Size multipartite Ramsey numbers for stripes versus stripes

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Abstract

For graphs G and H, the size balanced Ramsey multipartite number $m_j(G,H)$ is defined as the smallest positive integer s such that any arbitrary red/blue coloring of the graph $K_{j\times s}$ forces the appearance of a red G or a blue H. In the main case of this paper we generalize methods used in finding bipartite Ramsey numbers for $b(nK_2, mK_2)$ to finding the balanced Ramsey multipartite number $m_j(nK_2, mK_2)$.

Introduction

All graphs considered in this paper are without loops and multiple edges. Let $K_{j\times s}$ denote complete balanced multipartite graph with each of the j partite sets consisting of exactly s elements. The size multipartite Ramsey number denoted by $m_j(G,H)$, is the smallest natural number s such that any two colouring (red and blue) of $K_{j\times s}$ forces a red G or a blue H as a subgraph.

Over time mathematicians have attempted to find size multipartite Ramsey number for various graphs. Burger et al [1], investigated $m_j(K_{n\times l}, K_{s\times t})$ where $n, s \geqslant 2$. They were successful in finding some small size numbers and lower and upper bounds for some large size numbers. Syafrizal and et al [3], established the exact value of $m_j(P_s, P_n)$ where $s = 2, 3$, $j \geqslant 2$ and $n \geqslant 3$. They (see [4]) also obtained exact values for $m_j(P_s, G)$ with $s = 2, 3$ and $j \geqslant 2$ where G is a wheel W_n, a star S_n, a fan F_n or a windmill M_{2n} with $n \geqslant 6$, in addition to some lower bounds for $m_j(P_n, K_{j\times b})$ where $b \geqslant 2$ and $j, n \geqslant 2$. Extending his work on cycles Syafrizal (see [5]) determined $m_j(P_s, C_3)$ where $s \geqslant 2$, $j \geqslant 3$ and $m_j(P_s, C_4)$ where $s \geqslant 4$.

Not many results have been found on size multipartite Ramsey number with regard to stripes. However, stripes and trees in the two colouring of a complete bipartite graph have been studied by Christou et al (see [2]). They were successful in obtaining the Ramsey numbers $R_b(mP_2, nP_2)$, $R_b(T_m, T_n)$, $R_b(S_m, nP_2)$, $R_b(T_2, nP_2)$ and (S_m, T_n). In this paper we obtain the exact value for $m_j(nK_2, mK_2)$ where $j \geqslant 2$ and $m \leqslant n$.

1 Notation

A *matching* of the graph $G = (V, E)$ is a set of edges such that no two edges share a common vertex. Given a matching M of a graph $G(V, E)$ of size t, let $V(M)$ denote the $2t$ vertices adjacent to the edges of matching M and let $V(M)^c$ denote the vertices outside this matching of size $|V(G)| - 2t$.

Consider any red/blue coloring of $K_{j\times s}$. Let H_R denote the graph having the vertex set $V(K_{j\times s})$ and the edge set consisting of all the red edges. Similarly let H_B denote the graph having the vertex set $V(K_{j\times s})$ and the edge set consisting of all the blue edges. We denote such a edge coloring by $K_{j\times s} = H_R \oplus H_B$.

2 Size multipartite Ramsey numbers for Stripes vs. Stripes

The main aim of this paper is to prove Theorem 8. Theorem 8 is proved using induction. Lemma 4 and lemma 7 are needed to prove the inductive step. Lemma 1 and the other propositions are used as a supporting tool to prove these two pivotal lemmas namely, lemma 4 and lemma 7, needed to prove the main theorem.

Lemma 1. $m_j(nK_2, mK_2) \geqslant \left\lceil \frac{2n+m-1}{j} \right\rceil$ *where $j \geqslant 3$ and $m \leqslant n$.*

Proof. Let $j \geqslant 3$ and $m \leqslant n$. Consider the red/blue coloring given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{2n+m-1}{j} \right\rceil - 1$, such that $K_{j\times s}$ consists of any $2n-1$ vertices, connected to each other by red edges whenever, two of these vertices lie in distinct partite sets and all the other edges by blue. Note that, $s = \left\lceil \frac{2n+m-1}{j} \right\rceil - 1 < \frac{2n+m-1}{j}$. Therefore, H_B will not contain a blue mK_2 as we would get $sj - (2n-1) < m$. Clearly the graph has no red nK_2. Hence we get, $m_j(nK_2, mK_2) \geqslant \left\lceil \frac{2n+m-1}{j} \right\rceil$. □

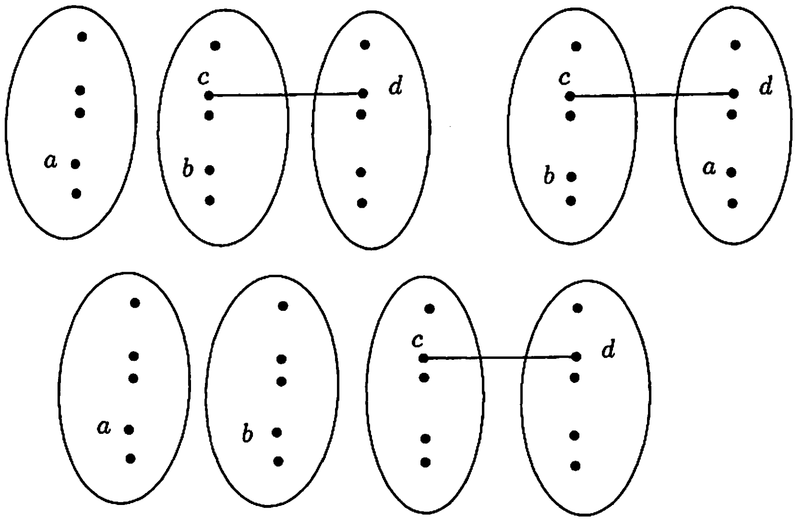

Figure 1: Three possible cases if a and b belong to distinct partite sets

Proposition 2. *Consider a red/blue coloring of $K_{j\times s}$ such that it has no red nK_2. Suppose M is a red matching of size $n-1$ of $K_{j\times s}$. If a and b are two vertices of $V(M)^c$ belonging to distinct partite sets then, given any edge (c,d) of M there exists a vertex of $\{a,b\}$, such that it is incident to a vertex of $\{c,d\}$ in blue.*

Proof. Let a, b be any two vertices outside M. The four vertices a, b, c and d can fall in to one of the three cases as illustrated in Figure 1, up to reordering of vertex a with vertex b, vertex c with vertex d.

Suppose that both (b,d) and (a,c) are red. Then, we could replace the red edge (c,d) of M by the two red edges (b,d) and (a,c) and obtain a red nK_2. Hence, the proposition follows. □

Proposition 3. *Consider a red/blue coloring of $K_{j\times s}$ such that it has no red nK_2. Suppose M is a red matching of size $n-1$ of $K_{j\times s}$. If a and b are two vertices of $V(M)^c$ belonging to to the same partite set (say V_1) then, given any edge (c,d) of M not incident to V_1, there exists a vertex of $\{a,b\}$ such that it is incident to a vertex of $\{c,d\}$ in blue.*

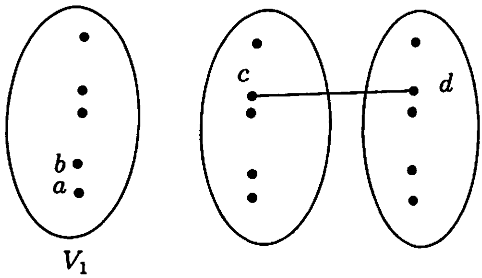

Figure 2: If a and b belong to the same partite set V_1

Proof. Suppose the proposition is false. Then (b, d) and (a, c) must be red. However, as seen in Figure 2 if this happens, we could replace the red edge (c, d) of M by the two red edges (b, d) and (a, c), and obtain a red nK_2. □

Lemma 4. *Suppose $j \geqslant 3$. If $m_j((n-1)K_2, mK_2) = \left\lceil \frac{2n+m-3}{j} \right\rceil$ then it follows that $m_j(nK_2, mK_2) = \left\lceil \frac{2n+m-1}{j} \right\rceil$, for all $m \in N$ such that $m \leqslant n-1$.*

Proof. Suppose $m_j((n-1)K_2, mK_2) = \left\lceil \frac{2n+m-3}{j} \right\rceil$ for $j \geqslant 3$ is true and $m \leqslant n-1$.

By Lemma 1, it suffices to show $m_j(nK_2, mK_2) \leqslant \left\lceil \frac{2n+m-1}{j} \right\rceil$, where $j \geqslant 3$. Consider any two colouring of $K_{j \times s}$ where $s = \left\lceil \frac{2n+m-1}{j} \right\rceil$.

Assume that the coloring is red nK_2 free. If $K_{j\times s}$ has a blue mK_2, then we are done with the proof. So assume $K_{j\times s}$ has no blue mK_2. Then the subgraph $K_{j\times s_0}$ where $s_0 = \left\lceil \frac{2n+m-3}{j} \right\rceil$ has no blue mK_2, so it has a red $(n-1)K_2$.

Let M^* consist of the set of all the red matchings of size $n-1$. Note that, $s = \left\lceil \frac{2n+m-1}{j} \right\rceil \geqslant \frac{2n+m-1}{j}$. Therefore as $sj - 2(n-1) \geqslant m+1$, we get that for any $M \in M^*$, $|V(M)^c| = sj - 2(n-1) \geqslant m+1$. This is illustrated in the following figure.

For each $M \in M^*$ we can construct a blue matching pK_2 using proposition 2 and proposition 3, where each edge of pK_2 connects a vertex of $V(M)$ to

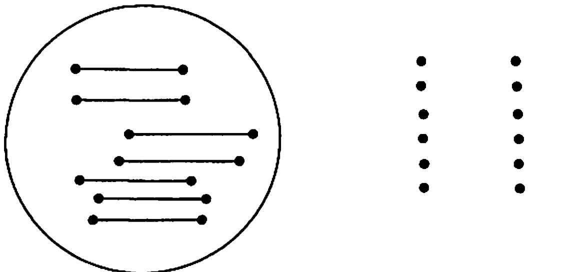

Figure 3: The $m+1$ vertices outside the red matchings of size $n-1$

a vertex belonging to $V(M)^c$ such that each edge of M is connected to at most one vertex of pK_2.
If $M \in M^*$, let $n_M = \max\{p : pK_2$ is a blue matching consisting of edges from $V(M)$ to $V(M)^c\}$.
Let $k = \max\{ n_M : M \in M^*\}$. As $j \geqslant 3$ and $m+1 \geqslant 2$, by proposition 2 and proposition 3 we get $k \geqslant 1$. Suppose kK_2 corresponds to some $M \in M^*$. Let V$(kK_2) = W$ and the edges of the blue kK_2 be denoted by (a_i, b_i), $i \in \{1, ..., k\}$ such that $a_i \in V(M)^c$ and $b_i \in V(M)$. By the definition each of the b_is are incident to a edge in M. Let c_i be the vertex of M such that (b_i, c_i), $i \in \{1, ..., k\}$ are elements of M. Since $k \leqslant m-1 \leqslant n-2$, there is a red edge $s' \in M$ that is not incident to any vertex in W. Thus we are left to consider two possible cases.

Case 1: $(V(M) \cup W)^c \subseteq V_1$ for some partite set V_1.

Case 2: $(V(M) \cup W)^c$ contained in more than one partite set of $K_{j\times s}$.

Suppose kK_2 comes under **Case 1**.
As, $sj - 2(n-1) - k \geqslant 2n + m - 1 - 2(n-1) - k = m + 1 - k \geqslant 2$, there are at least two points (say a, b) belonging to $(V(M) \cup W)^c$. Thus one of the following subcases must occur.

Subcase 1: Suppose that there exists $(b_i, c_i) \in M$ and a blue edge (b_i, a_i) of kK_2 such that $a_i, b_i, c_i \notin V_1$ (see Figure 4).
In the first scenario, if (b_i, b) is a blue edge keep the set M fixed and replace the blue edge (b_i, a_i) of the blue kK_2 by the blue edge (b_i, b). Then by applying proposition 2, to the two vertices a_i and a belonging to two distinct partite sets and the edge $s' \in M$ (found earlier such that it is not incident to any vertex in W), we will be able to increase the value of k which will contradict the maximality of k. In the second scenario, if (b_i, b) is a red edge, let $M_1 = (M \cup \{(b_i, b)\}) \backslash \{(b_i, c_i)\}$. Then as $M_1 \in M^*$, by applying

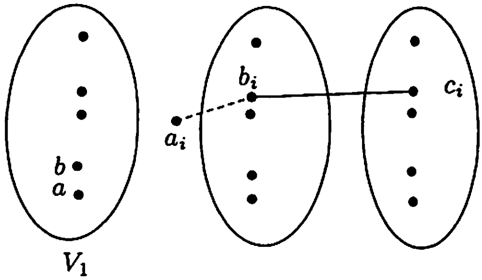

Figure 4: The graph corresponding to subcase 1

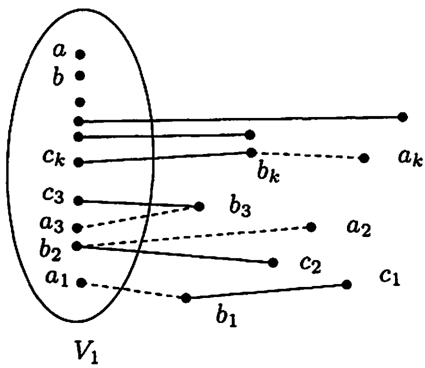

Figure 5: The graph corresponding to subcase 2

proposition 2, to the two vertices c_i and a belonging to two distinct partite sets and the edge $s' \in M$ (found earlier) we will be able to increase the value of k which will contradict the maximality of k.

Subcase 2: For each $i \in \{1, ..., k\}$, where $(b_i, c_i) \in M$ and (b_i, a_i) of kK_2 one of a_i, b_i, c_i will be in V_1. Also in order not to contradict the maximality of k, by proposition 3, every edge of M not incident to a vertex of W (there are $n-1-k$ such red edges) will be incident to some vertex of V_1.
As illustrated in the above figure, then every edge in red $(n-1)K_2$ will have a one to one correspondence with some vertex in $V_1 \backslash \{a, b\}$. Then $s-2 \geqslant n-1$. That is $\left\lceil \frac{2n+m-1}{j} \right\rceil \geqslant n+1$. Therefore, $\frac{m-1}{n} > j-2 \geqslant 1$ for any $n \geqslant m+1$. This is a contradiction.

Suppose kK_2 comes under **Case 2**. Let $a \in (V(M) \cup W)^c \cap V_1$ and let $b \in (V(M) \cup W)^c \cap V_2$ where V_1 and V_2 are distinct partite sets.
By applying proposition 2, to the two vertices a and b belonging to two

distinct partite sets and the edge $s' \in M$, we will be able to increase the value of k which will contradict the maximality of k.

□

Proposition 5. *Consider a red/blue coloring of $K_{j\times s}$ such that it has no blue nK_2. Suppose M is a blue matching of size $n-1$ of $K_{j\times s}$. If a and b are two vertices of $V(M)^c$ belonging to distinct partite sets then, given any edge (c,d) of M there exists a vertex of $\{a,b\}$, such that it is incident to a vertex of $\{c,d\}$ in red.*

We skip the proof as its similar to the proof of proposition 2.

Proposition 6. *Consider a red/blue coloring of $K_{j\times s}$ such that it has no blue nK_2. Suppose M is a blue matching of size $n-1$ of $K_{j\times s}$. If a and b are two vertices of $V(M)^c$ belonging to to the same partite set (say V_1) then, given any edge (c,d) of M not incident to V_1, there exists a vertex of $\{a,b\}$ such that it is incident to a vertex of $\{c,d\}$ in red.*

We skip the proof as its similar to the proof of proposition 3.

Lemma 7. *Suppose $j \geqslant 3$. Given that $m_j(nK_2,(n-1)K_2) = \left\lceil \frac{3n-2}{j} \right\rceil$ it follows that $m_j(nK_2,nK_2) = \left\lceil \frac{3n-1}{j} \right\rceil$, for all $n \in N$.*

Proof. Assume that $m_j(nK_2,(n-1)K_2) = \left\lceil \frac{3n-2}{j} \right\rceil$, where $j \geqslant 3$. By Lemma 1, it suffices to show $m_j(nK_2,nK_2) \leqslant \left\lceil \frac{3n-1}{j} \right\rceil$, where $j \geqslant 3$. Consider any two colouring (red and blue) of $K_{j\times s}$ where $s = \left\lceil \frac{3n-1}{j} \right\rceil$. If $K_{j\times s}$ has a red nK_2, then we are done with the proof. So assume $K_{j\times s}$ has no red nK_2. Then the subgraph $K_{j\times s_0}$ where $s_0 = \left\lceil \frac{3n-2}{j} \right\rceil$ has no red nK_2. So it has a blue $(n-1)K_2$. We now assume $K_{j\times s}$ has no blue nK_2. Let M_1^* consist of the set of all blue matchings of size $n-1$. Note that $s = \left\lceil \frac{3n-1}{j} \right\rceil \geqslant \frac{3n-1}{j}$. Therefore as $sj-2(n-1) \geqslant n+1$, we get that for any $M \in M_1^*$ $|V(M)^c| = sj-2(n-1) \geqslant n+1$. This is illustrated in the following figure.

For each $M \in M_1^*$ we can construct a red matching qK_2 using proposition 5 and proposition 6, where each edge of qK_2 connects a vertex of $V(M)$ to a vertex belonging to $V(M)^c$ such that each edge in M is connected to at most one vertex of qK_2.

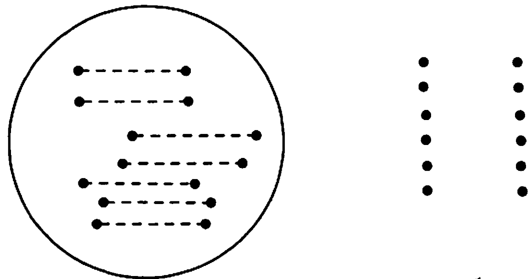

Figure 6: The $n+1$ vertices outside the blue matchings of size $n-1$

If $M \in M_1^*$, let $n'_M = \max\{q : qK_2$ is a red matching consisting of edges from $V(M)$ to $V(M)^c\}$. Let $k_1 = \max\{ n'_M : M \in M_1^*\}$. As $j \geqslant 3$ and $n+1 > 2$ by proposition 5 and proposition 6 we get $k_1 \geqslant 1$. Suppose k_1K_2 corresponds to some $M \in M_1^*$. Let V$(k_1K_2) = W_1$ and the edges of the red k_1K_2 be denoted by (a_i, b_i), $i \in \{1, ..., k_1\}$ such that $a_i \in V(M)^c$ and $b_i \in V(M)$. By the definition each of the b_is are incident to a vertex in M. Let c_i be the vertex of M such that (b_i, c_i), $i \in \{1, ..., k_1\}$ are elements of M.

Case 1: $(V(M) \cup W_1)^c \subseteq V_1$ for some partite set V_1.
Case 2: $(V(M) \cup W_1)^c$ contained in more than one partite set of $K_{j \times s}$.

Suppose k_1K_2 comes under **Case 1**. Then there are at least two points (say a, b) belonging to $(V(M) \cup W_1)^c$ since $sj - 2(n-1) - k_1 \geqslant n+1-k_1 \geqslant 2$. Thus one of the following subcases must occur.

Subcase 1: Suppose that there exists $(b_i, c_i) \in M$ and a red edge (b_i, a_i) of k_1K_2 such that $a_i, b_i, c_i \notin V_1$ (see Figure 7).
Suppose that $k_1 = n-1$. First note that, (a, a_i) and (b, a_i) have to be red in order to avoid a blue nK_2. Next (a, c_i) and (b, c_i) have to be blue in order to avoid a red nK_2. But then (a, b_i) cannot be a red edge as it will force a red nK_2 and (a, b_i) cannot be a blue edge as it will force a blue nK_2. Therefore $k_1 = n-1$ cannot occur.
Therefore, we may assume that, $k_1 < n-1$. Then there is a blue edge $s' \in M$ that is not incident to any vertex in W_1. In the first scenario, if (b_i, b) is a red edge keep the set M fixed and replace the red edge (b_i, a_i) of the red k_1K_2 by the red edge (b_i, b).
Then by applying proposition 5 to the two vertices a_i and a belonging to two distinct partite sets and the edge $s' \in M$ (found earlier such that it is not incident to any vertex in W_1), we will be able to increase the value of k_1 which will contradict the maximality of k_1. In the second scenario, if

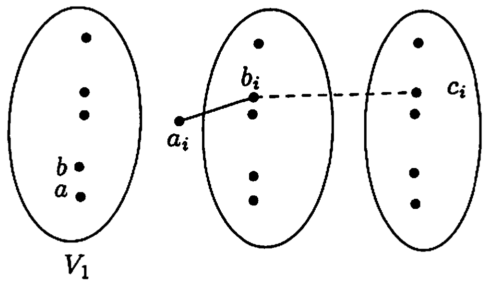

Figure 7: The graph corresponding to subcase 1

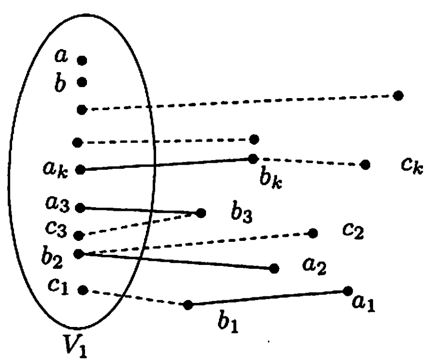

Figure 8: The graph corresponding to subcase 2

(b_i, b) is a blue edge, let $M_1 = (M \cup \{(b_i, b)\}) \backslash \{(b_i, c_i)\}$. Then as $M_1 \in M_1^*$, by applying proposition 5, to the two vertices c_i and a belonging to two distinct partite sets and the edge $s' \in M$ (found earlier), we will be able to increase the value of k_1 which will contradict the maximality of k_1.

Subcase 2: For each $i \in \{1, ..., k_1\}$, where $(b_i, c_i) \in M$ and (b_i, a_i) of $k_1 K_2$ one of a_i, b_i, c_i will be in V_1. Also in order not to contradict the maximality of k_1, by proposition 6, every edge of M not incident to a vertex of W_1 (there are $n - 1 - k_1$ such blue edges) will be incident to some vertex of V_1. That is every edge in M will have a one to one correspondence with some vertex in $V_1 \backslash \{a, b\}$. This is illustrated in the above figure. Then $s - 2 \geqslant n - 1$. That is $\left\lceil \frac{3n-1}{j} \right\rceil \geqslant n + 1$. Therefore, $\frac{3n-1}{n} > j \geqslant 3$. This is a contradiction.

Suppose $k_1 K_2$ comes under **Case 2.**
Then let a and b be two points of $(V(M) \cup W_1)^c$ belonging to two partite sets of $K_{j \times s}$ namely V_1 and V_2 respectively.
Suppose that $k_1 = n - 1$. Then (a, b) cannot be a red edge as it will force a red nK_2 and (a, b) cannot be a blue edge as it will force a blue nK_2. Therefore $k_1 = n - 1$ cannot occur.
Therefore, $k_1 < n-1$. Then there is a blue edge $s' \in M$ that is not incident to any vertex in W_1. Applying proposition 5 to the blue edge $s' \in M$ with respect to the points a and b we can increase the value of k_1, which will contradict the maximality of k_1. □

Theorem 8. *If* $m \leqslant n$ *then,*

$$m_j(nK_2, mK_2) = \begin{cases} n + m - 1 & \text{if } j = 2 \\ \left\lceil \dfrac{2n + m - 1}{j} \right\rceil & \text{if } j \geqslant 3 \end{cases}$$

Proof. The result corresponding to $j = 2$ follows from [2]. So we are left to prove $m_j(nK_2, mK_2) = \left\lceil \dfrac{2n + m - 1}{j} \right\rceil$ for $j \geqslant 3$. When $j \geqslant 3$ clearly result is true for $n = m = 1$ as $m_j(K_2, K_2) = 1$. By induction on n (using lemma 4 and lemma 7) the result follows. □

References

[1] A.P. Burger and J.H. van Vuuren, *Ramsey numbers in Complete Balanced Multipartite Graphs. Part II: Size Numbers*, Discrete Math. **283** (2004), 45-49.

[2] M. Christou, S. Iliopoulos and M. Miller *Bipartite Ramsey numbers involving stars, stripes and trees*, Electronic Journal of Graph Theory and Applications. **1(2)** (2013), 89-99.

[3] Syafrizal Sy, E.T. Baskoro and S. Uttunggadewa, *The size multipartite Ramsey number for paths*, Journal Combin. Math. Combin. Comput. **55** (2005), 103-107.

[4] Syafrizal Sy, E.T. Baskoro and S. Uttunggadewa, *The size multipartite Ramsey numbers for small paths versus other graphs*, Far East Journal Appl. Math. **28**, Issue 1 (2007), 131-138.

[5] Syafrizal Sy, *On size multipartite Ramsey numbers for paths versus cycles of three or four vertices*, Far East Journal Appl. Math. Volume **44** No. 2 July Issue (2010), 109-116.