The generalized characteristic polynomial of the subdivision-vertex and subdivision-edge coronae*

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Abstract

The subdivision graph $\mathcal{S}(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. Let G_1 and G_2 be two vertex disjoint graphs. The *subdivision-vertex corona* of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex-disjoint, by joining the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2. The *subdivision-edge corona* of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex-disjoint, by joining the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2, where $I(G_1)$ is the set of inserted vertices of $\mathcal{S}(G_1)$. In this paper we determine the generalized characteristic polynomial of $G_1 \odot G_2$ (respectively, $G_1 \ominus G_2$). As applications, the results on the spectra of $G_1 \odot G_2$ (respectively, $G_1 \ominus G_2$) enable us to construct infinitely many pairs of Φ-cospectral graphs. The adjacency spectra of $G_1 \odot G_2$ (respectively, $G_1 \ominus G_2$) help us to construct many infinite families of integral graphs. By using the Laplacian spectra, we also obtain the number of spanning trees and Kirchhoff index of $G_1 \odot G_2$ and $G_1 \ominus G_2$, respectively.

Keywords: Spectrum, Cospectral graphs, Integral graphs, Spanning trees, Kirchhoff index, Subdivision-vertex corona, Subdivision-edge corona

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1 Introduction

Throughout this paper, we consider simple graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. Let $d_G(v_i)$ be the degree of vertex v_i in G. Denote $D(G)$ to be the diagonal matrix with diagonal entries $d_G(v_1), \ldots, d_G(v_n)$. The *Laplacian matrix* of G and the *signless Laplacian matrix* of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively. Let $\phi(A(G); x) = \det(xI_n - A(G))$, or simply $\phi(A(G))$, be the *adjacency characteristic polynomial* of G. Similarly, $\phi(L(G))$ (respectively, $\phi(Q(G))$) denotes the *Laplacian* (respectively, *signless Laplacian*) *characteristic polynomial* of G. Denote the eigenvalues of $A(G), L(G)$ and $Q(G)$ by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, $\nu_1(G) \leq \nu_2(G) \leq \cdots \leq \nu_n(G)$, respectively. The eigenvalues (together with the multiplicities) of $A(G), L(G)$ and $Q(G)$ are called the *A-spectrum*, *L-spectrum* and *Q-spectrum* of G, respectively. Graphs with the same A-spectra (respectively, L-spectra, Q-spectra) are called *A-cospectral* (respectively, *L-cospectral*, *Q-cospectral*) graphs. Undefined terminology is consistent with [5,6].

Consider a graph G with adjacency matrix $A(G)$ and diagonal matrix $D(G)$. In [5], Cvetković et al. introduced a bivariate polynomial, denoted by $\Phi_G(x,t) = \det(xI_n - (A(G) - tD(G)))$ (or Φ_G or simply Φ if no confusion arises), which were defined as the generalized characteristic polynomial of G in [14]. The polynomial $\Phi_G(x,t)$ generalizes some well known characteristic polynomials of graph G, e.g. the characteristic polynomials of $A(G)$, $L(G)$ and $Q(G)$ are equal to $\Phi_G(x, 0)$, $(-1)^{|V(G)|}\Phi_G(-x, 1)$ and $\Phi_G(x, -1)$, respectively. Two graphs G and H are called Φ-cospectral if $\Phi_G(x,t) = \Phi_H(x,t)$. If G and H are called Φ-cospectral, then they are A-cospectral, L-cospectral and Q-cospectral.

The *corona* of two graphs was first introduced by R. FRUCHT and F. HARARY in [8] with the goal of constructing a graph whose automorphism group is the wreath product of the two component automorphism groups. Then, I. Gutman [11] and V. R. Rosenfeld [21] studied rooted product of some graphs. It is known that the A-spectrum (respectively, L-spectrum, Q-spectrum) of the corona of any two graphs can be expressed by that of the two factor graphs [1,4,20,22]. Similarly, the A-spectrum (respectively, L-spectrum, Q-spectrum) of the *edge corona* [13] of two graphs, which is a variant of the corona operation, was completely computed in [4,13,22]. Another variant of the corona operation, the *neighbourhood corona*, was introduced in [9] recently. The A-spectrum (respectively, L-spectrum, Q-spectrum) of such operation was investigated in [9,19].

The *subdivision graph* $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G [6]. We denote the set of such new vertices by $I(G)$. In [15], two new graph operations based on subdivision graphs: *subdivision-vertex join* and *subdivision-edge join* were introduced, and the A-spectrum of subdivision-vertex join (respectively, subdivision-edge join) of two regular graphs were computed in terms of that of the two graphs. More work on their L-spectra and Q-spectra were presented in [17]. In [18], the authors defined two new graph operations based on subdivision graphs: *subdivision-vertex neighbourhood corona* and *subdivision-edge neighbourhood corona*. They determined the A-spectrum, the L-spectrum and the Q-spectrum of the subdivision-vertex neighbourhood corona (respectively, subdivision-edge neighbourhood corona) of a regular graph and an arbitrary graph.

Motivated by the work above, we define two new graph operations based on subdivision graphs as follows.

Definition 1.1. The *subdivision-vertex corona* of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex-disjoint, by joining the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2.

Definition 1.2. The *subdivision-edge corona* of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $S(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex-disjoint, by joining the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2.

Let P_n denote a path on n vertices. Figure 1 depicts the subdivision-vertex corona $P_4 \odot P_2$ and subdivision-edge corona $P_4 \ominus P_2$, respectively. Note that if G_1 is a graph on n_1 vertices and m_1 edges and G_2 is a graph on n_2 vertices and m_2 edges, then the subdivision-vertex corona $G_1 \odot G_2$ has $n_1(1+n_2)+m_1$ vertices and $2m_1+n_1(n_2+m_2)$ edges, and the subdivision-edge corona $G_1 \ominus G_2$ has $m_1(1+n_2)+n_1$ vertices and $m_1(2+n_2+m_2)$ edges.

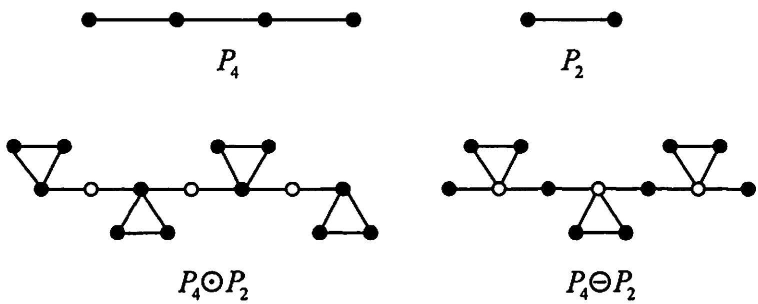

Fig. 1: An example of subdivision-vertex and subdivision-edge coronae.

In this paper, we will determine the A-spectra, the L-spectra and the Q-spectra of $G_1 \odot G_2$ (respectively, $G_1 \ominus G_2$) with the help of the *coronal* of a matrix and the *Kronecker product.* The M*-coronal* $\Gamma_M(x)$ of an $n \times n$ matrix M is defined [4, 20] to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is, $\Gamma_M(x) = \mathbf{1}_n^T(xI_n - M)^{-1}\mathbf{1}_n$, where $\mathbf{1}_n$ denotes the column vector of size n with all the entries equal one. It is well known [4, Proposition 2] that, if M is an $n \times n$ matrix with each row sum equal to a constant t, then

$$\Gamma_M(x) = \frac{n}{x - t}. \tag{1.1}$$

In particular, since for any graph G_2 with n_2 vertices, each row sum of $L(G_2)$ is equal to 0, we have

$$\Gamma_{L(G_2)}(x) = n_2/x. \tag{1.2}$$

The *Kronecker product* $A \otimes B$ of two matrices $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This is an associative operation with the property that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ whenever the products AC and BD exist. The latter implies $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ for nonsingular matrices A and B. Moreover, if A and B are $n \times n$ and $p \times p$ matrices, then $\det(A \otimes B) = (\det A)^p (\det B)^n$. The reader is referred to [12] for other properties of the Kronecker product not mentioned here.

The paper is organized as follows. In Section 2, we compute the generalized characteristic polynomial of subdivision-vertex coronae, and obtain the A-spectra, the L-spectra and the Q-spectra of the subdivision-vertex corona $G_1 \odot G_2$ for a regular graph G_1 and an arbitrary graph G_2 (see

Theorems 2.1, 2.3). Section 3 mainly investigates the generalized characteristic polynomial of subdivision-edge coronae, and obtain the A-spectra, the L-spectra and the Q-spectra of the subdivision-edge corona $G_1 \ominus G_2$ for a regular graph G_1 and an arbitrary graph G_2 (see Theorems 3.1, 3.3). As we will see in Corollaries 2.2, 3.2, our results on the spectra of $G_1 \odot G_2$ and $G_1 \ominus G_2$ enable us to construct infinitely many pairs of Φ-cospectral graphs. Our constructions of infinite families of integral graphs are stated in Corollaries 2.6, 2.7, 3.6, 3.7. In Corollaries 2.8, 2.9, 3.8, 3.9, we compute the number of spanning trees and the Kirchhoff index of $G_1 \odot G_2$ (respectively, $G_1 \ominus G_2$) for a regular graph G_1 and an arbitrary graph G_2.

2 Subdivision-vertex coronae

Let G_1 be an arbitrary graphs on n_1 vertices and m_1 edges, and G_2 an arbitrary graphs on n_2 vertices, respectively. We first label the vertices of $G_1 \odot G_2$ as follows. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $i = 1, 2, \ldots, n_1$, let $u_1^i, u_2^i, \ldots, u_{n_2}^i$ denote the vertices of the ith copy of G_2, with the understanding that u_j^i is the copy of u_j for each j. Denote $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n_1}\}$ for $j = 1, 2, \ldots, n_2$. Then $V(G_1) \cup I(G_1) \cup [W_1 \cup W_2 \cup \cdots \cup W_{n_2}]$ is a partition of $V(G_1 \odot G_2)$. The adjacency matrix of $G_1 \odot G_2$ can be written as

$$A(G_1 \odot G_2) = \begin{bmatrix} 0_{n_1 \times n_1} & R & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ R^T & 0_{m_1 \times m_1} & 0_{m_1 \times n_1 n_2} \\ \mathbf{1}_{n_2} \otimes I_{n_1} & 0_{n_1 n_2 \times m_1} & A(G_2) \otimes I_{n_1} \end{bmatrix}$$

where $0_{s \times t}$ denotes the $s \times t$ matrix with all entries equal to zero, R is the incidence matrix [5] of G_1, I_n is the identity matrix of order n, $\mathbf{1}_n$ is the column vector with all entries equal to 1. It is clear that the degrees of the vertices of $G_1 \odot G_2$ are: $d_{G_1 \odot G_2}(v_i) = n_2 + d_{G_1}(v_i)$ for $i = 1, 2, \ldots, n_1$, $d_{G_1 \odot G_2}(e_i) = 2$ for $i = 1, 2, \ldots, m_1$, and $d_{G_1 \odot G_2}(u_j^i) = d_{G_2}(u_j) + 1$ for $i = 1, 2, \ldots, n_1$, $j = 1, 2, \ldots, n_2$. Then the degree matrix of subdivision-vertex coronae can be written as follows

$$D(G_1 \odot G_2) = \begin{bmatrix} (r_1 + n_2) I_{n_1} & 0_{n_1 \times m_1} & 0_{n_1 \times n_1 n_2} \\ 0_{m_1 \times n_1} & 2I_{m_1} & 0_{m_1 \times n_1 n_2} \\ 0_{n_1 n_2 \times n_1} & 0_{n_1 n_2 \times m_1} & (D(G_2) + I_{n_2}) \otimes I_{n_1} \end{bmatrix}.$$

where $D(G_2)$ denotes the degree matrix of G_2.

2.1 Generalized characteristic polynomial of $G_1 \odot G_2$

Theorem 2.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then the generalized characteristic polynomial of subdivision-vertex coronae can be obtained as follows:*
$$\Phi_{G_1 \odot G_2}(x,t) = (x+2t)^{m_1-n_1} \cdot (\Phi_{G_2}(x+t,t))^{n_1} \cdot \prod_{i=1}^{n_1} (x^2 + (2t+tr_1+tn_2)x - (x+2t)\Gamma_{A(G_2)-tD(G_2)}(x+t) + 2t^2(r_1+n_2) - \lambda_i(G_1) - r_1).$$

Proof. Let R be the incidence matrix [5] of G_1. Then, with respect to the adjacent matrix and degree matrix of $V(G_1 \odot G_2)$, the generalized matrix of $G_1 \odot G_2$ is given by
$$A(G_1 \odot G_2) - tD(G_1 \odot G_2) = \begin{bmatrix} -(tr_1+tn_2)I_{n_1} & R & \mathbf{1}_{n_2}^T \otimes I_{n_1} \\ R^T & -2tI_{m_1} & 0_{m_1 \times n_1 n_2} \\ \mathbf{1}_{n_2} \otimes I_{n_1} & 0_{n_1 n_2 \times m_1} & (A(G_2) - tD(G_2) - tI_{n_2}) \otimes I_{n_1} \end{bmatrix}.$$
Thus the generalized characteristic polynomial of $G_1 \odot G_2$ can be obtained:
$$\begin{aligned} \Phi_{G_1 \odot G_2}(x,t) &= \det \begin{bmatrix} (x+tr_1+tn_2)I_{n_1} & -R & -\mathbf{1}_{n_2}^T \otimes I_{n_1} \\ -R^T & (x+2t)I_{m_1} & 0_{m_1 \times n_1 n_2} \\ -\mathbf{1}_{n_2} \otimes I_{n_1} & 0_{n_1 n_2 \times m_1} & ((x+t)I_{n_2} - (A(G_2) - tD(G_2))) \otimes I_{n_1} \end{bmatrix} \\ &= \det\left(((x+t)I_{n_2} - (A(G_2) - tD(G_2))) \otimes I_{n_1}\right) \cdot \det(S) \\ &= (\Phi_{G_2}(x+t,t))^{n_1} \cdot \det(S), \end{aligned}$$
where
$$\begin{aligned} S &= \begin{pmatrix} (x+tr_1+tn_2)I_{n_1} & -R \\ -R^T & (x+2t)I_{m_1} \end{pmatrix} - \begin{pmatrix} -\mathbf{1}_{n_2}^T \otimes I_{n_1} \\ 0_{m_1 \times n_1 n_2} \end{pmatrix} \\ &\quad \cdot \left(((x+t)I_{n_2} - (A(G_2) - tD(G_2))) \otimes I_{n_1}\right)^{-1} \begin{pmatrix} -\mathbf{1}_{n_2} \otimes I_{n_1} & 0_{n_1 n_2 \times m_1} \end{pmatrix} \\ &= \begin{pmatrix} \left(x+tr_1+tn_2 - \Gamma_{A(G_2)-tD(G_2)}(x+t)\right) I_{n_1} & -R \\ -R^T & (x+2t)I_{m_1} \end{pmatrix} \end{aligned}$$
is the Schur complement [24] of $((x+t)I_{n_2} - (A(G_2) - tD(G_2))) \otimes I_{n_1}$. It is well known [6] that $RR^T = A(G_1) + r_1 I_{n_1}$. Thus, the result follows from
$$\begin{aligned} \det(S) &= (x+2t)^{m_1} \cdot \det\left(\left(x+tr_1+tn_2 - \Gamma_{A(G_2)-tD(G_2)}(x+t)\right) I_{n_1} - \frac{RR^T}{x+2t} \right) \\ &= (x+2t)^{m_1-n_1} \cdot \prod_{i=1}^{n_1} (x^2 + (2t+tr_1+tn_2)x - (x+2t) \\ &\quad \cdot \Gamma_{A(G_2)-tD(G_2)}(x+t) + 2t^2(r_1+n_2) - \lambda_i(G_1) - r_1). \end{aligned}$$
$\square$

Corollary 2.2. *(a) If G_1 and G_2 are Φ-cospectral r-regular graphs, and H is an arbitray graph, then $G_1 \odot H$ and $G_2 \odot H$ are Φ-cospectral. (b) If G is an regular graph, and H_1 and H_2 are Φ-cospectral graphs with $\Gamma_{A(H_1)-tD(H_1)}(x) = \Gamma_{A(H_2)-tD(H_2)}(x)$, then $G \odot H_1$ and $G \odot H_2$ are Φ-cospectral.*

Theorem 2.3. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then*

(1) *The adjacency characteristic polynomial as follows:* $\phi(A(G_1 \odot G_2); x) = x^{m_1-n_1} \cdot (\phi(A(G_2); x))^{n_1} \cdot \prod_{i=1}^{n_1} \left(x^2 - \Gamma_{A(G_2)}(x)x - r_1 - \lambda_i(G_1)\right)$.

(2) *The Laplacian characteristic polynomial as follows:* $\phi(L(G_1 \odot G_2); x) = (x-2)^{m_1-n_1} \cdot \prod_{i=2}^{n_2} (x - 1 - \mu_i(G_2))^{n_1} \cdot \prod_{i=1}^{n_1} \Big(x^3 - (3 + r_1 + n_2)x^2 + (2 + r_1 + \mu_i(G_1) + 2n_2)x - \mu_i(G_1)\Big)$.

(3) *The signless Laplacian characteristic polynomial as follows:* $\phi(Q(G_1 \odot G_2); x) = (x-2)^{m_1-n_1} \cdot \prod_{i=1}^{n_2} (x - 1 - \nu_i(G_2))^{n_1} \cdot \prod_{i=1}^{n_1} \Big(x^2 - (2 + r_1 + n_2 + \Gamma_{Q(G_2)}(x-1))x + 2(r_1 + n_2 + \Gamma_{Q(G_2)}(x-1)) - \nu_i(G_1)\Big)$.

Proof. (1) By the equation $\phi(A(G_1 \odot G_2); x) = \Phi_{G_1 \odot G_2}(x, 0)$ and Theorem 2.1
$\phi(A(G_1 \odot G_2); x)$

$$= x^{m_1-n_1} \cdot (\Phi_{G_2}(x,0))^{n_1} \cdot \prod_{i=1}^{n_1} \left(x^2 - \Gamma_{A(G_2)}(x)x - r_1 - \lambda_i(G_1)\right)$$

$$= x^{m_1-n_1} \cdot (\phi(A(G_2); x))^{n_1} \cdot \prod_{i=1}^{n_1} \left(x^2 - \Gamma_{A(G_2)}(x)x - r_1 - \lambda_i(G_1)\right).$$

(2) By the equation $\phi(L(G_1 \odot G_2); x) = (-1)^{|V(G_1 \odot G_2)|} \Phi_{G_1 \odot G_2}(-x, 1)$, for an arbitrary graph G_2 with n_2 vertices , we have the coronal of Laplacian matrix $\Gamma_{L(G_2)} = \Gamma_{-L(G_2)} = \frac{n_2}{x}$, the minmun eigenvalues of Laplacian matrix is 0, as we all know $\lambda_i(G_1) = r_1 - \mu_i(G_1) (i = 1, 2, ..., n_1)$ and by virtue of Theorem 2.1:

$$\begin{aligned}
&\phi(L(G_1 \odot G_2); x) \\
&= (-1)^{n_1(1+n_2)+m_1} \cdot (2-x)^{m_1-n_1} \cdot \left(\Phi_{G_2}(-x+1, 1)\right)^{n_1} \\
&\quad \cdot \prod_{i=1}^{n_1} \Big(x^2 - (2+r_1+n_2)x + 2r_1 + 2n_2 - (-x+2)\Gamma_{-L(G_2)}(-x+1) \\
&\quad + \mu_i(G_1) - 2r_1\Big) \\
&= (x-2)^{m_1-n_1} \cdot \left(\phi(L(G_2); x-1)\right)^{n_1} \cdot (x-1)^{-n_1} \\
&\quad \cdot \prod_{i=1}^{n_1} \Big(x^3 - (3+r_1+n_2)x^2 + (2+r_1+2n_2+\mu_i(G_1))x - \mu_i(G_1)\Big) \\
&= (x-2)^{m_1-n_1} \cdot \prod_{i=2}^{n_1} \Big(x - 1 - \mu_i(G_1)\Big)^{n_1} \\
&\quad \cdot \prod_{i=1}^{n_1} \Big(x^3 - (3+r_1+n_2)x^2 + (2+r_1+2n_2+\mu_i(G_1))x - \mu_i(G_1)\Big)
\end{aligned}$$

(3)By the equation $\phi(Q(G_1 \odot G_2); x) = \Phi_{G_1 \odot G_2}(x, -1)$, as we know, $\lambda_i(G_1) = \nu_i(G_1) - r_1 (i = 1, 2, ..., n_1)$ and by virtue of Theorem 2.1:

$$\begin{aligned}
&\phi(Q(G_1 \odot G_2); x) \\
&= (x-2)^{m_1-n_1} \cdot \left(\Phi_{G_2}(x-1, -1)\right)^{n_1} \cdot \prod_{i=1}^{n_1} \Big(x^2 - (2+r_1+n_2)x + 2r_1 \\
&\quad + 2n_2 - (x-2)\Gamma_{Q(G_2)}(x-1) - \lambda_i(G_1) - r_1\Big) \\
&= (x-2)^{m_1-n_1} \cdot \left(\phi(Q(G_2); x-1)\right)^{n_1} \cdot \prod_{i=1}^{n_1} \Big(x^2 - (2+r_1+n_2)x + 2r_1 \\
&\quad + 2n_2 - (x-2)\Gamma_{Q(G_2)}(-x+1) - \nu_i(G_1)\Big)
\end{aligned}$$

2.2 Some conclusions of A-spectrum of $G_1 \odot G_2$

Theorem 2.3 enables us to compute the A-spectra of many subdivision-vertex coronae, if we can determine the $A(G_2)$-coronal $\Gamma_{A(G_2)}(x)$. Fortunately, we have known the $A(G_2)$-coronal for some graph G_2. For example, if G_2 is an r_2-regular graph on n_2 vertices, then [4, 20] $\Gamma_{A(G_2)}(x) = n_2/(x - r_2)$, and if $G_2 \cong K_{p,q}$ which is the complete bipartite graph with $p, q \geqslant 1$ vertices in the two parts of its bipartition, then [20] $\Gamma_{A(G_2)}(x) = ((p+q)x + 2pq)/(x^2 - pq)$. Thus, Theorem 2.3 implies the following results immediately.

Corollary 2.4. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices. Then the A-spectrum of $G_1 \odot G_2$ consists of: (a) $\lambda_i(G_2)$, repeated n_1 times, for each $i = 2, 3, \ldots, n_2$; (b) 0, repeated $m_1 - n_1$ times; (c) three roots of the equation $x^3 - r_2x^2 - (r_1 + \lambda_j(G_1) + n_2)x + r_2(r_1 + \lambda_j(G_1)) = 0$, for each $j = 1, 2, \ldots, n_1$.*

Corollary 2.5. *Let G be an r-regular graph on n vertices and m edges with $m \geqslant n$, and let $p, q \geqslant 1$ be integers. Then the A-spectrum of $G \odot K_{p,q}$ consists of: (a) 0, repeated $m + (p + q - 3)n$ times; (b) four roots of the equation $x^4 - (pq + p + q + r + \lambda_j(G))x^2 - 2pqx + pq(r + \lambda_j(G)) = 0$, for each $j = 1, 2, \ldots, n$.*

A graph whose A-spectrum consists of entirely of integers is called an *A-integral graph.* The question of "Which graphs have A-integral spectra?" was first posed by F. HARARY and A.J. SCHWENK in 1973 [7]. A-integral graphs are very rare and difficult to be found. For more properties and constructions on A-integral graphs, please refer to an excellent survey [2].

The *complement* $\overline{G}$ of a graph G is the graph with the same vertex set as G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Note that the complete graph K_n is $(n-1)$-regular with the A-spectrum $(n-1)^1$, $(-1)^{n-1}$, where a^b denotes that the multiplicity of a is b. Then, by Corollary 2.4, the A-spectrum of $K_{n_1} \odot \overline{K_{n_2}}$ consists of $\left(\pm\sqrt{2n_1 - 2 + n_2}\right)^1$, $\left(\pm\sqrt{n_1 - 2 + n_2}\right)^{n_1-1}$, $0^{n_1(n_2-1)+m_1}$, which implies that $K_{n_1} \odot \overline{K_{n_2}}$ is A-integral if and only if $\sqrt{2n_1 - 2 + n_2}$ and $\sqrt{n_1 - 2 + n_2}$ are integers. Now we present our first construction of an infinite family of A-integral graphs (see Figure 2 for an example).

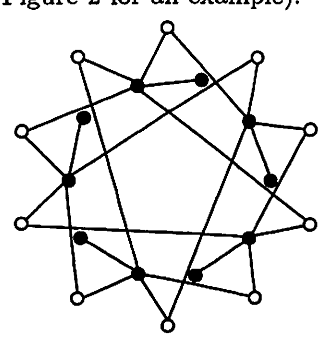

Fig. 2: $K_5 \odot \overline{K_1}$ with A-spectrum $(\pm 3)^1$, $(\pm 2)^4$, 0^{10}.

Corollary 2.6. *$K_{n_1} \odot \overline{K_{n_2}}$ is A-integral if and only if $n_1 = s^2 - h^2$ and $n_2 = 2h^2 - s^2 + 2$ for $h = 2, 3, \ldots$, $s = 3, 4, \ldots$, and $h^2 < s^2 < 2h^2 + 2$.*

Proof. From the above statement, $K_{n_1} \odot \overline{K_{n_2}}$ is A-integral if and only if $\sqrt{2n_1 - 2 + n_2}$ and $\sqrt{n_1 - 2 + n_2}$ are integers. Let $\sqrt{2n_1 - 2 + n_2} = s$ and $\sqrt{n_1 - 2 + n_2} = h$, where s, h are nonnegative integers. Solving these two equations, we obtain that $n_1 = s^2 - h^2$, $n_2 = 2h^2 - s^2 + 2$. Since $n_1 > 0$ and $n_2 > 0$, we have that $h \geq 2$, $s \geq 3$ and $h^2 < s^2 < 2h^2 + 2$. □

Notice that the complete bipartite graph $K_{n,n}$ is n-regular with the A-spectrum $(\pm n)^1$, 0^{2n-2}. By Corollary 2.4, the A-spectrum of $K_{n_1,n_1} \odot \overline{K_{n_2}}$ consists of $\left(\pm\sqrt{2n_1 + n_2}\right)^1$, $\left(\pm\sqrt{n_1 + n_2}\right)^{2n_1-2}$, $\left(\pm\sqrt{n_2}\right)^1$, $0^{2n_1n_2-2n_1+m_1}$, which implies that $K_{n_1,n_1} \odot \overline{K_{n_2}}$ is A-integral if and only if $\sqrt{2n_1 + n_2}$, $\sqrt{n_1 + n_2}$ and $\sqrt{n_2}$ are integers. Here, we present our second construction of an infinite family of A-integral graphs.

Corollary 2.7. *$K_{n_1,n_1} \odot \overline{K_{n_2}}$ is A-integral if $n_1 = 4st^2(2s^2 + 3s + 1)$ and $n_2 = t^2(2s^2 - 1)^2$ for $s = 1, 2, 3, \ldots$, $t = 1, 2, 3, \ldots$.*

Proof. From the above statement, $K_{n_1,n_1} \odot \overline{K_{n_2}}$ is A-integral if and only if $\sqrt{2n_1 + n_2}$, $\sqrt{n_1 + n_2}$ and $\sqrt{n_2}$ are integers. Let $\sqrt{2n_1 + n_2} = a$, $\sqrt{n_1 + n_2} = b$ and $\sqrt{n_2} = c$, where a, b, c are nonnegative integers. Solving these equations, we obtain that $n_1 = a^2 - b^2$ and $n_2 = 2b^2 - a^2 = c^2$. Notice that $2b^2 - a^2 = c^2$ is equivalent to $(a+b)(a-b) = (b+c)(b-c)$. Let $u = a+b$, $v = a - b$, $x = b + c$ and $y = b - c$. Then we obtain that $uv = xy$ and $u - v = x + y$. Combining these two equations and eliminating u, we have $2v^2 = (x-v)(y-v)$. Let $x - v = 2vs$ and $s(y-v) = v$ for $s = 1, 2, \ldots$. Then $x = 2vs + v = b + c$ and $y = \frac{v}{s} + v = b - c$. Thus, $a = b + v = vs + 2v + \frac{v}{2s}$, $b = vs + v + \frac{v}{2s}$ and $c = vs - \frac{v}{2s}$. Let $v = 2st$ for $t = 1, 2, \ldots$. Then we have $a = 2s^2t + 4st + t$, $b = 2s^2t + 2st + t$ and $c = 2s^2t - t$, which imply that $n_1 = 4st^2(2s^2 + 3s + 1)$ and $n_2 = c^2 = t^2(2s^2 - 1)^2$. □

2.3 Some conclusions of L-spectrum of $G_1 \odot G_2$

Let $t(G)$ denote the number of spanning trees of G. It is well known [5] that if G is a connected graph on n vertices with Laplacian spectrum $0 = \mu_1(G) < \mu_2(G) \leq \cdots \leq \mu_n(G)$, then $t(G) = \frac{\mu_2(G)\cdots\mu_n(G)}{n}$. By Theorem 2.3, we can readily obtain the following result.

Corollary 2.8. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then* $t(G_1 \odot G_2) = \frac{2^{m_1-n_1}\cdot(2+r_1+2n_2)n_1\cdot t(G_1)\cdot\prod_{i=2}^{n_2}\left(\mu_i(G_2)+1\right)^{n_1}}{n_1+m_1+n_1n_2}$.

The *Kirchhoff index* of a graph G, denoted by $Kf(G)$, is defined as the sum of resistance distances between all pairs of vertices [3, 16]. At almost exactly the same time, Gutman et al. [10] and Zhu et al. [23] proved that the Kirchhoff index of a connected graph G with $n\,(n \geq 2)$ vertices can be expressed as $Kf(G) = n\sum_{i=2}^{n} \frac{1}{\mu_i(G)}$, where $\mu_2(G), \ldots, \mu_n(G)$ are the non-zero Laplacian eigenvalues of G. Theorem 2.3 implies the following result.

Corollary 2.9. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then* $Kf(G_1 \odot G_2) = \Big(n_1(1 + n_2) + m_1\Big) \times \left(\frac{m_1+n_1-2}{2} + \frac{3+r_1+n_2}{2+r_1+2n_2} + \frac{2+r_1+2n_2}{n_1} \cdot Kf(G_1) + \sum_{i=2}^{n_2} \frac{n_1}{1+\mu_i(G_2)}\right)$.

3 Subdivision-edge coronae

We label the vertices of $G_1 \ominus G_2$ as follows. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $i = 1, 2, \ldots, m_1$, let $u_1^i, u_2^i, \ldots, u_{n_2}^i$ denote the vertices of the ith copy of G_2, with the understanding that u_j^i is the copy of u_j for each j. Denote $W_j = \{u_j^1, u_j^2, \ldots, u_j^{m_1}\}$ for $j = 1, 2, \ldots, n_2$. Then $V(G_1) \cup I(G_1) \cup [W_1 \cup W_2 \cup \cdots \cup W_{n_2}]$ is a partition of $V(G_1 \ominus G_2)$. The adjacency matrix of $G_1 \ominus G_2$ can be written as

$$A(G_1 \ominus G_2) = \begin{bmatrix} 0_{n_1 \times n_1} & R & 0_{n_1 \times m_1 n_2} \\ R^T & 0_{m_1 \times m_1} & \mathbf{1}_{n_2}^T \otimes I_{m_1} \\ 0_{m_1 n_2 \times n_1} & \mathbf{1}_{n_2} \otimes I_{m_1} & A(G_2) \otimes I_{m_1} \end{bmatrix}.$$

It is clear that the degrees of the vertices of $G_1 \ominus G_2$ are: $d_{G_1 \ominus G_2}(v_i) = d_{G_1}(v_i)$ for $i = 1, 2, \ldots, n_1$, $d_{G_1 \ominus G_2}(e_i) = 2 + n_2$ for $i = 1, 2, \ldots, m_1$, and $d_{G_1 \ominus G_2}(u_j^i) = d_{G_2}(u_j) + 1$ for $i = 1, 2, \ldots, m_1$, $j = 1, 2, \ldots, n_2$. The degree matrix of $G_1 \ominus G_2$ can be written as

$$D(G_1 \ominus G_2) = \begin{bmatrix} r_1 I_{n_1} & 0_{n_1 \times m_1} & 0_{n_1 \times m_1 n_2} \\ 0_{m_1 \times n_1} & (2 + n_2) I_{m_1} & 0_{m_1 \times m_1 n_2} \\ 0_{m_1 n_2 \times n_1} & 0_{m_1 n_2 \times m_1} & \big(D(G_2) + I_{n_2}\big) \otimes I_{m_1} \end{bmatrix}.$$

3.1 Generalized characteristic polynomial of $G_1 \ominus G_2$

Theorem 3.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then the generalized character-*

istic polynomial of subdivision-edge coronae can be obtained as follows

$$\begin{aligned}
&\Phi_{G_1 \ominus G_2}(x,t) \\
&= \left(x + 2t + n_2 t - \Gamma_{A(G_2)-tD(G_2)}(x+t)\right)^{m_1-n_1} \cdot (\Phi_{G_2}(x+t,t))^{m_1} \\
&\cdot \prod_{i=1}^{n_1} \Big(x^2 + (2t + tr_1 + tn_2)x - (x + tr_1)\Gamma_{A(G_2)-tD(G_2)}(x+t) \\
&+ t^2(2r_1 + n_2 r_1) - \lambda_i(G_1) - r_1\Big).
\end{aligned}$$

Proof. Let R be the incidence matrix [5] of G_1. Then, with respect to the adjacent matrix and degree matrix of $V(G_1 \ominus G_2)$, the generalized matrix of $G_1 \ominus G_2$ is given by

$$\begin{aligned}
&A(G_1 \ominus G_2) - tD(G_1 \ominus G_2) \\
&= \begin{bmatrix} -tr_1 I_{n_1} & R & 0_{n_1 \times m_1 n_2} \\ R^T & -(2t + n_2 t)I_{m_1} & 1_{n_2}^T \otimes I_{m_1} \\ 0_{m_1 n_2 \times n_1} & 1_{n_2} \otimes I_{m_1} & (A(G_2) - tD(G_2) - tI_{n_2}) \otimes I_{m_1} \end{bmatrix}
\end{aligned}$$

Thus, the generalized characteristic polynomial of $G_1 \ominus G_2$ can be obtained:

$$\begin{aligned}
&\Phi_{G_1 \ominus G_2}(x,t) \\
&= \det \begin{bmatrix} (x + tr_1)I_{n_1} & -R & 0_{n_1 \times m_1 n_2} \\ -R^T & (x + 2t + n_2 t)I_{m_1} & -1_{n_2}^T \otimes I_{m_1} \\ 0_{m_1 n_2 \times n_1} & -1_{n_2} \otimes I_{m_1} & ((x+t)I_{n_2} - (A(G_2) - tD(G_2))) \otimes I_{m_1} \end{bmatrix} \\
&= \det\left(((x+t)I_{n_2} - (A(G_2) - tD(G_2))) \otimes I_{m_1}\right) \cdot \det(S) \\
&= \left(\det((x+t)I_{n_2} - (A(G_2) - tD(G_2)))\right)^{m_1} \cdot \left(\det(I_{m_1})\right)^{n_2} \cdot \det(S) \\
&= (\Phi_{G_2}(x+t,t))^{m_1} \cdot \det(S),
\end{aligned}$$

where

$$\begin{aligned}
S &= \begin{pmatrix} (x + tr_1)I_{n_1} & -R \\ -R^T & (x + 2t + t + tn_2)I_{m_1} \end{pmatrix} \\
&- \begin{pmatrix} 0_{n_1 \times m_1 n_2} \\ -1_{n_2}^T \otimes I_{m_1} \end{pmatrix} \left(((x+t)I_{n_2} - (A(G_2) - tD(G_2))) \otimes I_{m_1}\right)^{-1} \\
&\cdot \begin{pmatrix} 0_{m_1 n_2 \times n_1} & -1_{n_2} \otimes I_{m_1} \end{pmatrix} \\
&= \begin{pmatrix} (x + tr_1)I_{n_1} & -R \\ -R^T & (x + 2t + tn_2 - \Gamma_{A(G_2)-tD(G_2)}(x+t))\, I_{m_1} \end{pmatrix}
\end{aligned}$$

is the Schur complement [24] of $\left((x+t)I_{n_2} - (A(G_2) - tD(G_2))\right) \otimes I_{m_1}$. It is well known [6] that $RR^T = A(G_1) + r_1 I_{n_1}$. Thus, the result follows from

$\det(S)$

$$= (x+tr_1)^{n_1} \cdot \det\left(\left(x+2t+n_2t-\Gamma_{A(G_2)-tD(G_2)}(x+t)\right) I_{m_1} - \frac{RR^T}{x+tr_1}\right)$$

$$= \left(x+2t+n_2t-\Gamma_{A(G_2)-tD(G_2)}(x+t)\right)^{m_1-n_1} \cdot \prod_{i=1}^{n_1} \left(x^2+(2t+tn_2+tr_1)x\right.$$

$$\left. - (x+tr_1)\Gamma_{A(G_2)-tD(G_2)}(x+t) + t^2(2r_1+n_2) - \lambda_i(G_1) - r_1\right). \qquad \square$$

Corollary 3.2. *(a) If G_1 and G_2 are Φ-cospectral r-regular graphs, and H is an arbitrary graph, then $G_1 \ominus H$ and $G_2 \ominus H$ are Φ-cospectral. (b) If G is an regular graph, and H_1 and H_2 are Φ-cospectral graphs with $\Gamma_{A(H_1)-tD(H_1)}(x) = \Gamma_{A(H_2)-tD(H_2)}(x)$, then $G \odot H_1$ and $G \odot H_2$ are Φ-cospectral.*

Theorem 3.3. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then*

(1) *The adjacency characteristic polynomial as follows:*

$$\phi\left(A(G_1 \ominus G_2); x\right) = \left(\phi(A(G_2); x)\right)^{m_1} \cdot \left(x - \Gamma_{A(G_2)}(x)\right)^{m_1-n_1} \cdot \prod_{i=1}^{n_1} \left(x^2 - \Gamma_{A(G_2)}(x)x - \lambda_i(G_1) - r_1\right).$$

(2) *The Laplacian characteristic polynomial as follows:*
$\phi\left(L(G_1 \ominus G_2); x\right) = \left(x^2-(3+n_2)x+2\right)^{m_1-n_1} \cdot \prod_{i=2}^{n_2}\left(x-1-\mu_i(G_2)\right)^{m_1} \cdot \prod_{i=1}^{n_1}\left(x^3-(3+r_1+n_2)x^2+(2+r_1+r_1n_2+\mu_i(G_1))x-\mu_i(G_1)\right)$.

(3) *The signless Laplacian characteristic polynomial as follows:*
$\phi\left(Q(G_1 \ominus G_2); x\right) = \left(x-2-n_2-\Gamma_{Q(G_2)}(x-1)\right)^{m_1-n_1} \cdot \prod_{i=1}^{n_2}\left(x-1-\nu_i(G_2)\right)^{m_1} \cdot \prod_{i=1}^{n_1}\left(x^2-(2+r_1+n_2+\Gamma_{Q(G_2)}(x-1))x+r_1(2+n_2+\Gamma_{Q(G_2)}(x-1))-\nu_i(G_1)\right)$.

Proof. (1) By the equation $\phi\left(A(G_1 \ominus G_2); x\right) = \Phi_{G_1 \ominus G_2}(x, 0)$ and Theorem 3.1

$$\phi\left(A(G_1 \ominus G_2); x\right) = \left(x-\Gamma_{A(G_2)}(x)\right)^{m_1-n_1} \cdot \left(\Phi_{G_2}(x,0)\right)^{m_1} \cdot \prod_{i=1}^{n_1}\left(x^2-\Gamma_{A(G_2)}(x)x-\lambda_i(G_1)-r_1\right)$$

$$= \left(x-\Gamma_{A(G_2)}(x)\right)^{m_1-n_1} \cdot \left(\phi(A(G_2); x)\right)^{m_1} \cdot \prod_{i=1}^{n_1}\left(x^2-\Gamma_{A(G_2)}(x)x-\lambda_i(G_1)-r_1\right).$$

(2) By the equation $\phi\left(L(G_1 \ominus G_2);x\right) = (-1)^{|V(G_1 \ominus G_2)|}\Phi_{G_1 \ominus G_2}(-x,1)$, for an arbitrary graph G_2 with n_2 vertices , we have the coronal of Laplacian matrix $\Gamma_{L(G_2)} = \Gamma_{-L(G_2)} = \frac{n_2}{x}$, the minmun eigenvalues of Laplacian matrix is 0, as we know $\lambda_i(G_1) = r_1 - \mu_i(G_1)(i = 1, 2, ..., n_1)$ and by Theorem 3.1

$$\begin{aligned}
&\phi\left(L(G_1 \ominus G_2);x\right) \\
&= (-1)^{m_1(1+n_2)+n_1} \cdot (\Phi_{G_2}(-x+1,1))^{m_1} \cdot (2 - x + n_2 x - \Gamma_{-L(G_2)}(-x+1))^{m_1-n_1} \\
&\quad \cdot \prod_{i=1}^{n_1}(x^2 - (2 + r_1 + n_2)x - (r_1 - x)\Gamma_{-L(G_2)}(-x+1) + 2r_1 + n_2 r_1 - \lambda(G_1) - r_1) \\
&= (x^2 - (3 + n_2)x + 2)^{m_1-n_1} \cdot \prod_{i=2}^{n_2}\left(x - 1 - \mu_i(G_2)\right)^{m_1} \\
&\quad \cdot \prod_{i=1}^{n_1}\left(x^3 - (3 + r_1 + n_2)x^2 + (2 + r_1 + r_1 n_2 + \mu_i(G_1))x - \mu_i(G_1)\right)
\end{aligned}$$

(3) By the equation $\phi\left(Q(G_1 \ominus G_2);x\right) = \Phi_{G_1 \ominus G_2}(x,-1)$, as we know $\lambda_i(G_1) = \nu_i(G_1) - r_1(i = 1, 2, ..., n_1)$ and by Theorem 3.1

$$\begin{aligned}
&\phi\left(Q(G_1 \ominus G_2);x\right) \\
&= (x - 2 - n_2 x - \Gamma_{Q(G_2)}(x-1))^{m_1-n_1} \cdot (\Phi_{G_2}(x-1,-1))^{m_1} \cdot \prod_{i=1}^{n_1}\Big(x^2 \\
&\quad - (2 + r_1 + n_2)x - (x - r_1)\Gamma_{Q(G_2)}(x-1) + 2r_1 + n_2 r_1 - \lambda_i(G_1) - r_1\Big) \\
&= (x - 2 - n_2 x - \Gamma_{Q(G_2)}(x-1))^{m_1-n_1} \cdot (\phi(Q(G_2); x-1))^{m_1} \cdot \prod_{i=1}^{n_1}\Big(x^2 \\
&\quad - (2 + r_1 + n_2)x - (x - r_1)\Gamma_{Q(G_2)}(x-1) + 2r_1 + n_2 r_1 - \nu_i(G_1)\Big)
\end{aligned}$$

3.2 Some conclusions of A-spectrum of $G_1 \ominus G_2$

Corollary 3.4. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an r_2-regular graph on n_2 vertices. Then the A-spectrum of $G_1 \ominus G_2$ consists of: (a) $\lambda_i(G_2)$, repeated m_1 times, for each $i = 2, 3, \ldots, n_2$; (b) two roots of the equation $x^2 - r_2 x - n_2 = 0$, each root repeated $m_1 - n_1$ times; (c) three roots of the equation $x^3 - r_2 x^2 - (r_1 + \lambda_j(G_1) + n_2)x + r_2(r_1 + \lambda_j(G_1)) = 0$, for each $j = 1, 2, \ldots, n_1$.*

Corollary 3.5. *Let G be an r-regular graph on n vertices and m edges with $m \geqslant n$, and let $p, q \geqslant 1$ be integers. Then the A-spectrum of $G \ominus K_{p,q}$ consists of : (a) 0, repeated $m(p+q-2)$ times; (b) three roots of the equation $x^3 - (pq + p + q)x - 2pq = 0$, each root repeated $m - n$ times; (c) four roots of the equation $x^4 - (pq + p + q + r + \lambda_i(G))x^2 - 2pqx + pq(r + \lambda_i(G)) = 0$, for each $i = 1, 2, \ldots, n$.*

Similar to Corollary 2.6, the subdivision-edge coronae enable us to construct infinite families of A-integral graphs by using Corollary 3.4. Note that the A-spectrum of $K_{n_1} \ominus \overline{K_{n_2}}$ is $\left(\pm\sqrt{2n_1-2+n_2}\right)^1$, $\left(\pm\sqrt{n_1-2+n_2}\right)^{n_1-1}$, $\left(\pm\sqrt{n_2}\right)^{m_1-n_1}$, $0^{m_1(n_2-1)+n_1}$. Then $K_{n_1} \ominus \overline{K_{n_2}}$ is A-integral if and only if $\sqrt{2n_1-2+n_2}$, $\sqrt{n_1-2+n_2}$ and $\sqrt{n_2}$ are integers.

Corollary 3.4 implies that the A-spectrum of $K_{n_1,n_1} \ominus \overline{K_{n_2}}$ consists of $\left(\pm\sqrt{2n_1+n_2}\right)^1$, $\left(\pm\sqrt{n_1+n_2}\right)^{2n_1-2}$, $\left(\pm\sqrt{n_2}\right)^{m_1-2n_1+1}$, $0^{2n_1+m_1n_2-m_1}$. Thus $K_{n_1,n_1} \ominus \overline{K_{n_2}}$ is A-integral if and only if $\sqrt{2n_1+n_2}$, $\sqrt{n_1+n_2}$ and $\sqrt{n_2}$ are integers. Then we have the following two constructions of A-integral graphs (see Figure 3 for an example of Corollary 3.7).

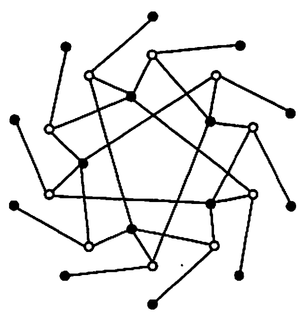

Fig. 3: $K_5 \ominus \overline{K_1}$ with A-spectrum $(\pm 3)^1$, $(\pm 2)^4$, $(\pm 1)^5$, 0^5.

Corollary 3.6. *$K_{n_1} \ominus \overline{K_{n_2}}$ is A-integral if $n_1 = 2t+3$ and $n_2 = t^2$ for $t = 1, 2, \ldots$.*

Corollary 3.7. *$K_{n_1,n_1} \ominus \overline{K_{n_2}}$ is A-integral if $n_1 = 4st^2(2s^2+3s+1)$ and $n_2 = t^2(2s^2-1)^2$ for $s = 1, 2, 3, \ldots$, $t = 1, 2, 3, \ldots$.*

3.3 Some conclusions of L-spectrum of $G_1 \ominus G_2$

Corollary 3.8. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then* $t(G_1 \ominus G_2) = \frac{2^{m_1-n_1}\cdot(2+r_1+r_1n_2)n_1\cdot t(G_1)\cdot\prod_{i=2}^{n_2}\left(\mu_i(G_2)+1\right)^{m_1}}{n_1+m_1+m_1n_2}$.

Corollary 3.9. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 an arbitrary graph on n_2 vertices. Then* $Kf(G_1 \ominus G_2) = (m_1(1+n_2)+n_1)\times\left(\frac{(3+n_2)m_1-(n_2+1)n_1-2}{2}+\frac{3+r_1+n_2}{2+r_1+r_1n_2}+\frac{2+r_1+r_1n_2}{n_1}\cdot Kf(G_1)+\sum_{i=2}^{n_2}\frac{m_1}{1+\mu_i(G_2)}\right)$.

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