New inequalities on the hyperbolicity constant of line graphs

Walter Carballosa[1], José M. Rodríguez[1], and José M. Sigarreta[2]

[1]Departamento de Matemáticas
Universidad Carlos III de Madrid,
Av. de la Universidad 30, 28911 Leganés, Madrid, Spain
waltercarb@gmail.com, jomaro@math.uc3m.es

[2]Facultad de Matemáticas
Universidad Autónoma de Guerrero,
Carlos E. Adame 5, Col. La Garita, Acapulco, Guerrero, México.
jsmathguerrero@gmail.com

Abstract

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is *δ-hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) := \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. The main result of this paper is the inequality $\delta(G) \leq \delta(\mathcal{L}(G))$ for the line graph $\mathcal{L}(G)$ of every graph G. We prove also the upper bound $\delta(\mathcal{L}(G)) \leq 5\delta(G) + 3l_{max}$, where l_{max} is the supremum of the lengths of the edges of G. Furthermore, if every edge of G has length k, we obtain $\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\delta(G) + 5k/2$.

Keywords: Infinite Graphs; Line Graphs; Geodesics; Gromov Hyperbolicity.

AMS Subject Classification numbers: 05C69; 05A20; 05C50

1 Introduction.

Hyperbolic spaces play an important role in the geometric group theory and in the geometry of negatively curved spaces (see [1, 22, 23]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 22, 23]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory;

see, for instance [3, 4, 5, 9, 10, 11, 20, 28, 29, 30, 31, 32, 33, 36, 37, 39, 40, 41, 42, 46, 47, 48, 50, 52].

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [23] and the references therein), where it was demonstrated to have practical importance. This theory was applied principally to the study of automatic groups (see [38]), which play a role in the science of computation. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [49] that the internet topology embeds with better accuracy into a hyperbolic space than into a Euclidean space of comparable dimension. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [14, 18, 21, 35]). Another important application of these spaces is secure transmission of information by internet (see [28, 29, 30]). In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see [29, 30]). The hyperbolicity is also useful in the study of DNA data (see [9]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [25]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [2, 6, 26, 27, 42, 43, 44, 47, 48]. In particular, in [42, 47, 48, 50] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

In our study on hyperbolic graphs we use the notations of [22]. Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We say that γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L denotes the length of a curve. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If the metric space X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, we must identify any edge $[u, v] \in E(G)$ with the real interval $[0, l]$ (if $l := L([u, v])$); therefore, any point in the interior of any edge is a point of G and, if we consider the edge $[u, v]$ as a graph with just one edge, then it is isometric to $[0, l]$. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Throughout the paper we just consider simple (without loops and multiple edges) connected and locally finite (i.e., in each ball there are just a finite number of edges) graphs; these properties guarantee that the graphs are geodesic metric spaces. Note that the edges can have arbitrary lengths. We want to remark that by [4, Theorems 8 and 10] the study of the hyperbolicity of graphs with loops and multiple edges can be reduced to the study of the hyperbolicity of simple graphs.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$; sometimes we write the geodesic triangle T as $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. One can check that every geodesic polygon in X with n sides is $(n-2)\delta(X)$-thin; in particular, any geodesic quadrilateral is $2\delta(X)$-thin.

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see, e.g., [7, 22]). We have chosen this definition since it has a deep geometric meaning (see, e.g., [22]).

The following are interesting examples of hyperbolic spaces. The real line $\mathbb{R}$ is 0-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that $\mathbb{R}$ is 0-hyperbolic. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter, so that $\mathbb{R}^2$ with the Euclidean metric is not hyperbolic. This argument can be generalized in a similar way to higher dimensions: a normed vector space E is hyperbolic if and only if $\dim E = 1$. Every metric tree with arbitrary length edges is 0-hyperbolic: in fact, all points of a geodesic triangle in a tree belongs simultaneously to two sides of the triangle. Every bounded metric space X is $(\operatorname{diam} X)/2$-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -c^2$, for some positive constant c, is hyperbolic. We refer to [7, 22] for more background and further results.

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces

X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [12]).

Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic. However, for a general graph or a general geodesic metric space deciding whether or not a space is hyperbolic is usually very difficult. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant.

It is a remarkable fact that the constants appearing in many results in the theory of hyperbolic spaces depend just on a small number of parameters (also, this is common in the theory of negatively curved surfaces). Usually, there is no explicit expression for these constants. Even though sometimes it is possible to estimate the constants, those explicit values obtained are, in general, far from being sharp (see, e.g., Theorem 2.1 and (1.1) below).

The main result of this paper is the inequality $\delta(G) \le \delta(\mathcal{L}(G))$ for the line graph $\mathcal{L}(G)$ of every graph G (see Theorem 3.10).

Line graphs were initially introduced in the papers [51] and [34], although the terminology of line graph was used in [24] for the first time.

There are previous results relating the hyperbolicity constant of the line graph $\mathcal{L}(G)$ with the hyperbolicity constant of the graph G. In [11, Theorem 2.4] the authors obtain the inequalities

$$\frac{1}{12}\,\delta(G) - \frac{3}{4} \le \delta(\mathcal{L}(G)) \le 12\,\delta(G) + 18, \tag{1.1}$$

for graphs G with edges of length 1. This result allows to obtain the main qualitative result of [11]: the line graph of G is hyperbolic if and only if G is hyperbolic. Although the multiplicative and additive constants appearing in (1.1) allow to prove this main result, it is a natural problem to improve the inequalities in (1.1). In this paper we also improve the second inequality; in fact, Theorem 3.10 states

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\,\delta(G) + 3 \sup_{e\in E(G)} L(e), \tag{1.2}$$

where here the edges of G can have arbitrary lengths. The second inequality in (1.2) can be improved for graphs with edges of length k (see Corollary 3.12) in the following way:

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 5k/2.$$

Also, we obtain for graphs with edges of length k other inequalities involving the hyperbolicity constant of $\mathcal{L}(G)$ (see Theorem 3.14 and Corollary 3.15).

2 Background and previous results.

Let G be a graph such that its edges $E(G) = \{e_i\}_{i\in\mathcal{I}}$ have arbitrary lengths. The *line graph* $\mathcal{L}(G)$ of G is a graph which has a vertex $V_{e_i} \in V(\mathcal{L}(G))$ for each edge e_i of G, and an edge joining V_{e_i} and V_{e_j} when $e_i \cap e_j \neq \varnothing$. Note that we have a complete subgraph K_n in $\mathcal{L}(G)$ corresponding to one vertex v of G with degree $\deg_G(v) = n$. Some authors define the edges of line graph with length 1 or another fixed constant, but we define the length of the edge $[V_{e_i}, V_{e_j}] \in E(\mathcal{L}(G))$ as $(L(e_i) + L(e_j))/2$. Note that if every edge in G has length k, then every edge in $\mathcal{L}(G)$ also has length k.

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α,β)-*quasi-isometric embedding*, with constants $\alpha \geq 1$, $\beta \geq 0$ if we have for every $x, y \in X$:

$$\alpha^{-1} d_X(x,y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x,y) + \beta.$$

We say that f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry* if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that f is a ε-full (α, β)-quasi-isometric embedding.

Two metric spaces X and Y are *quasi-isometric* if there exist a quasi-isometry $f : X \longrightarrow Y$.

A fundamental property of hyperbolic spaces is the following:

Theorem 2.1 (Invariance of hyperbolicity). *Let (X, d_X), (Y, d_Y) be two geodesic metric spaces and $f : X \longrightarrow Y$ an (α,β)-quasi-isometry embedding.*

i) If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which just depends on δ, α and β.

ii) If f is ε-full, then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if X is δ'-hyperbolic, then Y is δ-hyperbolic, where δ is a constant which just depends on δ', α, β and ε.

We will need the following result (see [47, Lemma 2.1]):

Lemma 2.2. *Let us consider a geodesic metric space X. If every geodesic triangle in X which is a simple closed curve, is δ-thin, then X is δ-thin.*

This lemma has the following direct consequence. As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices in a graph, except for the last one, which is equal to the first vertex.

Corollary 2.3. *In any graph G,*

$$\delta(G) = \sup\{\delta(T) : T \textit{ is a geodesic triangle that is a cycle}\}.$$

The next result follows from $\delta(X) \le (\operatorname{diam} X)/2$ (see [46, Theorem 11] for a detailed proof).

Theorem 2.4. *The cycle graphs with every edge of length* 1 *verify* $\delta(C_n) = n/4$ *for every* $n \ge 3$.

This theorem has the following direct consequence.

Corollary 2.5. *Any cycle graph C verifies* $\delta(C) = L(C)/4$.

In this work, $PMV(G)$ will denote the set of points of the graph G which are either vertices or midpoints of its edges.

We will use the following result (see [3, Theorem 2.7]).

Theorem 2.6. *For any hyperbolic graph G with edges of length k, there exists a geodesic triangle* $T = \{x, y, z\}$ *that is a cycle with* $\delta(T) = \delta(G)$ *and* $x, y, z \in PMV(G)$.

3 Inequalities involving the hyperbolicity constant of line graphs.

We obtain in this section the results on the hyperbolicity constant of a line graph with edges of arbitrary lengths. The main result in this section is Theorem 3.10, which states

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 3l_{max},$$

with $l_{max} = \sup_{e \in E(G)} L(e)$.

For the sake of clarity and readability, we have opted to state and prove several preliminary lemmas. This makes the proof of Theorem 3.10 much more understandable.

Let us consider $Pm(e)$ the midpoint of $e \in E(G)$; also, we denote by $PM(G)$ the set of the midpoints of the edges of G, i.e., $PM(G) := \{Pm(e)/e \in E(G)\}$. Besides, let us consider $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ the point in $[V_{e_i}, V_{e_j}] \in E(\mathcal{L}(G))$ with $L([V_{e_i} Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])]) = L(e_i)/2$ (and then

$L([Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])V_{e_j}]) = L(e_j)/2)$. Analogously, we denote $PM_{\mathcal{L}}(\mathcal{L}(G))$ the set of these points in each edge of $\mathcal{L}(G)$, i.e., $PM_{\mathcal{L}}(\mathcal{L}(G)) := \{Pm_{\mathcal{L}}(e)/e \in E(\mathcal{L}(G))\}$. Note that $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ is the midpoint of $[V_{e_i}, V_{e_j}]$ when $L(e_i) = L(e_j)$; thus, if every edge of G has the same length then $PM_{\mathcal{L}}(\mathcal{L}(G))$ is the set of midpoints of the edges of $\mathcal{L}(G)$.

Let us consider the sets $PMV(G) := PM(G) \cup V(G)$ and $PM_{\mathcal{L}}V(\mathcal{L}(G)) := PM_{\mathcal{L}}(\mathcal{L}(G)) \cup V(\mathcal{L}(G))$.

We define a function $h : PM_{\mathcal{L}}V(\mathcal{L}(G)) \longrightarrow PMV(G)$ as follows: for every vertex V_e of $V(\mathcal{L}(G))$, the image via h of V_e is $Pm(e)$, and for every $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ in $PM_{\mathcal{L}}(\mathcal{L}(G))$, the image via h of $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ is the vertex $e_i \cap e_j$ in $V(G)$, i.e.,

$$h(x) := \begin{cases} Pm(e), & \text{if } x = V_e \in V(\mathcal{L}(G)), \\ e_i \cap e_j, & \text{if } x = Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}]) \in PM_{\mathcal{L}}(\mathcal{L}(G)). \end{cases} \tag{3.3}$$

Remark 3.1. *If $x \in PM(G)$, then $h^{-1}(x)$ is a single point, but otherwise, $h^{-1}(x)$ can have more than one point.*

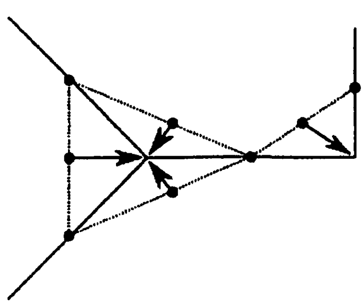

Figure 1: Graphical view of h.

The function h defined in (3.3) can be extended to $\mathcal{L}(G)$. Note that every point $x_0 \in \mathcal{L}(G) \setminus PM_{\mathcal{L}}V(\mathcal{L}(G))$ is located in $\mathcal{L}(G)$ between one vertex V_e and one point $Pm_{\mathcal{L}}(V_eV_{e_0})$. For each $x_0 \in \text{int}([V_ePm_{\mathcal{L}}([V_eV_{e_0}])])$ we define $h(x_0)$ as the point $x \in \text{int}[Pm(e)h(Pm_{\mathcal{L}}([V_eV_{e_0}]))]$ such that $L([xPm(e)]) = L([x_0V_e])$; hence, $L([x_0V_e]) = L([h(x_0)h(V_e)])$ and $L([x_0Pm_{\mathcal{L}}(V_eV_{e_0})]) = L([h(x_0)h(Pm_{\mathcal{L}}(V_eV_{e_0}))])$.

In what follows we denote by h this extension.

We call *half-edge* in G a geodesic contained in an edge with an endpoint in $V(G)$ and an endpoint in $PM(G)$; similarly, a *half-edge* in $\mathcal{L}(G)$ is a geodesic contained in an edge with an endpoint in $V(\mathcal{L}(G))$ and an endpoint in $PM_{\mathcal{L}}(\mathcal{L}(G))$.

Proposition 3.2. *h is an 1-Lipschitz continuous function, i.e.,*

$$d_G(h(x), h(y)) \leq d_{\mathcal{L}(G)}(x, y)\,, \quad \forall\, x, y \in \mathcal{L}(G). \tag{3.4}$$

Proof. First of all note that, by definition of $\mathcal{L}(G)$, we have for every $x' \in h(\mathcal{L}(G)) \cap PMV(G)$,

$$|h^{-1}(x')| = \begin{cases} 1, & \text{if } x' \in PM(G), \\ \deg_G(x')(\deg_G(x') - 1)/2, & \text{if } x' \in V(G). \end{cases}$$

In order to prove (3.4), we verify that

$$d_G(x', y') = d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y'))\,, \quad \forall\, x', y' \in h(\mathcal{L}(G)) \cap PMV(G). \tag{3.5}$$

We study separately the different cases of $x', y' \in h(\mathcal{L}(G)) \cap PMV(G)$.

Case 1 $x', y' \in PM(G)$.

Let us consider $x' := Pm(e_i)$ and $y' := Pm(e_j)$ with $e_i, e_j \in E(G)$, and define $d := d_G(Pm(e_i), Pm(e_j)) \geq 0$.

If $d = 0$, then $e_i = e_j$, so, $h^{-1}(Pm(e_i)) = h^{-1}(Pm(e_j))$ and $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = 0$.

If $d > 0$, then $e_i \neq e_j$ and $d_G(Pm(e_i), Pm(e_j)) = (L(e_i) + L(e_j))/2 + d_G(e_i, e_j)$. Note that, if $d_G(e_i, e_j) = 0$ and $e_i \neq e_j$, then $d_G(x', y') = (L(e_i) + L(e_j))/2 = d_{\mathcal{L}(G)}(V_{e_i}, V_{e_j}) = d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y'))$. If $d_G(e_i, e_j) > 0$, then a geodesic γ joining e_i and e_j in G contains the edges $e_{i_1}, \dots, e_{i_r}$ in this order, with $r \geq 1$. Now, we have that $d_G(e_i, e_j) = \sum_{k=1}^{r} L(e_{i_k})$; hence, $V_{e_i} V_{e_{i_1}} \dots V_{e_{i_r}} V_{e_j}$ is a path joining V_{e_i} and V_{e_j} with length d. So, $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) \leq d$.

We prove now that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d$. Seeking for a contradiction, assume that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_{\mathcal{L}(G)}(V_{e_i}, V_{e_j}) < d$. Hence, there exists $V_{e_{j_1}}, \dots, V_{e_{j_m}}$ such that $V_{e_i} V_{e_{j_1}} \dots V_{e_{j_m}} V_{e_j}$ is a geodesic in $\mathcal{L}(G)$ joining V_{e_i} and V_{e_j} with length $(L(e_i) + L(e_j))/2 + \sum_{k=1}^{m} L(e_{j_k}) < d$. Since $d = (L(e_i) + L(e_j))/2 + d_G(e_i, e_j)$, we have $\sum_{k=1}^{m} L(e_{j_k}) < d_G(e_i, e_j)$. By definition of $\mathcal{L}(G)$ we have that $\gamma^* := e_{j_1} \cup \dots \cup e_{j_m}$ is a path in G joining e_i and e_j with length $\sum_{k=1}^{m} L(e_{j_k}) < d_G(e_i, e_j)$. This is the contradiction we were looking for; so we have $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_G(x', y')$.

Case 2 $x' \in PM(G)$ and $y' \in V(G)$.

Let us consider $x' := Pm(e)$ with $e \in E(G)$ and $y' \in V(G) \setminus \{w \in V(G)/\deg_G(w) = 1\}$, and define $d := d_G(e, y')$; then $d_G(Pm(e), v) = d + L(e)/2$. Note that if $y' \in V(G)$ and $\deg_G(y') = 1$, then $y' \notin h(\mathcal{L}(G))$.

If $d = 0$, then y is an endpoint of e and $d_{\mathcal{L}(G)}(V_e, h^{-1}(y')) = L(e)/2$; note that $|h^{-1}(y')| = \deg_G(y')[\deg_G(y') - 1]/2$, where $|A|$ denotes the cardinality of the set A.

If $d_G(e, y') = d > 0$, then there exist $e_{i_1}, \ldots, e_{i_r} \in E(G)$ such that $\gamma := e_{i_1} \cup \ldots \cup e_{i_r}$ is a geodesic joining e and y' in G with length $d = \sum_{k=1}^{r} L(e_{i_k})$. Note that e, e_{i_1} are different and adjacent edges. So, we have that $V_e V_{e_{i_1}} \ldots V_{e_{i_r}}$ is a path in $\mathcal{L}(G)$ joining V_e and $V_{e_{i_r}}$ with length $L(e)/2 + \sum_{k=1}^{r} L(e_{i_k}) - L(e_{i_r})/2$. Since y' is an endpoint of e_{i_r}, we have $d_{\mathcal{L}(G)}(h^{-1}(y'), V_{e_{i_r}}) = L(e_{i_r})/2$ and $d_{\mathcal{L}(G)}(h^{-1}(y'), V_e) \leq d + L(e)/2$.

We prove now that $d_{\mathcal{L}(G)}(h^{-1}(y'), V_e) = d + L(e)/2$. Seeking for a contradiction, assume that $d_{\mathcal{L}(G)}(h^{-1}(y'), V_e) < d + L(e)/2$. Hence, there exists $V_{e_{j_1}}, \ldots, V_{e_{j_m}}$ such that $V_e V_{e_{j_1}} \ldots V_{e_{j_m}} \cup [V_{e_{j_m}} z]$ is a geodesic of $\mathcal{L}(G)$ joining V_e and $z \in h^{-1}(y')$ with length $L(e)/2 + \sum_{k=1}^{m} L(e_{j_k}) < d + L(e)/2$. We have $z = Pm_{\mathcal{L}}([V_{e_{j_m}}, V_{e_s}])$ with e_{j_m}, e_s edges in G starting in y'. By definition of $\mathcal{L}(G)$ we have that $\gamma^* := e_{j_1} \cup \ldots \cup e_{j_m}$ contains a path in G joining e and y' with length at most $\sum_{k=1}^{m} L(e_{j_k}) < d$. This is the contradiction we were looking for; so we have $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_G(x', y')$.

Case 3 $x', y' \in V(G)$.

Let us consider $x', y' \in V(G) \setminus \{v \in V(G)/\deg_G(v) = 1\}$, and define $d := d_G(x', y') \geq 0$.

If $d = 0$, then $x' = y'$, so $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = 0$.

If $d_G(x', y') = d > 0$, then there exists $e_{i_1}, \ldots, e_{i_r} \in E(G)$ such that $\gamma := e_{i_1} \cup \ldots \cup e_{i_r}$ is a geodesic joining x' and y' in G with length $d = \sum_{k=1}^{r} L(e_{i_k})$. So, we have that there exist $a \in h^{-1}(x')$ and $b \in h^{-1}(y')$ such that $[aV_{e_{i_1}}] \cup V_{e_{i_1}} \ldots V_{e_{i_r}} \cup [V_{e_{i_r}} b]$ is a path in $\mathcal{L}(G)$ joining a and b with length $\sum_{k=1}^{r} L(e_{i_k}) = d$. Then, we have that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) \leq d$.

We prove now that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d$. Seeking for a contradiction, assume that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) < d$. Hence, there exist $\alpha \in h^{-1}(x')$, $\beta \in h^{-1}(y')$ and $V_{e_{j_1}}, \ldots, V_{e_{j_m}}$ vertices in $\mathcal{L}(G)$ such that $[\alpha V_{e_{j_1}}] \cup V_{e_{j_1}} \ldots V_{e_{j_m}} \cup [V_{e_{j_m}} \beta]$ is a geodesic joining α and β in $\mathcal{L}(G)$ with length $\sum_{k=1}^{m} L(e_{j_k}) < d$. We have $\alpha = Pm_{\mathcal{L}}([V_{e_s^1}, V_{e_{j_1}}])$ with e_{j_1}, e_s^1 edges in G starting in x', and $\beta = Pm_{\mathcal{L}}([V_{e_{j_m}}, V_{e_s^2}])$ with

e_{j_m}, e_s^2 edges in G starting in y'. By definition of $\mathcal{L}(G)$ we have that $\gamma^* := e_{j_1} \cup \ldots \cup e_{j_m}$ contains a path in G joining x' and y' with length at most $\sum_{k=1}^{m} L(e_{j_k}) < d$. This is the contradiction we were looking for; so we have $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_G(x', y')$.

This prove (3.5) and guarantees (3.4) for $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$ when we take $x := h(x')$ and $y := h(y')$. We know that there exist $X_1, X_2, Y_1, Y_2 \in PM_{\mathcal{L}}V(\mathcal{L}(G))$ with $x \in [X_1, X_2]$ and $y \in [Y_1, Y_2]$ such that $d_{\mathcal{L}(G)}(x, X_1) = \varepsilon_x$, $d_{\mathcal{L}(G)}(x, X_2) = \delta_x$, $d_{\mathcal{L}(G)}(y, Y_1) = \varepsilon_y$, $d_{\mathcal{L}(G)}(y, Y_2) = \delta_y$ and $[X_1 X_2]$, $[Y_1 Y_2]$ are two half-edges in $\mathcal{L}(G)$. Hence, we have $h(x) \in [h(X_1)h(X_2)]$, $h(y) \in [h(Y_1)h(Y_2)]$ with $d_G(h(x), h(X_1)) = \varepsilon_x$, $d_G(h(x), h(X_2)) = \delta_x$, $d_G(h(y), h(Y_1)) = \varepsilon_y$ and $d_G(h(y), h(Y_2)) = \delta_y$; besides $[h(X_1)h(X_2)]$ and $[h(Y_1)h(Y_2)]$ are two half-edges in G.

Note that if $[X_1 X_2] = [Y_1 Y_2]$, then $d_G(h(x), h(y)) = d_{\mathcal{L}(G)}(x, y)$. Otherwise, we have

$$d_{\mathcal{L}(G)}(x,y) = \min \left\{ \begin{array}{l} d_{\mathcal{L}(G)}(X_1, Y_1) + \varepsilon_x + \varepsilon_y, \\ d_{\mathcal{L}(G)}(X_1, Y_2) + \varepsilon_x + \delta_y, \\ d_{\mathcal{L}(G)}(X_2, Y_1) + \delta_x + \varepsilon_y, \\ d_{\mathcal{L}(G)}(X_2, Y_2) + \delta_x + \delta_y \end{array} \right\} \qquad (3.6)$$

and

$$d_G(h(x), h(y)) = \min \left\{ \begin{array}{l} d_G(h(X_1), h(Y_1)) + \varepsilon_x + \varepsilon_y, \\ d_G(h(X_1), h(Y_2)) + \varepsilon_x + \delta_y, \\ d_G(h(X_2), h(Y_1)) + \delta_x + \varepsilon_y, \\ d_G(h(X_2), h(Y_2)) + \delta_x + \delta_y \end{array} \right\}. \qquad (3.7)$$

Let us consider X_i, Y_j with $i, j \in \{1, 2\}$, $\alpha \in \{\varepsilon_x, \delta_x\}$ and $\beta \in \{\varepsilon_y, \delta_y\}$ such that $d_{\mathcal{L}(G)}(x, y) = d_{\mathcal{L}(G)}(X_i, Y_j) + \alpha + \beta$. Hence, by (3.5) we have

$$\begin{aligned} d_{\mathcal{L}(G)}(x,y) &= d_{\mathcal{L}(G)}(X_i, Y_j) + \alpha + \beta, \\ &\geq d_G(h(X_i), h(Y_j)) + \alpha + \beta, \\ &\geq d_G(h(x), h(y)). \end{aligned}$$

□

The following result is a consequence of Proposition 3.2.

Remark 3.3. *Let x and y be in $V(\mathcal{L}(G))$, then we have that*

$$d_{\mathcal{L}(G)}(x,y) = d_G(h(x), h(y)).$$

We also have a kind of reciprocal of Proposition 3.2.

Lemma 3.4. *For every $x, y \in \mathcal{L}(G)$ we have*

$$d_{\mathcal{L}(G)}(x,y) \le d_G(h(x),h(y)) + 2l_{max}, \tag{3.8}$$

where $l_{max} := \sup_{e\in E(G)} L(e)$.

Proof. First of all, we prove (3.8) for $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$. In order to prove it, we can assume that $\operatorname{diam}_{\mathcal{L}(G)} h^{-1}(h(x)), \operatorname{diam}_{\mathcal{L}(G)} h^{-1}(h(y)) > 0$ (i.e., $h(x), h(y) \in V(G)$ and $\deg_G(h(x)), \deg_G(h(x)) > 2$), since otherwise the argument is easier. Thus, by definition of $\mathcal{L}(G)$ we have a complete subgraph $K_{\deg(v)}$ associated to $v \in V(G)$ and $h^{-1}(v) = PM_{\mathcal{L}}(\mathcal{L}(G)) \cap K_{\deg(v)}$. Let us choose $x'' \in h^{-1}(h(x))$, $y'' \in h^{-1}(h(y))$ with $d_{\mathcal{L}(G)}(x'',y'') = d_{\mathcal{L}(G)}(h^{-1}(h(x)), h^{-1}(h(y)))$. Consider a geodesic γ joining x'' and y'' in $\mathcal{L}(G)$. Let V_1 (respectively, V_2) be the closest vertex to x'' (respectively, y'') in γ. It is easy to check that, since $h^{-1}(v) = PM_{\mathcal{L}}(\mathcal{L}(G)) \cap K_{\deg(v)}$ and $L([V_{e_i}, V_{e_j}]) = (L(e_i) + L(e_j))/2$, we have

$$d_{\mathcal{L}(G)}(V_1,x) \le d_{\mathcal{L}(G)}(V_1,x'') + \sup_{e\in E(G)} L(e),$$

$$d_{\mathcal{L}(G)}(V_2,y) \le d_{\mathcal{L}(G)}(V_2,y'') + \sup_{e\in E(G)} L(e),$$

and since

$$d_{\mathcal{L}(G)}(x'',V_1) + d_{\mathcal{L}(G)}(V_1,V_2) + d_{\mathcal{L}(G)}(V_2,y'') = d_{\mathcal{L}(G)}(x'',y''),$$

we deduce (3.8) for $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$.

Now, let us consider $X_{i'}, Y_{j'}$ with $i', j' \in \{1,2\}$, $\alpha' \in \{\varepsilon_x, \delta_x\}$ and $\beta' \in \{\varepsilon_y, \delta_y\}$ such that $d_G(h(x),h(y)) = d_G(h(X_{i'}), h(Y_{j'})) + \alpha' + \beta'$. Hence, we have

$$\begin{aligned} d_G(h(x),h(y)) &= d_G(h(X_{i'}),h(Y_{j'})) + \alpha' + \beta', \\ &\ge d_{\mathcal{L}(G)}(X_{i'},Y_{j'}) - 2l_{max} + \alpha' + \beta', \end{aligned}$$

finally, (3.6) gives the condition. □

It is easy to see that $G \setminus h(\mathcal{L}(G))$ is the union of the half-edges of G such that one of its vertices has degree 1; thus the following fact holds.

Remark 3.5. *h is a $(l_{max}/2)$-full $(1, 2l_{max})$-quasi-isometry with $l_{max} = \sup_{e\in E(G)} L(e)$.*

Now, let us consider a cycle C in G. We define $g_C : C \longrightarrow \mathcal{L}(G)$ in the following way; $g_C(Pm(e)) := V_e$ for $e \in E(G) \cap C$; if C^* is the cycle in $\mathcal{L}(G)$ with vertices $\cup_{e \in E(G)} g_C(Pm(e))$, then one can check that $h|_{C^*} : C^* \longrightarrow C$ is a bijection; we define

$$g_C := (h|_C^*)^{-1} : C \longrightarrow C^*. \tag{3.9}$$

Corollary 3.6. *Let C be a geodesic polygon in a graph G that is a cycle and let g_C be the function defined by (3.9). Then, $C^* := g_C(C)$ is a geodesic polygon in $\mathcal{L}(G)$ with the same number of edges than C.*

Furthermore, if γ is a geodesic in C, then $g_C(\gamma)$ is a geodesic in $\mathcal{L}(G)$ with $L(g_C(\gamma)) = L(\gamma)$.

Proof. First of all, note that $L(C) = L(C^*)$ since if $E(C) = \{e_1, \dots, e_n\}$ with $e_1 \cap e_n \neq \varnothing$ and $e_i \cap e_{i+1} \neq \varnothing$ for $1 \leq i < n$, then $L(C) = \sum_{i=1}^{n} L(e_i)$ and $L(C^*) = L(e_1)/2 + \sum_{i=1}^{n-1}(L(e_i) + L(e_{i+1}))/2 + L(e_n)/2$.

Now, let us consider a geodesic γ in C joining x and y. Since $g_C(\gamma)$ is a path joining $g_C(x)$ and $g_C(y)$, we have that $d_{\mathcal{L}(G)}(g_C(x), g_C(y)) \leq d_{C^*}(g_C(x), g_C(y)) = d_G(x, y)$; Proposition 3.2 gives $d_{\mathcal{L}(G)}(g_C(x), g_C(y)) \geq d_G(h(g_C(x)), h(g_C(y))) = d_G(x, y)$. Then we obtain that

$$d_{\mathcal{L}(G)}(g_C(x), g_C(y)) = d_G(x, y).$$

Since γ is an arbitrary geodesic in C we obtain that g_C maps geodesics in G (contained in C) in geodesics in $\mathcal{L}(G)$ (contained in C^*). □

Now, we deal with the geodesics in $\mathcal{L}(G)$.

Lemma 3.7. *Let γ^* be a geodesic joining x and y in $\mathcal{L}(G)$. Then $h(\gamma^*)$ is a path in G joining $h(x)$ and $h(y)$, which is the union of three geodesics γ_1, γ_2, γ_3 in G, with $h(x) \in \gamma_1$, $h(y) \in \gamma_3$ and $0 \leq L(\gamma_1), L(\gamma_3) < \sup_{e \in E(G)} L(e)$.*

Proof. Note that if x, y are contained in one edge $[V_1, V_2]$ of $\mathcal{L}(G)$, then $\gamma^* \subset [V_1, V_2]$ and $h(\gamma^*)$ is a geodesic in G joining $h(x)$ and $h(y)$, since $h(\gamma^*) \subset \gamma := [h(V_1)h(Pm_{\mathcal{L}}([V_1, V_2]))] \cup [h(Pm_{\mathcal{L}}([V_1, V_2]))h(V_2)]$ and γ is a geodesic in G by Remark 3.3.

If x, y are not contained in the same edge of $\mathcal{L}(G)$, then let us consider V_α as the closest vertex in γ^* to α, for $\alpha \in \{x, y\}$ (it is possible to have $V_x = x$ or $V_y = y$). By Remark 3.3, we have that $h([V_x V_y]) = [h(V_x)h(V_y)]$ is a geodesic joining $h(V_x)$ and $h(V_y)$ in G; moreover, $h(\gamma^*) = [h(x)h(V_x)] \cup [h(V_x)h(V_y)] \cup [h(V_y)h(y)]$ where $[h(x)h(V_x)]$ and $[h(V_y)h(y)]$ are geodesics in G since x, V_x (respectively y, V_y) are contained in the same edge of

$\mathcal{L}(G)$. This finishes the proof, since $L(e^*) \le \sup_{e\in E(G)} L(e)$ for every $e^* \in E(\mathcal{L}(G))$. □

The arguments in the proof of Lemma 3.7 give the following result.

Lemma 3.8. *Let G be a graph with edges of length k and γ^* be a geodesic of $\mathcal{L}(G)$ joining x and y with $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$. Then $h(\gamma^*)$ is the union of three geodesics γ_1^*, γ_2^*, γ_3^* in G with $h(x) \in \gamma_1^*$, $h(y) \in \gamma_3^*$ and $0 \le L(\gamma_1^*), L(\gamma_3^*) \le k/2$.*

Also, we shall need a property of geodesic quadrilaterals in G.

Lemma 3.9. *For every x, y, u, v in the graph G, let us define $\Gamma := [xu] \cup [uv] \cup [vy]$. If $L([xu]), L([vy]) \le \sup_{e\in E(G)} L(e)$, then*

$$\forall\, \alpha \in \Gamma\ ,\ \exists\, \beta \in [xy]\ :\ d_G(\alpha, \beta) \le 2\delta(G) + \sup_{e\in E(G)} L(e). \qquad (3.10)$$

Proof. Let us consider the geodesic quadrilateral $Q = \{[xy], [xu], [uv], [vy]\}$ and $\alpha \in \Gamma$. If $\alpha \in [xu] \cup [vy]$, then there exists $\beta \in \{x, y\} \subset [xy]$ such that $d_G(\alpha, \beta) \le \max\{L([xu]), L([vy])\} \le \sup_{e\in E(G)} L(e)$. If $\alpha \in [uv]$, then there exists $\alpha' \in [xy] \cup [xu] \cup [vy]$ such that $d_G(\alpha, \alpha') \le 2\delta(G)$ since Q is a geodesic quadrilateral in G. So, there exists $\beta \in [xy]$ such that $d_G(\alpha', \beta) \le \sup_{e\in E(G)} L(e)$. Then, we obtain that $d_G(\alpha, \beta) \le d_G(\alpha, \alpha') + d_G(\alpha', \beta) \le 2\delta(G) + \sup_{e\in E(G)} L(e)$. □

Theorem 3.10. *Let G be a graph and consider $\mathcal{L}(G)$ the line graph of G. Then*

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 3l_{max}, \qquad (3.11)$$

with $l_{max} = \sup_{e\in E(G)} L(e)$. Furthermore, the first inequality is sharp: the equality is attained by every cycle graph.

Proof. First, let us consider a geodesic triangle $T = [xy] \cup [yz] \cup [zx]$ in G that is a cycle. Hence, if g_T is defined by (3.9), then Corollary 3.6 gives that $T^* = [g_T(x)g_T(y)] \cup [g_T(y)g_T(z)] \cup [g_T(z)g_T(x)]$ is a geodesic triangle in $\mathcal{L}(G)$; besides, by Proposition 3.2 we have that $d_G(u, v) \le d_{\mathcal{L}(G)}(g_T(u), g_T(v))$ for every $u, v \in T$.

Let $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$ be a permutation of $([xy], [yz], [zx])$. So, by Proposition 3.2 we have

$$\begin{aligned}\sup_{a\in\gamma_1} \inf_{b\in\gamma_2\cup\gamma_3} d_G(a, b) &\le \sup_{a\in\gamma_1} \inf_{b\in\gamma_2\cup\gamma_3} d_{\mathcal{L}(G)}(g_T(a), g_T(b))\\ &\le \sup_{a^*\in g_T(\gamma_1)} \inf_{b^*\in g_T(\gamma_2)\cup g_T(\gamma_3)} d_{\mathcal{L}(G)}(a^*, b^*).\end{aligned}$$

Since Γ is an arbitrary permutation, we obtain

$$\delta(T) \le \delta(T^*) \le \delta(\mathcal{L}(G)).$$

This finishes the proof of the first inequality by Corollary 2.3.

Now, let us consider a geodesic triangle $T^* = \{[x^*y^*], [y^*z^*], [z^*x^*]\}$ in $\mathcal{L}(G)$ that is a cycle, and a permutation $\Gamma = (\gamma_1^*, \gamma_2^*, \gamma_3^*)$ of $([x^*y^*], [y^*z^*], [z^*x^*])$. So, by Lemma 3.4 we have

$$\begin{aligned}\sup_{a^*\in\gamma_1^*} \inf_{b^*\in\gamma_2^*\cup\gamma_3^*} d_{\mathcal{L}(G)}(a^*, b^*) &\le \sup_{a^*\in\gamma_1^*} \inf_{b^*\in\gamma_2^*\cup\gamma_3^*} d_G(h(a^*), h(b^*)) + 2l_{max} \\ &\le \sup_{a\in h(\gamma_1^*)} \inf_{b\in h(\gamma_2^*)\cup h(\gamma_3^*)} d_G(a, b) + 2l_{max}, \\ \sup_{a^*\in\gamma_1^*} d_{\mathcal{L}(G)}(a^*, \gamma_2^*\cup\gamma_3^*) &\le \sup_{a\in h(\gamma_1^*)} d_G(a, h(\gamma_2^*)\cup h(\gamma_3^*)) + 2l_{max}.\end{aligned} \tag{3.12}$$

By Lemma 3.7 we know that $h([x^*y^*])$ is the union of three geodesics $[\alpha_z^1 P_{\alpha_z^1}]$, $[P_{\alpha_z^1} P_{\alpha_z^2}]$ and $[P_{\alpha_z^2} \alpha_z^2]$ in G:

$$h([x^*y^*]) = [\alpha_z^1 P_{\alpha_z^1}] \cup [P_{\alpha_z^1} P_{\alpha_z^2}] \cup [P_{\alpha_z^2} \alpha_z^2].$$

Analogously, $h([y^*z^*])$ and $h([z^*x^*])$ are the union of three geodesics in G:

$$h([y^*z^*]) = [\alpha_x^1 P_{\alpha_x^1}] \cup [P_{\alpha_x^1} P_{\alpha_x^2}] \cup [P_{\alpha_x^2} \alpha_x^2],$$

$$h([z^*x^*]) = [\alpha_y^1 P_{\alpha_y^1}] \cup [P_{\alpha_y^1} P_{\alpha_y^2}] \cup [P_{\alpha_y^2} \alpha_y^2].$$

Now, let us consider a geodesic triangle $T := \{[h(x^*)h(y^*)], [h(y^*)h(z^*)], [h(z^*)h(x^*)]\}$ in G. Without loss of generality we can assume that $\gamma_1^* = [x^*y^*]$, $\gamma_2^* = [y^*z^*]$ and $\gamma_3^* = [z^*x^*]$. Hence, by Lemma 3.9 we have that, if $\alpha \in h(\gamma_1^*)$ then there exists $\beta \in [h(x^*)h(y^*)]$ such that

$$d_G(\alpha, \beta) \le 2\delta(G) + l_{max}.$$

Since $\beta \in [h(x^*)h(y^*)]$, there exists $\beta' \in [h(y^*)h(z^*)] \cup [h(z^*)h(x^*)]$ such that

$$d_G(\beta, \beta') \le \delta(G).$$

Without loss of generality we can assume that $\beta' \in [h(y^*)h(z^*)]$. If we consider the geodesic quadrilateral $\{[\alpha_x^1 \alpha_x^2], [\alpha_x^1 P_{\alpha_x^1}], [P_{\alpha_x^1} P_{\alpha_x^2}], [P_{\alpha_x^2} \alpha_x^2]\}$, then there exists $\alpha' \in h([y^*z^*])$ such that

$$d_G(\beta', \alpha') \leq 2\delta(G).$$

Thus, since $d_G(\alpha, h(\gamma_2^*) \cup h(\gamma_3^*)) \leq d_G(\alpha, \beta) + d_G(\beta, \beta') + d_G(\beta', \alpha')$ we obtain that

$$d_G(\alpha, h(\gamma_2^*) \cup h(\gamma_3^*)) \leq 5\delta(G) + l_{max}. \tag{3.13}$$

Then, by (3.12) and (3.13) we obtain

$$\sup_{a^* \in \gamma_1^*} d_{\mathcal{L}(G)}(a^*, \gamma_2^* \cup \gamma_3^*) \leq 5\delta(G) + 3l_{max}.$$

Finally, since Γ is an arbitrary permutation of any triangle that is a cycle, Corollary 2.3 gives

$$\delta(\mathcal{L}(G)) \leq 5\delta(G) + 3l_{max}.$$

Corollary 2.5 gives that $\delta(G) = \delta(\mathcal{L}(G)) = L(G)/4$ for every cycle graph G. □

Remark 3.11. *The cycle graphs are not the only graphs G with $\delta(\mathcal{L}(G)) = \delta(G)$, as the following example shows. Let C_n be the cycle graph with n vertices and every edge with length k, and $u, v \in V(C_n)$ with $d_{C_n}(u, v) = 2k$; if G is the graph obtained by adding the edge $[u, v]$ (also with length k) to C_n, one can check that $\delta(\mathcal{L}(G)) = \delta(G) = kn/4$.*

Let us consider now graphs with edges of length k. We will improve Theorem 3.10 in this case.

Corollary 3.12. *Let G be any graph such that every edge has length k and consider $\mathcal{L}(G)$ the line graph of G. Then*

$$\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\delta(G) + \frac{5k}{2}.$$

Proof. We just need to prove the second inequality. By Theorem 2.6 it suffices to consider geodesic triangles in $\mathcal{L}(G)$ with vertices in $PM_{\mathcal{L}}V(\mathcal{L}(G)) = PMV(\mathcal{L}(G))$. Then the arguments in the proof of Theorem 3.10, replacing Lemma 3.7 by Lemma 3.8, give the result. □

In [46, Corollary 20] we find the following result.

Lemma 3.13. *Let G be any graph with m edges such that every edge has length k. Then $\delta(G) \leq km/4$, and the equality is attained if and only if G is a cycle graph.*

Theorem 3.14. *Let G be any graph such that every edge has length k, with n vertices and maximum degree Δ. Then*

$$\delta(\mathcal{L}(G)) \leq nk\Delta(\Delta - 1)/8,$$

and the equality is attained if and only if G is a cycle graph.

Proof. It is well known that $2(m(\mathcal{L}(G)) + m(G)) = \sum_{i=1}^{n}(\deg_G(v_i))^2$, where $\deg_G(v_i)$ are the degrees of the vertices of G. Since $2m(G) = \sum_{i=1}^{n}\deg_G(v_i)$, Lemma 3.13 gives the inequality, and the equality is attained if and only if G is a cycle graph. □

Using the argument in the proof of Theorem 3.14 we also obtain the following inequality.

Corollary 3.15. *If G is any graph such that every edge has length k, with n vertices $v_1, \dots, v_n$, then*

$$\delta(\mathcal{L}(G)) + \delta(G) \leq \frac{k}{8}\sum_{i=1}^{n}(\deg_G(v_i))^2,$$

and the equality is attained if and only if G is a cycle graph.

4 Acknowledgements

We would like to thank the referee for a careful reading of the manuscript and for some helpful suggestions.

This work was partly supported by the Spanish Ministry of Science and Innovation through project MTM2009-07800 and a grant from CONACYT (CONACYT-UAG I0110/62/10), México.

References

[1] Alonso, J., Brady, T., Cooper, D., Delzant, T., Ferlini, V., Lustig, M., Mihalik, M., Shapiro, M. and Short, H., Notes on word hyperbolic groups, in: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), Group Theory from a Geometrical Viewpoint, World Scientific, Singapore, 1992.

[2] Balogh, Z. M. and Buckley, S. M., Geometric characterizations of Gromov hyperbolicity, *Invent. Math.* **153** (2003), 261-301.

[3] Bermudo, S., Rodríguez, J. M. and Sigarreta, J. M., Computing the hyperbolicity constant, *Comput. Math. Appl.* **62** (2011), 4592-4595.

[4] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic graphs. Submitted. Preprint in http://gama.uc3m.es/index.php/jomaro.html

[5] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Tourís, E., Hyperbolicity and complement of graphs, *Appl. Math. Letters* **24** (2011), 1882-1887.

[6] Bonk, M., Heinonen, J. and Koskela, P., Uniformizing Gromov hyperbolic spaces. Astérisque 270, 2001.

[7] Bowditch, B. H., Notes on Gromov's hyperbolicity criterion for path-metric spaces. Group theory from a geometrical viewpoint, Trieste, 1990 (ed. E. Ghys, A. Haefliger and A. Verjovsky; World Scientific, River Edge, NJ, 1991) 64-167.

[8] Bowers, P. L., Negatively curved graph and planar metrics with applications to type, *Michigan Math. J.* **45** (1998), 31-53.

[9] Brinkmann, G., Koolen J. and Moulton ,V., On the hyperbolicity of chordal graphs, *Ann. Comb.* **5** (2001), 61-69.

[10] Carballosa, W., Pestana, D., Rodríguez, J. M. and Sigarreta, J. M., Distortion of the hyperbolicity constant of a graph, *Electr. J. Comb.* **19** (2012), P67.

[11] Carballosa, W., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolicity of line graphs, *Electr. J. Comb.* **18** (2011), P210.

[12] Chen, B., Yau, S.-T. and Yeh, Y.-N., Graph homotopy and Graham homotopy, *Discrete Math.* **241** (2001), 153-170.

[13] Chepoi, V., Dragan, F. F., Estellon, B., Habib, M. and Vaxes Y., Notes on diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs, *Electr. Notes Discr. Math.* **31** (2008), 231-234.

[14] Chepoi, V. and Estellon, B., Packing and covering δ-hyperbolic spaces by balls, APPROX-RANDOM 2007 pp. 59-73.

[15] DeVos, M. and Mohar, B., An analogue of the Descartes-Euler formula for infinite graph and Higuchi's conjecture, *Trans. Amer. Math. Soc.* **359** (2007), 3275-3286.

[16] Dirac, G .A., Généralisation du théorème de Menger, *C .R. Acad. Sci. Paris* **250** (26) (1960), 4252-4253.

[17] Diestel, R., Graph Theory, Heidelberg Graduate Texts in Mathematics 173, Springer-Verlag, 2010.

[18] Eppstein, D., Squarepants in a tree: sum of subtree clustering and hyperbolic pants decomposition, SODA' 2007.

[19] Fiedler, M., A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, *Czech. Math. J.* **25** (100) (1975) 619-633.

[20] Frigerio, R. and Sisto, A., Characterizing hyperbolic spaces and real trees, *Geom. Dedicata* **142** (2009), 139-149.

[21] Gavoille, C. and Ly, O., Distance labeling in hyperbolic graphs, In ISAAC 2005 pp. 171–179.

[22] Ghys, E. and de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. Progress in Mathematics 83, Birkhäuser Boston Inc., Boston, MA, 1990.

[23] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[24] Harary, F. and Norman, R. Z., Some properties of line digraphs, *Rend. Circ. Math. Palermo* **9** (1960), 161-169.

[25] Hästö, P. A., Gromov hyperbolicity of the j_G and $\tilde{j}_G$ metrics, *Proc. Amer. Math. Soc.* **134** (2006), 1137–1142.

[26] Hästö, P. A., Lindén, H., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Denjoy domains with hyperbolic and quasihyperbolic metrics, *J. Math. Soc. Japan* **64** (2012), 247–261.

[27] Hästö, P. A., Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolic equivalence of the hyperbolic and quasihyperbolic metrics in Denjoy domains, *Bull. London Math. Soc.* **42** (2010), 282-294.

[28] Jonckheere, E. and Lohsoonthorn, P., A hyperbolic geometry approach to multipath routing, Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002), Lisbon, Portugal, July 2002. FA5-1.

[29] Jonckheere, E. A., Contrôle du traffic sur les réseaux à géométrie hyperbolique–Vers une théorie géométrique de la sécurité l'acheminement de l'information, *J. Europ. Syst. Autom.* **8** (2002), 45-60.

[30] Jonckheere, E. A. and Lohsoonthorn, P., Geometry of network security, *Amer. Control Conf.* **ACC** (2004), 111-151.

[31] Jonckheere, E. A., Lohsoonthorn, P. and Ariaesi, F, Upper bound on scaled Gromov-hyperbolic delta, *Appl. Math. Comp.* **192** (2007), 191-204.

[32] Jonckheere, E. A., Lohsoonthorn, P. and Bonahon, F., Scaled Gromov hyperbolic graphs, *J. Graph Theory* **2** (2007), 157-180.

[33] Koolen, J. H. and Moulton, V., Hyperbolic Bridged Graphs, *Europ. J. Comb.* **23** (2002), 683-699.

[34] Krausz, J., Démonstration nouvelle d'un théorème de Whitney sur les réseaux, *Mat. Fiz. Lapok* **50** (1943), 75-85.

[35] Krauthgamer, R. and Lee, J. R., Algorithms on negatively curved spaces, FOCS 2006.

[36] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Hyperbolicity and parameters of graphs, *Ars Comb.* Volume C (2011), 43-63.

[37] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolicity in cartesian product graphs, *Proc. Indian Acad. Sci. Math. Sci.* **120** (2010), 1-17.

[38] Oshika, K., Discrete groups, AMS Bookstore, 2002.

[39] Pestana, D., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolic cubic graphs, *Central Europ. J. Math.* **10** (2012), 1141-1151.

[40] Portilla, A., Rodríguez, J. M., Sigarreta, J. M. and Tourís, E., Gromov hyperbolic directed graphs. Submitted.

[41] Portilla, A., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic tessellation graphs, to appear in *Utilitas Math.* Preprint in http://gama.uc3m.es/index.php/jomaro.html

[42] Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123-149.

[43] Portilla, A., Rodríguez, J. M. and Tourís, E., Stability of Gromov hyperbolicity, *J. Adv. Math. Studies* **2** (2009), 1-20.

[44] Portilla, A. and Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.* **53** (2009), 83-110.

[45] Rodríguez, J. M. and Sigarreta, Bounds on Gromov hyperbolicity constant in graphs, to appear in *Proc. Indian Acad. Sci. Math. Sci.* DOI 10.1007/s12044-012-0060-0

[46] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M. and Villeta, M., On the hyperbolicity constant in graphs, *Discr. Math.* **311** (2011), 211-219.

[47] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.* **103** (2004), 53-84.

[48] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica* **23** (2007), 209-228.

[49] Shavitt, Y., Tankel, T., On internet embedding in hyperbolic spaces for overlay construction and distance estimation, INFOCOM 2004.

[50] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011), 865-881.

[51] Whitney, H., Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), 150-168.

[52] Wu, Y. and Zhang, C., Chordality and hyperbolicity of a graph, *Electr. J. Comb.* **18** (2011), P43.