \geq k$, then $r_2(n,k) = R_2(n+1,k)$.

The relevant values of $R_2(n+1,k)$ are determined by [1], [8], Fact 1 and the following additional information from [7].

Fact 2 ([7], Theorem 2). Suppose $a+1 \leq m \leq 2a+1$. Then $R_2(m,a) = 1$ iff $m = a+1$. If $a+2 \leq m \leq 2a+1$, then $R_2(m,a) \in \{3,4,5\}$, and we have:

$R_2(m,a) = 3$ iff $m \leq \frac{3a}{2} + 1$ and $a \equiv m-1 \pmod 2$.

$R_2(m,a) = 4$ iff either:

(i) $m \leq \frac{3a}{2} + 1$ and $a \not\equiv m-1 \pmod 2$, or
(ii) $m > \frac{3a}{2} + 1$ and $a \equiv m-1 \pmod 3$.

$R_2(m,a) = 5$ iff $m > \frac{3a}{2} + 1$ and $a \not\equiv m-1 \pmod 3$.

If $n < k$. then Theorem 1 yields $r_2(n,k) = r_2(k,n) = R_2(k+1,n)$, but for the purposes of Theorem 3 (below) we will need to know that $r_2(n,k) = R_2(n+1,k)$ in most cases where $2n \geq k > n$.

Theorem 2. If $n \geq 2$ and $2n \geq k > n$, then $r_2(n,k) = R_2(n+1,k)$ except in the following cases:

$r_2(2,3) = 4$ while $R_2(3,3) = 9$,

$r_2(2,4) = 5$ while $R_2(3,4) = 10$,

$r_2(3,5) = 5$ while $R_2(4,5) = 9$, and

if $10 \leq k \leq 14$ then $r_2(k-5,k) = 5$ while $R_2(k-4,k) = 6$.

The proof of Theorem 2 relies on the following two results from [7].

Fact 3 ([7], Theorem 4). If $\frac{2a}{3} + 1 \leq m \leq a$, then:

for $a = 3$ we have $R_2(a,a) = 9$, and

for $a \geq 4$ we have

$$R_2(m,a) = 3 \text{ if } a \equiv m-1 \pmod 2 \text{ and}$$
$$R_2(m,a) = 4 \text{ if } a \not\equiv m-1 \pmod 2.$$

Fact 4 ([7], Theorem 5). If $\frac{a}{2}+1 \leq m < \frac{2a}{3}+1$ (so $a \geq 4$) then:

for $a \equiv m-1 \pmod 3$ we have $R_2(m,a) = 4$, and

for $a \not\equiv m-1 \pmod 3$ we have $R_2(m,a) = 5$ *except* that

$R_2(3,4) = 10$ and $R_2(4,5) = 9$, and
$R_2(m,a) = 6$ if $10 \leq a \leq 14$ and $m = a-4$.

We will show that Theorems 1 and 2 have the following consequence.

Theorem 3. Let $n \geq 2$, let $a_1, \ldots, a_\ell$ be positive integers, and let $A = a_1 + \cdots + a_\ell$. Then the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_n = a_1y_1 + \cdots + a_\ell y_\ell$$

is at least $r_2(n,A)$ and at most $R_2(n+1,A)$. If $n \geq A$, the 2-color Rado number is $R_2(n+1,A)$. If $2n \geq A > n$, the same conclusion holds provided that the pair (n,A) is none of $(2,3),(2,4),(3,5),(5,10),(6,11),(7,12),(8,13),(9,14)$.

For the case $A > 2n$ we have the following.

Theorem 4. If $n \geq 2$, $A = a_1 + \cdots + a_\ell$ and $A > 2n$, then the 2-color Rado number of

$$x_1 + x_2 + \cdots + x_n = a_1y_1 + \cdots + a_\ell y_\ell$$

is at least $\lceil \frac{A}{n} \lceil \frac{A}{n} \rceil \rceil$ and at most $R_2(n+1,A)$.

The values of $R_2(n+1,A)$ for $A > 2n$ are not provided by Facts 1 and 2, but the lower bound $\lceil \frac{A}{n} \lceil \frac{A}{n} \rceil \rceil$ might be useful in determining these values.

2. Proofs of the Theorems

Lemma. For any $n \geq k$, we have

$$C(n+1,k) \leq r_2(n,k) \leq R_2(n+1,k).$$

Proof. The second inequality is clear, since any monochromatic solution of equation (3) provides a monochromatic solution of equation (2).

To prove the first inequality it suffices to consider $n > k$ and exhibit a 2-coloring of $[C(n+1,k)-1]$ that yields no monochromatic solution of equation (2). Let the elements of $[\lceil \frac{n}{k} \rceil - 1]$ be colored red and let the remaining elements of $[C(n+1,k)-1]$ be colored blue. For any red solution of equation (2), the left side has total value at least n and the right side has total value at most $k(\lceil \frac{n}{k} \rceil - 1) < n$, so there is no red solution. For any blue solution the left side has total value at least $n\lceil \frac{n}{k} \rceil$ and the right side has total value at most $k(\lceil \frac{n}{k} \lceil \frac{n}{k} \rceil \rceil - 1) < n\lceil \frac{n}{k} \rceil$, so there is no blue solution. □

Proof of Theorem 1. Theorem 1 is obviously true when $k = 1$, so we assume $k \geq 2$.

If $3|k$ and $n \geq 2k$ or if $3 \nmid k$ and $n \geq 2k+1$ then by Fact 1 and [8] we have $R_2(n+1,k) = C(n+1,k)$, so by the Lemma we have $r_2(n,k) = R_2(n+1,k)$. To complete the proof, it suffices to consider the cases where $k \leq n \leq 2k$.

Note that $r_2(n,k) = 1$ iff $n = k$ iff $R_2(n+1,k) = 1$, so we can suppose that $k+1 \leq n \leq 2k$. Then by coloring 1 and 2 differently, we see that $r_2(n,k)$ cannot be 2, so $r_2(n,k) \geq 3$. We have $R_2(n+1,k) \in \{3,4,5\}$ by Fact 2. We now consider the mutually exclusive cases indicated in Fact 2.

First suppose that $n \leq \frac{3k}{2}$ and $k \equiv n \pmod 2$. Then by Fact 2 we have $R_2(n+1,k) = 3$. By the Lemma, $r_2(n,k)$ can only be 3.

Next suppose that $n \leq \frac{3k}{2}$ and $k \not\equiv n \pmod 2$. Then $R_2(n+1,k) = 4$ by Fact 2. By the Lemma, $r_2(n,k) = 3$ or 4. If we 2-color [3] by coloring 1 and 3 red and 2 blue, then since $k \not\equiv n \pmod 2$ there is no red solution of equation (2), and since $k \neq n$ there is no blue solution. So $r_2(n,k) = 4$.

Now suppose that $n > \frac{3k}{2}$ and $k \equiv n \pmod 3$. Then by Fact 2 we have $R_2(n+1,k) = 4$, so again $r_2(n,k) = 3$ or 4. Since $n > \frac{3k}{2}$, we have $C(n+1,k) = \lceil \frac{n}{k} \lceil \frac{n}{k} \rceil \rceil \geq \lceil \frac{n}{k} \cdot 2 \rceil$ and $\frac{n}{k} \cdot 2 > 3$, so it follows from the Lemma that $r_2(n,k) \geq 4$, and therefore $r_2(n,k) = 4$.

Finally, if $n > \frac{3k}{2}$ and $k \not\equiv n \pmod 3$, then $R_2(n+1,k) = 5$ by Fact 2. As in the preceding paragraph, we have $r_2(n,k) \geq 4$. If we 2-color [4] by coloring 1 and 4 red and 2 and 3 blue, then for any blue solution of equation (2) the left side has total value at least $2n$ and the right side has total value at most $3k$. Since $2n > 3k$, there is no blue solution. For any red solution the left side of the equation has total value congruent to $n \pmod 3$ and the right side has total value congruent to $k \pmod 3$. Since $k \not\equiv n \pmod 3$ there can be no red solution. So $r_2(n,k) = 5 = R_2(n+1,k)$. □

Proof of Theorem 2. Since $k > n$, we have $r_2(n,k) = r_2(k,n) = R_2(k+1,n)$, by Theorem 1. To evaluate $R_2(k+1,n)$ we will use Fact 2. Note that we can do so since the condition $a+2 \leq m \leq 2a+1$ of Fact 2, with $a = n$ and

$m = k+1$, becomes $n+2 \le k+1 \le 2n+1$, which holds since $2n \ge k > n$.

Case 1: $k > n \ge \frac{2k}{3}$.

In this case, if $k = 3$ then $n = 2$. We have $r_2(2,3) = R_2(3+1,2) = 4$ by [8], while $R_2(n+1,k) = R_2(3,3) = 9$ by Fact 3.

We claim that if $k \ge 4$ then $r_2(n,k) = R_2(n+1,k)$, i.e., $R_2(k+1,n) = R_2(n+1,k)$. Note that since $k \le \frac{3n}{2}$, Fact 2 (with $a = n$ and $m = k+1$) yields $R_2(k+1,n) = 3$ if $n \equiv k \pmod 2$ and $R_2(k+1,n) = 4$ if $n \not\equiv k \pmod 2$. To determine $R_2(n+1,k)$ we can use Fact 3 (with $a = k$ and $m = n+1$), since $\frac{2k}{3}+1 \le n+1 \le k$. We find that $R_2(n+1,k) = 3$ if $k \equiv n \pmod 2$ and $R_2(n+1,k) = 4$ if $k \not\equiv n \pmod 2$. This concludes the proof in Case 1.

Case 2: $\frac{2k}{3} > n \ge \frac{k}{2}$.

Using Fact 2 with $a = n$ and $m = k+1$, we note that since $k > \frac{3n}{2}$ we have $R_2(k+1,n) = 4$ if $n \equiv k \pmod 3$ and $R_2(k+1,n) = 5$ if $n \not\equiv k \pmod 3$. Using Fact 4 with $a = k$ and $m = n+1$ (which is legitimate since $\frac{k}{2}+1 \le n+1 < \frac{2k}{3}+1$) we find that $R_2(n+1,k) = 4$ if $k \equiv n \pmod 3$ and $R_2(n+1,k) = 5$ if $k \not\equiv n \pmod 3$ (and therefore $r_2(n,k) = R_2(n+1,k)$) *unless* the pair $(n+1,k)$ is $(3,4)$, $(4,5)$, or $(k-4,k)$ for some $10 \le k \le 14$, in which case $R_2(n+1,k)$ is 10, 9, or 6, respectively. □

Proof of Theorem 3. The 2-color Rado number of

$$x_1 + x_2 + \cdots + x_n = a_1 y_1 + \cdots + a_\ell y_\ell \tag{4}$$

is at least that of

$$x_1 + x_2 + \cdots + x_n = y_1 + \cdots + y_A \tag{5}$$

and at most that of

$$x_1 + x_2 + \cdots + x_n = Ay, \tag{6}$$

since any monochromatic solution of equation (4) yields a monochromatic solution of equation (5) and any monochromatic solution of equation (6) yields a monochromatic solution of equation (4). This proves the first assertion of the theorem. We have $r_2(n,A) = R_2(n+1,A)$ by Theorem 1 when $n \ge A$ and by Theorem 2, with the stated exceptions, when $2n \ge A > n$. Thus the 2-color Rado number of equation (4) is $R_2(n+1,A)$ if $n \ge A$ or if $2n \ge A > n$ and (n,A) is not one of the indicated exceptional pairs. □

Proof of Theorem 4. As above, the 2-color Rado number of equation (4) is at least that of equation (5), which is $R_2(A+1,n)$ by Theorem 1. Since $A > 2n$, we have $A+1 \ge 2n+2$, so $R_2(A+1,n) = \lceil \frac{A}{n} \lceil \frac{A}{n} \rceil \rceil$ by Fact 1.

The upper bound is established as in the proof of Theorem 3. □

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