ARC-TRANSITIVE PENTAVALENT GRAPHS OF ORDER FOUR TIMES A PRIME POWER

JIANGMIN PAN*, ZHAOHONG HUANG, AND CAI HENG LI

ABSTRACT. In this paper, we study arc-transitive pentavalent graphs of order $4p^n$ with p a prime and n a positive integer. It is proved that no such graph exists for each prime $p \geq 5$, and all such graphs with $p = 2$ or 3 which are G-basic (that is, G has no non-trivial normal subgroup such that the graph is a normal cover of the corresponding normal quotient graph) are determined. Moreover, as an application, arc-transitive pentavalent graphs of order $4p^2$ and $4p^3$ are determined.

KEYWORDS. arc-transitive, simple group, normal cover

1. INTRODUCTION

For a finite, simple and undirected graph Γ, let $V\Gamma$ denote its vertex set, and let $\mathsf{Aut}\Gamma$ denote its full automorphism group. If there is $G \leq \mathsf{Aut}\Gamma$ such that G is transitive on the vertex set, edge-set or arc-set of Γ, then Γ is called *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive*, respectively. If Γ is a regular graph (that is, each vertex of Γ is adjacent to the same number of vertices in Γ), denote by $\mathsf{val}(\Gamma)$ its valency. Then Γ is called a *pentavalent graph* if $\mathsf{val}(\Gamma) = 5$.

For a positive integer s, an *s-arc* of Γ is a sequence $v_0, v_1, \ldots, v_s$ of $s+1$ vertices such that v_{i-1}, v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. If $G \leq \mathsf{Aut}\Gamma$ is transitive on the set of s-arcs of Γ, then Γ is called *(G, s)-arc-transitive*; if Γ is

1991 MR Subject Classification 20B15, 20B30, 05C25.

This paper was partially supported by NNSF of China (No.11171292, 11231008, 11461007).

*Corresponding author.

(G, s)-arc-transitive but not $(G, s+1)$-arc-transitive, then Γ is simply called (G, s)*-transitive.*

One of the most important methods for studying transitive graphs is taking normal quotient graphs. Let Γ be a G-vertex-transitive graph with $G \leq \mathsf{Aut}\Gamma$. For an intransitive normal subgroup N of G, the *normal quotient graph* of Γ relative to N, denoted by Γ_N, is defined with vertices the orbits of N on $V\Gamma$ and two orbits B, C are adjacent if and only if some $\alpha \in B$ is adjacent in Γ to some $\beta \in C$. If Γ and Γ_N have the same valency, then Γ is called a *normal cover* of Γ_N, that is, the induced subgraph on $B \cup C$ for adjacent B and C is a perfect matching.

Transitive graphs of order a small number times a prime power have been received quite a lot attention with numerous references in the literature, based on that these graphs not only have independent interest but also (may be more important) their characterizations would be applied for the studying of many other more general families of transitive graphs since these graphs often appear as normal quotient graphs. For example, Chao [3] classified arc-transitive graphs of prime order, Cheng and Oxley [4] classifyied edge-transitive graphs of order twice a prime, Feng and Kwak [8] classified cubic symmetric graphs with order small number times a prime or a prime-square, and these papers have been high cited for studying a lot of other families of transitive graphs.

The main purpose of this paper is to characterize arc-transitive pentavalent graphs of order $4p^n$ for each prime p and positive integer n. We remark that, transitive graphs of order $4p$ have been extensively studied by [6, 9, 18, 22, 26], transitive graphs of order $4p^2$ and valency 3 and 4 have been investigated by [7, 10, 27], and there are rare results for the case where $n \geq 3$.

The first result of this paper is the following assertion, which reduces our discussion to the cases where $p = 2$ and 3.

Theorem 1.1. *There is no connected arc-transitive pentavalent graph of order* $4p^n$ *for each prime* $p \geq 5$ *and positive integer* n.

The proof of Theorem 1.1 depends on the classification of finite simple groups. Noticing that Theorem 1.1 is not true for $p = 2$ and 3, the 5-dimensional hypercube Q_5 with order 32 and $K_{6,6} - 6K_2$ with order 12 are examples.

Theorem 1.1 has the following consequence which classifies arc-transitive pentavalent graphs of order $4p^2$ and $4p^3$, where C_{16} denotes the *Clebsch graph*, a non-bipartite arc-transitive pentavalent graph of order 16 (refer to [1]), and $\mathcal{G}_{36}$ is an arc-transitive pentavalent graph of order 36 (see [13, Construction I]).

Corollary 1.2. *Let p be a prime. Then*

(1) C_{16} *and* $\mathcal{G}_{36}$ *are the only connected arc-transitive pentavalent graphs of order* $4p^2$*;*

(2) Q_5 *is the unique connected arc-transitive pentavalent graph of order* $4p^3$.

It seems difficult to classify arc-transitive pentavalent graphs of order $4p^n$ with $p = 2$ or 3 yet. We here classify such graphs that are G-basic, which will play important role for approaching a possible general classification. Recall that, a G-arc-transitive graph Γ is called *G-basic* if G has no non-trivial normal subgroup N such that Γ is a normal cover of Γ_N. Then, there naturally arises a 'two-steps' strategy for studying G-arc-transitive graphs: determining G-basic graphs, and reconstructing the original graphs from the basic graphs by using covering techniques.

The terminologies and notations used in this paper are standard, refer to [5] or [15]. For example, for a positive integer n, denote by $\mathbb{Z}_n$, D_{2n}, A_n and S_n the cyclic group of order n, the dihedral group of order $2n$, the alternating group and symmetric group of degree n, respectively. For two groups N and H, denote by $N \times H$ the direct product of N and H, by $N.H$ an extension of N by H, and by $N : H$ instead of $N.H$ if the extension is split.

The next result determines all G-basic arc-transitive pentavalent graphs of order four times a prime power, where I_{12} denotes the *Icosahedron* graph, a non-bipartite arc-transitive pentavalent graph of order 12 (refer to [13]), M_{10} denotes the stabilizer

	Γ	(G, G_α)	s	$\mathrm{Aut}\Gamma$
1	I_{12}	$(\mathrm{A}_5, \mathbb{Z}_5)$	1	$\mathrm{A}_5 \times \mathbb{Z}_2$
2	$\mathsf{K}_{6,6} - 6\mathsf{K}_2$	$(\mathrm{S}_5, \mathrm{D}_{10})$	1	$\mathrm{S}_6 \times \mathbb{Z}_2$
		$(\mathrm{S}_6, \mathrm{A}_5)$	2	
3	C_{16}	$(\mathbb{Z}_2^4 : \mathbb{Z}_5, \mathbb{Z}_5), (\mathbb{Z}_2^4 : \mathrm{D}_{10}, \mathrm{D}_{10})$	1	$\mathbb{Z}_2^4 : \mathrm{S}_5$
		$(\mathbb{Z}_2^4 : \mathrm{A}_5, \mathrm{A}_5), (\mathbb{Z}_2^4 : \mathrm{S}_5, \mathrm{S}_5)$	2	
4	$\mathcal{G}_{36}$	$(\mathrm{A}_6, \mathrm{D}_{10}), (\mathrm{PGL}(2,9), \mathrm{D}_{20})$	1	$\mathrm{A}_6.\mathbb{Z}_2^2$
		$(\mathrm{S}_6, \mathsf{F}_{20}), (\mathrm{A}_6.\mathbb{Z}_2^2, \mathsf{F}_{20} \times \mathbb{Z}_2)$	2	
		$(\mathsf{M}_{10}, \mathsf{F}_{20})$	2	

TABLE 1. G-basic arc-transitive pentavalent graphs of order $4p^n$

of the Mathieu simple group M_{11} acting naturally on 11 points, and F_{20} denotes the unique Frobenius group of order 20.

Theorem 1.3. *Let Γ be a connected G-basic arc-transitive pentavalent graph of order $4p^n$, where $G \leq \mathrm{Aut}\Gamma$, p is a prime and $n \geq 1$. Then $p = 2$ or 3, $n = 1$ or 2, and Γ is (G, s)-transitive with $s = 1$ or 2. Further, the triple (Γ, G, G_α) is listed in the following Table 1, where $\alpha \in V\Gamma$.*

We remark that a graph may be G-basic for some arc-transitive automorphism group G, but not H-basic for another arc-transitive automorphism group H. For example, $\Gamma = \mathsf{K}_{6,6} - 6\mathsf{K}_2$ is S_5-basic and S_6-basic arc-transitive, but it is not $\mathrm{Aut}\Gamma$-basic since $\mathrm{Aut}\Gamma$ has a normal subgroup $\mathbb{Z}_2$ such that Γ is the standard double cover of $\Gamma_{\mathbb{Z}_2} \cong \mathsf{K}_6$; similarly, I_{12} is A_5-basic arc-transitive but not $\mathrm{Aut}\Gamma$-basic arc-transitive.

This paper is organized as follows. After this introduction section, some preliminary results are presented in Section 2. Then, Theorems 1.1 and 1.3 are proved in Sections 3 and 4 respectively.

2. PRELIMINARIES

In this section, we collect certain preliminary results which will be used later.

Let G be a group and H a subgroup of G. We use $\mathsf{C}_G(H)$ and $\mathsf{N}_G(H)$ to denote the centralizer and normalizer of H in G, respectively.

Lemma 2.1. ([15, Chapter I, Theorem 4]) *The quotient group $\mathsf{N}_G(H)/\mathsf{C}_G(H)$ is isomorphic to a subgroup of the automorphism group of H.*

For a group G, the *Fitting subgroup* of G is the largest nilpotent normal subgroup of G. Clearly, the Fitting subgroup is a characteristic subgroup.

Lemma 2.2. ([21, P. 30, Corollary]) *Let F be the Fitting subgroup of a group G. If $G \neq 1$ is soluble, then $F \neq 1$ and $\mathsf{C}_G(F) \leq F$.*

The following is the well known Maschke's Theorem on group representation theory. For simplicity, we state it in terms of group theory.

Theorem 2.3. ([23, THEOREM 1.4]) *Let $G = N : H$ be a split extension, where N is an elementary abelian p-group with p a prime and not dividing the order of H. If H normalizes a nontrivial subgroup N_1 of N, then there exists a subgroup N_2 of N such that $N = N_1 \times N_2$ and H normalizes N_2.*

The next lemma is from [11, P. 12-14].

Lemma 2.4. *Let T be a simple group such that $|T|$ has exactly three distinct prime divisors. Then the couple $(T, |T|)$ is listed in the following table.*

T	A_5	A_6	$\mathrm{PSL}(2,7)$	$\mathrm{PSL}(2,8)$
$\lvert T\rvert$	$2^2 \cdot 3 \cdot 5$	$2^3 \cdot 3^2 \cdot 5$	$2^3 \cdot 3 \cdot 7$	$2^3 \cdot 3^2 \cdot 7$
T	$\mathrm{PSL}(2,17)$	$\mathrm{PSL}(3,3)$	$\mathrm{PSU}(3,3)$	$\mathrm{PSU}(4,2)$
$\lvert T\rvert$	$2^4 \cdot 3^2 \cdot 17$	$2^4 \cdot 3^3 \cdot 13$	$2^5 \cdot 3^3 \cdot 7$	$2^6 \cdot 3^4 \cdot 5$

In particular, if T is a $\{2, 3, 5\}$-simple group, then $T = \mathrm{A}_5, \mathrm{A}_6$ or $\mathrm{PSU}(4, 2)$.

The stabilizers of arc-transitive pentavalent graphs are known, which were first obtained by Weiss [24], and improved by [12, Theorem 1.1] and [28, Theorem 4.1].

Lemma 2.5. *Let Γ be a pentavalent (G, s)-transitive graph, where $G \leq \mathrm{Aut}\Gamma$ and $s \geq 1$. Then $|G_\alpha|$ divides $2^9 \cdot 3^2 \cdot 5$ with $\alpha \in V\Gamma$, and one of the following statements holds.*

(a) *If G_α is soluble, then $s \leq 3$ and the couple (s, G_α) is listed in the following table.*

s	1	2	3
G_α	$\mathbb{Z}_5$, D_{10}, D_{20}	F_{20}, $\mathrm{F}_{20} \times \mathbb{Z}_2$	$\mathrm{F}_{20} \times \mathbb{Z}_4$

(b) *If G_α is insoluble, then $2 \leq s \leq 5$, and the couple (s, G_α) is listed in the following table.*

s	2	3	4	5
G_α	$\mathrm{A}_5, \mathrm{S}_5$	$\mathrm{A}_4 \times \mathrm{A}_5, \mathrm{S}_4 \times \mathrm{S}_5$	$\mathbb{Z}_2^4{:}\mathrm{GL}(2,4)$	$\mathbb{Z}_2^6{:}\mathrm{GL}(2,4)$
		$(\mathrm{A}_4 \times \mathrm{A}_5).\mathbb{Z}_2$	$\mathbb{Z}_2^4{:}\Gamma\mathrm{L}(2,4)$	$\mathbb{Z}_2^6{:}\Gamma\mathrm{L}(2,4)$

For a graph Γ, let $\Gamma(\alpha) = \{\beta \in V\Gamma \mid \beta \textit{ is adjacent to } \alpha\}$, the neighbor set of a vertex α in Γ. Then Γ is called *G-locally-primitive* with $G \leq \mathrm{Aut}\Gamma$ if the vertex stabilizer $G_\alpha{:} = \{g \in G \mid \alpha^g = \alpha\}$ acts primitively on $\Gamma(\alpha)$ for each vertex α. Obviously, an arc-transitive graph with prime valency is locally-primitive.

A permutation group $G \leq \mathrm{Sym}(\Omega)$ is called *quasiprimitive* if each non-trivial normal subgroup of G is transitive, while G is called *bi-quasiprimitive* if each non-trivial normal subgroup of G has at most two orbits and there exists one which has two orbits on Ω. We claim that a bipartite graph cannot admit a vertex-quasiprimitive arc-transitive automorphism group G, for if not, the stabilizer of G on the biparts is normal in G with index 2 and has two orbits on the vertex set, which is a contradiction. Similarly, a non-bipartite graph cannot admit a vertex bi-quasiprimitive arc-transitive automorphism group.

The following theorem provides a reduction method for the studying of locally-primitive graphs (refer to [17, Lemma 2.4]), which slightly improves a remarkable result of Praeger [20, Theorem 4.1].

Theorem 2.6. *Let Γ be a G-vertex-transitive locally-primitive graph, and let $N \lhd G$ have at least three orbits on $V\Gamma$, where $G \leq \mathsf{Aut}\Gamma$. Then N is semiregular on $V\Gamma$, $G/N \leq \mathsf{Aut}\Gamma_N$, Γ_N is G/N-locally-primitive and Γ is a normal cover of Γ_N.*

In particular, each vertex-transitive locally-primitive graph is a normal cover of a vertex-quasiprimitive or vertex-biquasiprimitive locally-primitive graph.

Theorem 2.6 particularly says that a G-arc-transitive graph of prime valency is G-basic if and only if G is either quasiprimitive or bi-quasiprimitive on the vertex set of the graph.

The final lemma of this section collects certain characterizations of arc-transitive pentavalent graphs.

Lemma 2.7. (1) *The graphs I_{12} and $\mathsf{K}_{6,6} - 6\mathsf{K}_2$ are the only connected arc-transitive pentavalent graphs of order $4p$ with p a prime, see* [13, Proposition 3.2]*;*

(2) *Each connected arc-transitive pentavalent Cayley graph of an abelian group is isomorphic to K_6, $\mathsf{K}_{5,5}$, $\mathsf{K}_{6,6} - 6\mathsf{K}_2$, C_{16} or Q_5, see* [1, Theorem 1.1]*;*

(3) *The graph C_{16} is the unique connected arc-transitive pentavalent graph of order* 16*, see* [14, Theorem 1].

(4) *The graph $\mathcal{G}_{36}$ is the unique connected arc-transitive pentavalent graph of order* 36*, see* [13, Theorem 4.1].

3. Proofs of Theorem 1.1 and Corollary 1.2.

For a positive integer $m = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ with $p_1, p_2, \ldots, p_s$ distinct primes, we denote $m_{p_i} = p_i^{r_i}$, the p_i-part of m.

Lemma 3.1. *Let T be a nonabelian simple group. Suppose that $|T|$ divides $2^{11} \cdot 3^2 \cdot 5 \cdot p^n$, and $5p^n$ divides $|T|$, where p is an odd prime and $n \geq 2$. Then $p = 3$, and $T = \mathrm{A}_6$ or $\mathrm{PSU}(4,2)$.*

Proof. If $|T|$ has exactly three prime divisors, it follows directly from Lemma 2.4 that $p = 3$, and $T = \mathrm{A}_6$ or $\mathrm{PSU}(4,2)$. Thus, suppose that $|T|$ has four prime divisors in the following. Then $p > 5$, $|T|_5 = 5$ and $|T|_p = p^n$. We will prove that no example appears in this case, so the lemma is true.

First, by [11, P. 135-136], T is not a sporadic simple group. If $T = \mathrm{A}_m$ is an alternating group, since $|T|$ has four prime divisors, $7 \leq m \leq 10$. On the other hand, as $|T|_p = p^n$ with $p > 5$ and $n \geq 2$, we also have $m \geq 14$, which is a contradiction.

Now, assume that $T = X(q)$ is a simple group of Lie type, where X is one type of Lie groups, and $q = r^s$ is a prime power with r a prime and $s \geq 1$.

If $r = 5$, by [11, P. 135], 5^2 divides $|T|$ with the only exception $T = \mathrm{PSL}(2,5) \cong \mathrm{A}_5$, yielding a contradiction.

If $r = 3$, since $|T|_3$ divides 9, q^3 does not divide $|T|$, by [11, P. 135], the only possibility is $T = \mathrm{PSL}(2,q)$. Then, as $|T|_3 = q = 3^s$, we have $s = 1$ or 2. However, $\mathrm{PSL}(2,3)$ is not a simple group, and $|\mathrm{PSL}(2,9)| = |\mathrm{A}_6|$ has exactly three prime divisors, which is a contradiction.

Suppose $r = 2$. If $T = {}^2F_4(q)$, then 2^{12} divides $|{}^2F_4(2)|$, which is a contradiction as $|T|_2$ divides 2^{11}. If $T = \mathrm{Sz}(q)$, then $s \geq 3$ is odd, and as $|T|_2 = 2^{2s}$ divides 2^{11}, we have $s = 3$ or 5. However, $7 \cdot 13$ divides $|\mathrm{Sz}(2^3)|$, and 5^2 divides $|\mathrm{Sz}(2^5)|$, a contradiction occurs. For other Lie simple groups, since $3^3 \mid (2^6 - 1)(2^2 - 1)$ and $|T|_3$ divides 9, we have $(q^6 - 1)(q^2 - 1)$ does not divide $|T|$, then by [11, P. 135] and noting that $7 \cdot 31$ divides $|\mathrm{PSL}(5,2)|$, we conclude that the only possibilities are as following:

$T = \mathrm{PSL}(n, 2^s)$ with $n \leq 4$, $T = \mathrm{PSU}(n, 2^s)$ with $3 \leq n \leq 5$, or $T = \mathrm{PSp}(4, 2^s)$.

Since $q^2 - 1 = 2^{2s} - 1$ is an odd divisor of $|T|$, we have $(q^2 - 1) \mid 3^2 \cdot 5 \cdot p^n$. Since $(q-1, q+1) = 1$, at least one of $q-1$ and $q+1$ divides 45, implying $q = 2, 4, 8$ or 16. However, by checking the orders, all the corresponding simple groups do not satisfy the assumptions of Lemma 3.1, no example appears.

Suppose finally $r > 5$. By [11, P. 135], we always have $\frac{q(q^2-1)}{2}$ divides $|T|$. Since p is the unique prime divisor bigger than 5 of $|T|$, we have $r = p$, and so $\frac{q^2-1}{2}$ divides $2^{11} \cdot 3^2 \cdot 5$. If

$\frac{q-1}{2}$ is even, as $(\frac{q-1}{2}, \frac{q+1}{2}) = 1$, we obtain $\frac{q+1}{2} \mid 45$, it follows $q = 17, 29$ or 89. Further, as $7 \mid (29^2 - 1)$ and $11 \mid (89^2 - 1)$, $q = 29$ and 89 are not the cases. Hence $q = 17$. Similarly, if $\frac{q+1}{2}$ is even, we obtain $q = 7, 11$ or 31. Now, for each $q \in \{7, 11, 17, 31\}$, one easily checks that $G_2(q)$ and ${}^3D_4(q)$ give rise no example. Further, as $(q^2 - 1)(q^4 - 1)$ is a multiple of 3^4 or 5^2, $(q^2 - 1)(q^4 - 1)$ does not divide $|T|$. Then by [11, P. 135], we conclude that $T = \mathrm{PSL}(2, q), \mathrm{PSL}(3, q)$ or $\mathrm{PSU}(3, q)$. However, a simple computation shows that these groups for $q = 7, 11, 17$ and 31 cannot give rise to example. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that, on the contrary, Γ is a connected G-arc-transitive pentavalent graph of order $4p^n$ with the smallest order, where $G \leq \mathrm{Aut}\Gamma$, $p \geq 5$ is a prime and n is a positive integer. By Lemma 2.7(1), $n \geq 2$.

Let $\alpha \in V\Gamma$. By Lemma 2.5, $|G_\alpha|$ divides $2^9 \cdot 3^2 \cdot 5$, so $|G|$ divides $2^{11} \cdot 3^2 \cdot 5 \cdot p^n$. Let R be the soluble radical of G, that is, R is the largest soluble normal subgroup of G.

(i) Assume first $R = 1$. Let N be a minimal normal subgroup of G. Then $N = T^d$ with T nonabelian simple and $d \geq 1$, so $|N|$ does not divide $|V\Gamma| = 4p^n$. By Theorem 2.6, N has at most two orbits on $V\Gamma$, and so p^n divides $|N : N_\alpha|$. Moreover, since $|N : N_\alpha|$ divides $4p^n$, $N_\alpha \neq 1$, and since Γ is connected, we have $1 \neq N_\alpha^{\Gamma(\alpha)} \lhd G_\alpha^{\Gamma(\alpha)}$, so 5 divides $|N_\alpha|$ as $G_\alpha^{\Gamma(\alpha)}$ is transitive of degree 5. Hence, $5p^n$ divides $|N|$.

If $p = 5$, then T is a $\{2, 3, 5\}$-nonabelian simple group, by Lemma 3.1, $T = \mathrm{A}_5, \mathrm{A}_6$ or $\mathrm{PSU}(4, 2)$. Since 3^3 does not divide $|N|$, $T \neq \mathrm{PSU}(4, 2)$. If $T = \mathrm{A}_5$ or A_6, then $|T|_5 = 5$, and as 5^3 divides $|N| = |T|^d$, we have $d \geq 3$, and so 3^3 divides $|N|$, yielding a contradiction. Suppose $p > 5$. Then $(|T|^d)_5 = |N|_5 = 5$, so $d = 1$ and $N = T$. Noting that $|T|$ divides $2^{11} \cdot 3^2 \cdot 5 \cdot p^n$, and $5p^n$ divides $|T|$ with $n \geq 2$. By Lemma 3.1, we have $p = 3$, also yielding a contradiction.

(ii) Assume now $R \neq 1$. Let F be the Fitting subgroup of R. By Lemma 2.2, $1 \neq F \lhd G$ and $\mathsf{C}_R(F) \leq F$. Further, as $|V\Gamma| = 4p^n$, $F = \mathrm{O}_2(R) \times \mathrm{O}_p(R)$, where $\mathrm{O}_2(R)$ and $\mathrm{O}_p(R)$ denote the largest normal 2- and p-subgroups of R, respectively. Observing that both $\mathrm{O}_2(R)$ and $\mathrm{O}_p(R)$ have at least four orbits on $V\Gamma$, by Theorem 2.6, $\mathrm{O}_2(R)$ and $\mathrm{O}_p(R)$ are semiregular on $V\Gamma$, so $|\mathrm{O}_2(R)|$ divides 4, and $|\mathrm{O}_p(R)|$ divides p^n.

If $\mathrm{O}_p(R) \neq 1$, by Theorem 2.6, the normal quotient graph $\Gamma_{\mathrm{O}_p(R)}$ is a connected arc-transitive pentavalent graph of order $4p^m$ for some $m < n$, which contradicts the minimality of the order of Γ. So $F = \mathrm{O}_2(R)$. If $|F| = 4$, then Γ_F is arc-transitive of odd valency 5 and odd order p^n, which is not possible. Thus, $F \cong \mathbb{Z}_2$. By Lemma 2.2, $\mathsf{C}_R(F) = F$, then by Lemma 2.1, $R/F \leq \mathsf{Aut}(F) = 1$, implying $R = F \cong \mathbb{Z}_2$.

Now, Γ_R is a connected G/R-arc-transitive pentavalent graph. Since the soluble radical of G/R is trivial, $|V\Gamma_R| = 2p^n$ and $n \geq 2$, with almost the same argument as in part (i) above, one may also draw a contradiction. □

Proof of Corollary 1.2. Let Γ be a connected G-arc-transitive pentavalent graph of order $4p^2$ or $4p^3$, where $G \leq \mathsf{Aut}\Gamma$. By Theorem 1.1, we have $p = 2$ or 3.

Suppose $|V\Gamma| = 4p^2$. Then $|V\Gamma| = 16$ or 36, by Lemma 2.7(3-4), $\Gamma \cong \mathsf{C}_{16}$ or $\mathcal{G}_{36}$.

Suppose $|V\Gamma| = 4p^3$. Then $|V\Gamma| = 32$ or 108. By Magma [2], there is a unique connected arc-transitive pentavalent graph of order 32, and there is no connected arc-transitive pentavalent graph of order 108, so $\Gamma \cong \mathsf{Q}_5$. □

4. Proof of Theorem 1.3.

We first give a simple observation.

Lemma 4.1. *Let Γ be a connected G-arc-transitive graph of prime valency, where $G \leq \mathsf{Aut}\Gamma$. Suppose that G has a normal subgroup N which is not semiregular on $V\Gamma$. Then N_α is transitive on $\Gamma(\alpha)$ for $\alpha \in V\Gamma$, and either*

(i) *N is transitive on $V\Gamma$, and Γ is N-arc-transitive; or*
(ii) *N has two orbits on $V\Gamma$ and Γ is N-edge-transitive.*

Proof. Since Γ is G-arc-transitive of prime valency, $G_\alpha^{\Gamma(\alpha)}$ is primitive. Since $1 \neq N_\alpha \lhd G_\alpha$ and Γ is connected, we have $1 \neq N_\alpha^{\Gamma(\alpha)} \lhd G_\alpha^{\Gamma(\alpha)}$, it follows that $N_\alpha^{\Gamma(\alpha)}$ is transitive. Then, as $N \lhd G$ is not semiregular on $V\Gamma$, by Theorem 2.6, N has at most two orbits on $V\Gamma$, the lemma follows. □

A graph Γ is called a *Cayley graph* of a group H if there is a subset $S \subseteq H \setminus \{1\}$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$, such that $V\Gamma = H$ and x is adjacent to y if and only if $yx^{-1} \in S$. This Cayley graph is denoted by $\mathsf{Cay}(H, S)$. It is well known that a graph is isomorphic to a Cayley graph of a group H if and only if it has an automorphism group which is isomorphic to H and regular on its vertex set. For a G-arc-transitive graph Γ with $G \leq \mathsf{Aut}\Gamma$, if G contains a normal subgroup H which is regular on $V\Gamma$, then Γ is called a *G-normal arc-transitive Cayley graph* of H, see [16] for some nice properties of normal arc-transitive Cayley graphs.

Lemma 4.2. *Let Γ be a G-edge-transitive graph with $G \leq \mathsf{Aut}\Gamma$, and suppose that G has an abelian normal subgroup H which acts semiregularly and has two orbits on $V\Gamma$. Then Γ is a Cayley graph of a group $\langle H, \sigma\rangle$, where σ is an involution such that $h^\sigma = h^{-1}$ for each $h \in H$.*

In particular, if $H \cong \mathbb{Z}_2^m$ is an elementary abelian 2-group, then Γ is an abelian Cayley graph of $\mathbb{Z}_2^{m+1}$.

Proof. Let $H = \{h_1, h_2, \ldots, h_n\}$, and let Δ_1 and Δ_2 be the two orbits of H on $V\Gamma$. Then $\Delta_1 = \{u^{h_i} \mid 1 \leq i \leq n\}$ and $\Delta_2 = \{v^{h_i} \mid 1 \leq i \leq n\}$ for $u \in \Delta_1$ and $v \in \Delta_2$. Since Γ is G-edge-transitive and $H \lhd G$, it is easy to show that there is no edge in each of Δ_1 and Δ_2. Suppose $\mathsf{val}(\Gamma) = k$. Then $k \leq n$ and without loss of generality, we may assume $\Gamma(u) = \{v^{h_i} \mid 1 \leq i \leq k\}$. Then $\Gamma(v) = \{u^{h_i^{-1}} \mid 1 \leq i \leq k\}$. Define

$$\sigma :\ u^{h_i} \to v^{h_i^{-1}},\ v^{h_i} \to u^{h_i^{-1}}\ \mathit{for}\ 1 \leq i \leq n.$$

Clearly, σ is a permutation on $V\Gamma$ with order 2. Since N is abelian, it is routine to check that σ is an automorphism of Γ. Further, for each $h, h_i \in H$, we have $(u^{h_i})^{\sigma h \sigma} = (v^{h_i^{-1}})^{h\sigma} = (v^{h_i^{-1}h})^{\sigma} = u^{h^{-1}h_i} = (u^{h_i})^{h^{-1}}$, and similarly, $(v^{h_i})^{\sigma h \sigma} = (v^{h_i})^{h^{-1}}$. Hence, $h^{\sigma} = \sigma h \sigma = h^{-1}$, and $\langle H, \sigma \rangle \cong H : \mathbb{Z}_2$. Since $\langle H, \sigma \rangle$ is regular on $V\Gamma$, Γ is a Cayley graph of $\langle H, \sigma \rangle$.

The final statement of the lemma is now obviously true. □

For a group G, denote by $\mathsf{soc}(G)$ the *socle* of G, that is, the product of all minimal normal subgroups of G.

Proof of Theorem 1.3. We first prove that, for each triple (Γ, G, G_α) listed in Table 1 of Theorem 1.3, Γ is really a connected G-basic arc-transitive pentavalent graph. Noting that, by Lemma 2.7, all the graphs Γ in Table 1 are connected arc-transitive pentavalent graphs, so $(\mathsf{Aut}\Gamma)_\alpha$ is transitive on $\Gamma(\alpha)$. Also, it is easy to check that, for each graph in Table 1, the corresponding candidates of (G, G_α) exist.

Suppose first $(\Gamma, G, G_\alpha) = (\mathsf{I}_{12}, \mathrm{A}_5, \mathbb{Z}_5)$. Since $|G : G_\alpha| = 12 = |V(\mathsf{I}_{12})|$, G is transitive on $V(\mathsf{I}_{12})$, and as $G \lhd \mathsf{Aut}(\mathsf{I}_{12})$, Lemma 4.1 implies that I_{12} is G-arc-transitive. Clearly, G is quasiprimitive on $V(\mathsf{I}_{12})$. So I_{12} is G-basic arc-transitive.

Suppose $\Gamma = \mathsf{K}_{6,6} - 6\mathsf{K}_2$. If $(G, G_\alpha) = (\mathrm{S}_5, \mathrm{D}_{10})$, then G is transitive on $V\Gamma$, and as G_α contains a Sylow 5-subgroup of $(\mathsf{Aut}\Gamma)_\alpha \cong \mathrm{S}_5$, G_α is transitive on $\Gamma(\alpha)$ by [25, Theorem 3.4], so Γ is G-arc-transitive. Since $G = \mathrm{S}_5$ has a unique conjugate class isomorphic to D_{10}, we have $G_\alpha = \mathrm{D}_{10}$ is contained in $\mathsf{soc}(G) = \mathrm{A}_5$, so $(\mathsf{soc}(G))_\alpha = G_\alpha = \mathrm{D}_{10}$ and $|\mathsf{soc}(G) : (\mathsf{soc}(G))_\alpha| = 6$, that is, $\mathsf{soc}(G)$ has two orbits on $V\Gamma$. Hence, G is bi-quasiprimitive on $V\Gamma$, and Γ is S_5-basic arc-transitive. For the case where $(G, G_\alpha) = (\mathrm{S}_6, \mathrm{A}_5)$, noting that both G and $\mathsf{soc}(G) = \mathrm{A}_6$ have two conjugate classes isomorphic to A_5, we also have $G_\alpha \cong \mathrm{A}_5$ is contained in $\mathsf{soc}(G)$, then with similar discussion, one may show that Γ is S_6-basic arc-transitive.

Let $(\Gamma, G, G_\alpha) = (\mathsf{C}_{16}, \mathbb{Z}_2^4 : \mathbb{Z}_5, \mathbb{Z}_5)$. Then $\mathsf{Aut}\Gamma \cong \mathbb{Z}_2^4 : \mathrm{S}_5$ and $(\mathsf{Aut}(G))_\alpha \cong \mathrm{S}_5$. Since $|G : G_\alpha| = 16 = |V\Gamma|$, G is transitive on $V\Gamma$. Since $(\mathsf{Aut}\Gamma)_\alpha$ is transitive of degree 5, and $G_\alpha \cong \mathbb{Z}_5$

is a Sylow 5-subgroup of $(\mathsf{Aut}\Gamma)_\alpha$, by [25, Theorem 3.4], G_α is transitive on $\Gamma(\alpha)$, so Γ is G-arc-transitive. We now claim: G is primitive on $V\Gamma$. If the claim is not true, then G_α is not maximal in G, and so G_α normalizes a nontrivial subgroup of N, where $N \cong \mathbb{Z}_2^4$ is the normal Sylow 2-subgroup of G. By Theorem 2.3, we may assume $N = N_1 \times N_2$, where $N_1 \cong \mathbb{Z}_2^l$ and $N_2 \cong \mathbb{Z}_2^{4-l}$ with $1 \leq l \leq 3$, such that G_α normalizes both N_1 and N_2. Since $\mathsf{Aut}(\mathbb{Z}_2^i)$ with $i = 1, 2, 3$ are of order coprime to 5, by Lemma 2.1, $G_\alpha \cong \mathbb{Z}_5$ centralizes both N_1 and N_2. It follows that G is abelian, which contradicts that G is arc-transitive on Γ. Hence G is primitive on $V\Gamma$, and C_{16} is $(\mathbb{Z}_2^4 : \mathbb{Z}_5)$-basic arc-transitive. For the other three candidates of (G, G_α) in row 3 of Table 1, G is always an overgroup of the group $\mathbb{Z}_2^4 : \mathbb{Z}_5$ above, so G is also primitive on $V\Gamma$ and arc-transitive on Γ, that is, Γ is G-basic arc-transitive.

Let (Γ, G, G_α) satisfy row 4 of Table 1. Then, one easily checks that for all the candidates of (G, G_α) there, $\mathsf{soc}(G) = \mathsf{soc}(\mathsf{Aut}\Gamma) \cong \mathrm{A}_6$ is transitive on $V\Gamma$, so G is always quasiprimitive on Γ, and since $\mathsf{Aut}\Gamma$ is arc-transitive on Γ, by Lemma 4.1, Γ is A_6-arc-transitive. Hence, $\mathcal{G}_{36}$ is G-basic arc-transitive.

Now, assume that Γ is a connected G-basic arc-transitive pentavalent graph of order $4p^n$. By Theorem 1.1, $p = 2$ or 3, and by Theorem 2.6, G is quasiprimitive or bi-quasiprimitive on $V\Gamma$. Let N be a minimal normal subgroup of G. Then $N = T^d$ has at most two orbits on $V\Gamma$, where T is a simple group and $d \geq 1$. By Lemma 2.5, $|G_\alpha|$ divides $2^9 \cdot 3^2 \cdot 5$, so either $|G|$ divides $2^{n+11} \cdot 3^2 \cdot 5$ if $p = 2$, or $|G|$ divides $2^{11} \cdot 3^{n+2} \cdot 5$ if $p = 3$. In particular, 5 divides $|G|$ but 25 does not divide $|G|$.

Case 1. Assume $p = 2$.

Then $|V\Gamma| = 2^{n+2}$. Suppose N is nonabelian. Then T is a nonabelian simple group. Since $|N| = |T|^d$ divides $2^{n+11} \cdot 3^2 \cdot 5$, we have $d = 1$ and $N = T$ is a $\{2, 3, 5\}$-simple group. By Lemma 2.4, $N \cong \mathrm{A}_5, \mathrm{A}_6$ or $\mathrm{PSU}(4, 2)$. However, by [5], each of $\mathrm{A}_5, \mathrm{A}_6$ and $\mathrm{PSU}(4, 2)$ has no subgroup with index a 2-power, which is a contradiction.

Thus, N is abelian. Since $|V\Gamma| = 2^{n+2}$, we have $T \cong \mathbb{Z}_2$ and $N \cong \mathbb{Z}_2^d$. If $N_\alpha \neq 1$, by Lemma 4.1, N_α is transitive on $\Gamma(\alpha)$, so 5 divides $|N_\alpha|$, which is a contradiction. Hence $N_\alpha = 1$, and either $N \cong \mathbb{Z}_2^{n+2}$ if N is transitive on $V\Gamma$, or $N \cong \mathbb{Z}_2^{n+1}$ if N has exactly two orbits on $V\Gamma$.

Assume $N \cong \mathbb{Z}_2^{n+1}$ has exactly two orbits on $V\Gamma$. Then Γ is a bipartite graph, and by Lemma 4.2, Γ is a Cayley graph of an abelian group $N \times \mathbb{Z}_2 \cong \mathbb{Z}_2^{n+2}$. Since $|V\Gamma|$ is a 2-power and C_{16} is not a bipartite graph, by Lemma 2.7(2), the only possibility is $\Gamma \cong \mathrm{Q}_5$. Then $\mathsf{Aut}\Gamma = M : (\mathsf{Aut}\Gamma)_\alpha \cong \mathbb{Z}_2 \wr \mathrm{S}_5$, where $M \cong \mathbb{Z}_2^5$ is a normal subgroup of $\mathsf{Aut}\Gamma$. By the wreath action of S_5 on M, S_5 centralizes an involution $a \in M$. Since $N \cong \mathbb{Z}_2^4$ is normal in G and semiregular on $V\Gamma$, $G = N : H$, where $G_\alpha \subset H \cong G_\alpha.2$. If $M \cap N = 1$, then $\mathbb{Z}_2^4 \cong N \cong MN/M \leq \mathsf{Aut}\Gamma/M \cong \mathrm{S}_5$, which is a contradiction. Thus, $1 \neq M \cap N \lhd G$, and hence $M \cap N = N$ as N is a minimal normal subgroup of G. If $a \in G$, then $\langle a \rangle \cong \mathbb{Z}_2$ is normal in G and has 16 orbits on $V\Gamma$, which contradicts that G is bi-quasiprimitive on $V\Gamma$. Thus, $a \notin G$, $M = \langle N, a \rangle = N \times \langle a \rangle$ and $G \cap M = N$. Now, as $G_\alpha.2 \cong H \cong G/N \cong GM/M \leq \mathsf{Aut}\Gamma/M \cong \mathrm{S}_5$, and 5 divides $|G_\alpha|$, we conclude that $H \cong \mathrm{D}_{10}, \mathbb{Z}_5 : \mathbb{Z}_4$ or S_5. Suppose $H = \langle G_\alpha, b \rangle$ for some $b \in G$. Then we may write $b = a^i xy$, where $i = 1$ or 2, $x \in N$ and $y \in (\mathsf{Aut}\Gamma)_\alpha$. If $i = 2$, then as $x \in N \subseteq G$, we have $y \in G \cap (\mathsf{Aut}\Gamma)_\alpha = G_\alpha$, it follows $G = \langle N, H \rangle = \langle N, G_\alpha, xy \rangle = N : G_\alpha$, which is a contradiction. Thus, $i = 1$. Then we have $G = \langle N, H \rangle = \langle N, G_\alpha, a \rangle$, so $a \in G$, also yielding a contradiction.

Therefore, $N \cong \mathbb{Z}_2^{n+2}$ is transitive on $V\Gamma$, so G is quasiprimitive on $V\Gamma$ and Γ is a non-bipartite graph. By Lemma 2.7(2), $\Gamma \cong \mathrm{C}_{16}$. Then $N \cong \mathbb{Z}_2^4$, and $G = N : G_\alpha \leq \mathsf{Aut}(\mathrm{C}_{16}) = N : \mathrm{S}_5$, implying $G_\alpha \leq \mathrm{S}_5$. Since G_α is transitive on $\Gamma(\alpha)$, we have 5 divides $|G_\alpha|$, it follows that $G_\alpha = \mathbb{Z}_5, \mathrm{D}_{10}, \mathrm{A}_5$ or S_5, as in row 3 of Table 1 of Theorem 1.3.

Case 2. Assume $p = 3$.

Since G is quasiprimitive or bi-quasiprimitive on $V\Gamma$, $|N : N_\alpha| = 4 \cdot 3^n$ or $2 \cdot 3^n$, so T is a nonabelian simple group. Since

$|G|$ divides $2^{11} \cdot 3^{n+2} \cdot 5$, so is $|N| = |T|^d$, hence $d = 1$, and by Lemma 2.4, $N \cong \mathrm{A}_5, \mathrm{A}_6$ or $\mathrm{PSU}(4,2)$. If G has another minimal normal subgroup M, then we also have $M \cong \mathrm{A}_5, \mathrm{A}_6$ or $\mathrm{PSU}(4,2)$, so 25 divides $|M \times N|$, which is a contradiction. Thus, $N = T$ is the unique minimal normal subgroup of G, and G is almost simple. Hence $G = T.o$ with $o \le \mathsf{Out}(T)$, the outer automorphism group of T.

Suppose first that G is quasiprimitive on $V\Gamma$. Then Γ is a non-bipartite graph, T is transitive on $V\Gamma$, and $|T : T_\alpha| = 4 \cdot 3^n$. Since $T \cong \mathrm{A}_5, \mathrm{A}_6$ or $\mathrm{PSU}(4,2)$, by [5], all the possibilities of (T, T_α) are as following:

$$(T, T_\alpha) = (\mathrm{A}_5, \mathbb{Z}_5), (\mathrm{A}_6, \mathrm{D}_{10}), (\mathrm{PSU}(4,2), \mathbb{Z}_2^4{:}\mathbb{Z}_5), (\mathrm{PSU}(4,2), \mathrm{S}_6).$$

Assume $(T, T_\alpha) = (\mathrm{A}_5, \mathbb{Z}_5)$. Then $|V\Gamma| = 12$, and since Γ is non-bipartite, by Lemma 2.7(1), we have $\Gamma \cong \mathsf{I}_{12}$, and $G \le \mathsf{Aut}(\mathsf{I}_{12}) = \mathrm{A}_5 \times \mathbb{Z}_2$. Since $G \le \mathsf{Aut}(\mathrm{A}_5) = \mathrm{S}_5$, we further conclude that $(G, G_\alpha) = (\mathrm{A}_5, \mathbb{Z}_5)$, as in row 1 of Table 1 of Theorem 1.3.

Assume $(T, T_\alpha) = (\mathrm{A}_6, \mathrm{D}_{10})$. Then $|V\Gamma| = 36$. By Lemma 2.7(4), $\Gamma \cong \mathcal{G}_{36}$. Since $\mathsf{Out}(\mathrm{A}_6) \cong \mathbb{Z}_2^2$, by [5], we conclude that $G \cong \mathrm{A}_6, \mathrm{S}_6, \mathrm{PGL}(2,9), \mathrm{M}_{10}$ or $\mathsf{Aut}(\mathrm{A}_6)$, and $G_\alpha \cong \mathrm{D}_{10}, \mathsf{F}_{20}, \mathrm{D}_{20}, \mathsf{F}_{20}$ or $\mathsf{F}_{20} \times \mathbb{Z}_2$ respectively, as in row 4 of Table 1 of Theorem 1.3.

Assume $(T, T_\alpha) = (\mathrm{PSU}(4,2), \mathrm{S}_6)$. Then $G_\alpha \ge \mathrm{S}_6$, which is not possible by Lemma 2.5(b). Similarly, for the case $(T, T_\alpha) = (\mathrm{PSU}(4,2), \mathbb{Z}_2^4 : \mathbb{Z}_5)$, as $\mathsf{Out}(\mathrm{PSU}(4,2)) = \mathbb{Z}_2$, we have $G \le \mathrm{PSU}(4,2).\mathbb{Z}_2$, and $\mathbb{Z}_2^4 : \mathbb{Z}_5 \le G_\alpha \le \mathbb{Z}_2^4 : \mathbb{Z}_5.\mathbb{Z}_2$, which is impossible by Lemma 2.5(a).

Suppose now G is bi-quasiprimitive on $V\Gamma$. Then $T < G$, Γ is a bipartite graph, and $|T : T_\alpha| = 2 \cdot 3^n$. By [5], we have

$$(T, T_\alpha) = (\mathrm{A}_5, \mathrm{D}_{10}), (\mathrm{A}_6, \mathrm{A}_5) \text{ or } (\mathrm{PSU}(4,2), \mathbb{Z}_2^4 : \mathrm{D}_{10}).$$

For the former two cases, $|V\Gamma| = 12$, by Lemma 2.7(1), $\Gamma \cong \mathsf{K}_{6,6} - 6\mathsf{K}_2$. If $(T, T_\alpha) = (\mathrm{A}_5, \mathrm{D}_{10})$, since $\mathsf{Out}(\mathrm{A}_5) = \mathbb{Z}_2$, we have $G = \mathrm{S}_5$, and $G_\alpha = T_\alpha = \mathrm{D}_{10}$; if $(T, T_\alpha) = (\mathrm{A}_6, \mathrm{A}_5)$, then $\mathrm{A}_6 < G \le \mathsf{Aut}\Gamma = \mathrm{S}_6 \times \mathbb{Z}_2$ and $G \le \mathsf{Aut}(T) = \mathrm{A}_6.\mathbb{Z}_2^2$, we conclude $G = S_6$, and $G_\alpha = T_\alpha = A_5$, as in row 2 of Table 1 of Theorem 1.3.

For the last case, as $\mathrm{Out}(\mathrm{PSU}(4,2)) = \mathbb{Z}_2$, we have $G = T.2$ and $G_\alpha = T_\alpha = \mathbb{Z}_2^4 : D_{10}$, which is not possible by Lemma 2.5(a). □

REFERENCES

[1] M. Alaeiyan, A. A. Talebi, K. Paryab, Arc-transitive Cayley graphs of valency five on abelian groups, *SEAMS Bull. Math.* **32** (2008), 1029-1035.

[2] W. Bosma, C. Cannon, C. Playoust, The Magma algebra system I: The user languages, *J. Symbolic Comput.* **24** (2007), 235-265.

[3] C. Y. Chao, On the classification of symmetric graphs with a prime number of vertices, *Trans. Amer. Math. Soc.* **158** (1971), 247-256.

[4] Y. Cheng, J. Oxley, On weakly symmetric graphs of order twice a prime, *J. Combin. Theory Ser. B* **42** (1987), 196-211.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of Finite Groups, Oxford Univ. Press, London/New York, 1985.

[6] R. M. Darafshen, A. Assari, Normal edge-transitive Cayley graphs on non-abelian groups of order $4p$, where p is a prime number, *Sci. China Ser. A* **56** (2013), 213-219.

[7] Y. Q. Feng, K. Kutnar, D. Marušič, C. Zhang, Tetravalent one-regular graphs of order $4p^2$, *Filomat* **28** (2014), 285-303.

[8] Y. Q. Feng, J. H. Kwak, Cubic symmetric graphs of order a small number times a prime or a prime square, *J. Combin. Theory Ser. B* **97** (2007), 627-646.

[9] Y. Q. Feng, K. S. Wang, C. X. Zhou, Tetravalent half-transitive graphs of order $4p$, *Europ. J. Combin.* **28** (2007), 726-733.

[10] M. Ghasemi, J. X. Zhou, Tetravalent s-transitive graphs of order $4p^2$, *Graphs Combin.* **29** (2013), 87-97.

[11] D. Gorenstein, Finite Simple Groups, Plenum Press, New York, 1982.

[12] S. T. Guo, Y. Q. Feng, A note on pentavalent s-transitive graphs, *Discrete Math.* **312** (2012), 2214-2216.

[13] S. T. Guo, J. X. Zhou, Y. Q. Feng, Pentavalent symmetric graphs of order $12p$, *Electronic J. Combin.* **18** (2011), #P233.

[14] X. H. Hua, Y. Q. Feng, Pentavalent symmetric graphs of order $8p$, *J. Beijing Jiaotong University* **35** (2011), 132-135.

[15] B. Huppert, Eudiche Gruppen I, Springer-Verlag, Berlin, 1967.

[16] C. H. Li, Finite edge-transitive Cayley graphs and rotary Cayley maps, *Trans. Amer. Math. Soc.* **358** (2006), 4605-4635.

[17] C. H. Li, J. M. Pan, Finite 2-arc-transitive abelian Cayley graphs, *Europ. J. Combin.* **29** (2007), 148-158.
[18] K. Kutnar, D. Marušič, P. Šparl, R. J. Wang, M. Y. Xu, Classification of half-arc-transitive graphs of order $4p$, *Europ. J. Combin.* **34** (2013), 1158-1176.
[19] J. M. Pan, Locally primitive Cayley graphs of dihedral groups, *Europ. J. Combin.* **36** (2014), 39-52.
[20] C. E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc-transitive graphs, *J. London Math. Soc.* **47** (1992), 227-239.
[21] M. Suzuki, Group Theroy II, Springer-Verlag, New York, 1985.
[22] X. Y. Wang, Y. Q. Feng, Hexavalent half-arc-transitive graphs of order $4p$, *Europ. J. Combin.* **30** (2009), 1263-1270.
[23] P. Webb, Finite group reprentations, Plenum Press, New York, 1980.
[24] R. M. Weiss, s-arc-transitive graphs, in: Algebraic Methods in Graph Theory, **2** (1981), 827-847.
[25] H. Wielandt, Finite permutation groups, Academic Press, New York and London, 1964.
[26] M. Y. Xu, Q. H. Zhang, J. X. Zhou, Arc-transitive cubic graphs of order $4p$, *Sci. Ann. Math.* **25** (2004), 545-554.
[27] J. X. Zhou, Y. T. Li, Cubic non-normal Cayley graphs of order $4p^2$, *Discrete Math.* **312** (2012), 1940-1946.
[28] J. X. Zhou, Y. Q. Feng, On symmetric graphs of valency five, *Discrete Math.* **310** (2010), 1725-1732.

J. M. PAN, SCHOOL OF STATISTICS AND MATHEMATICS, YUNNAN UNIVERSITY OF FINANCE AND ECONOMICS, KUNMING, P. R. CHINA
E-mail address: jmpan@ynu.edu.cn

Z. H. HUANG, SCHOOL OF MATHEMATICS AND STATISTICS, YUNNAN UNIVERSITY, KUNMING, P. R. CHINA

C. H. LI, SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF WESTERN AUSTRALIA, CRAWLEY 6009 WA, AUSTRALIA