On The Detour Monophonic Number of a Graph*

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Abstract

For a connected graph $G = (V, E)$ of order at least two, a *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ *detour monophonic path.* A set S of vertices of G is a *detour monophonic set* of G if each vertex v of G lies on an $x - y$ detour monophonic path for some x and y in S. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. For any two vertices u and v in G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a $u - v$ detour monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is the maximum monophonic distance from v to a vertex of G. The *monophonic radius* rad_mG of G is the minimum monophonic eccentricity among the vertices of G, while the *monophonic diameter* $diam_mG$ of G is the maximum monophonic eccentricity among the vertices of G. It is shown that for positive integers r, d and $n \geq 4$ with $r < d$ there exists a connected graph G with $rad_mG = r$, $diam_mG = d$ and $dm(G) = n$. Also, if p, d, n are integers with $2 \leq n \leq p - d + 1$ and $d \geq 3$, there is a connected graph G of order p, monophonic diameter d and detour monophonic number n. Further, we study how the detour monophonic number of a graph is affected by adding some pendant edges to the graph.

Keywords: monophonic distance, detour monophonic set, detour monophonic number.
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1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [5]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

The *detour distance* $D(u, v)$ between two vertices u and v in G is the length of a longest $u - v$ path in G. An $u - v$ path of length $D(u, v)$ is called an $u - v$ *detour*. It is known that D is a metric on the vertex set V of G. The closed detour interval $I_D[x, y]$ consists of x, y, and all the vertices in some $x - y$ detour of G. For $S \subseteq V$, $I_D[S]$ is the union of the sets $I_D[x, y]$ for all $x, y \in S$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. The concept of detour distance, detour number was introduced [1, 3] and further studied in [4, 2].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m G$ of G is $rad_m G = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m G$ of G is $diam_m G = \max \{e_m(v) : v \in V(G)\}$. A vertex u in G is a monophonic eccentric vertex of a vertex v in G if $e_m(u) = d_m(u, v)$. The monophonic distance was introduced in [6] and further studied in [7].

A set S of vertices of a graph G is a *detour monophonic set* if each vertex v of G lies on an $x - y$ detour monophonic path, for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$ [8].

For the graph G given in Figure 1.1, S_1=$\{x, y, z\}$, S_2=$\{x, w, z\}$, S_3=$\{u, z, y\}$, S_4=$\{x, u, z\}$, S_5=$\{y, w, z\}$ and $S_6 = \{u, w, z\}$ are the minimum detour monophonic sets of G and so $dm(G)$=3.

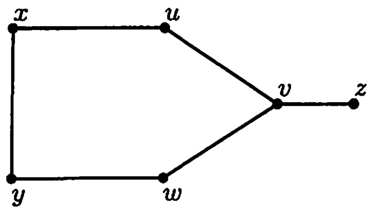

Figure 1.1: A graph G with $dm(G) = 3$.

The following theorems will be used in the sequel.

Theorem 1.1. [8] *Each extreme vertex of a connected graph G belongs to every detour monophonic set of G. Moreover, if the set S of all extreme vertices of G is a detour monophonic set, then S is the unique minimum detour monophonic set of G.*

Theorem 1.2. [8] *No cut vertex of G belongs to any minimum detour monophonic set of G.*

Theorem 1.3. [8] *If T is a tree with k end vertices, then $dm(T) = k$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2 Bounds and some realization results for the detour monophonic number of a graph

It is shown in [8] that if G is a connected graph of order $p \geq 2$, then $2 \leq dm(G) \leq p$. Also we have a graph G is complete if and only if $dm(G) = p$. Also, it is proved that for a connected graph $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$ if and only if $dm(G) = p - 1$. In the following theorem we give an improved upper bound for the detour monophonic number of a graph in terms of its order and monophonic diameter.

Theorem 2.1. *If G is a non-trivial connected graph of order p and monophonic diameter d, then $dm(G) \leq p - d + 1$.*

Proof. Let $x, y \in V(G)$ such that G contains an $x - y$ detour monophonic path P of length $diam_m G = d$. Let $S = (V(G) - V(P)) \bigcup \{x, y\}$. Since S is a detour monophonic set of G, it follows that $dm(G) \leq |S| \leq p - d+1$. □

Theorem 2.2. *For every non-trivial tree T of order p and monophonic diameter d, $dm(T) = p - d + 1$ if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : u = v_0, v_1, ..., v_d$ be a monophonic diametral path. Let k be the number of end vertices of T and l be the number of internal vertices of T other than $v_1, v_2, ..., v_{d-1}$. Then $d-1+l+k = p$. By Theorem 1.3, $dm(T) = k$ and so $dm(T) = p-d-l+1$. Hence $dm(T) = p-d+1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic diametral path P, if and only if T is a caterpillar. □

For any connected graph G, $rad_m G \leq diam_m G$. It is shown in [6] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the detour monophonic number can be prescribed when $rad_m G < diam_m G$.

Theorem 2.3. *For positive integers r, d and $n \geq 4$ with $r < d$, there exists a connected graph G with $rad_m G = r$, $diam_m G = d$ and $dm(G) = n$.*

Proof. We prove this theorem by considering two cases.
Case 1. $r = 1$. Then $d \geq 2$. Let $C_{d+2} : v_1, v_2, ..., v_{d+2}, v_1$ be the cycle of order $d+2$. Let G be the graph obtained by adding $n-2$ new vertices $u_1, u_2, ..., u_{n-2}$ to C_{d+2} and join each vertex $x \in \{u_1, u_2, ..., u_{n-2}, v_3, v_4, ..., v_{d+1}\}$ to the vertex v_1. The graph G is shown in Figure 2.1. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = 1$, $e_m(v_2) = d$. Then $rad_m G = 1$ and $diam_m G = d$. Let $S = \{u_1, u_2, ..., u_{n-2}, v_2, v_{d+2}\}$ be the set of all extreme vertices of G. Since S is a detour monophonic set of G, it follows from Theorem 1.1 that $dm(G) = n$.

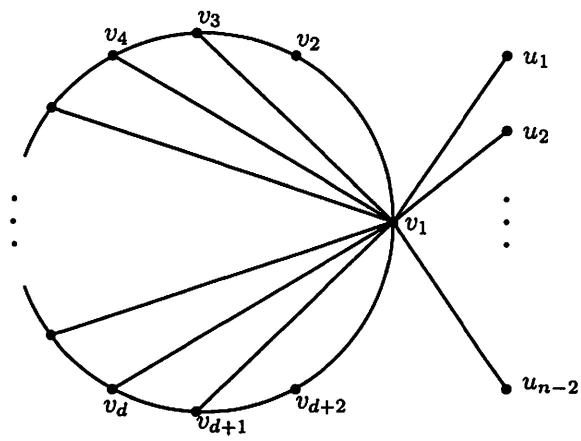

Figure 2.1: G

Case 2. $r \geq 2$. Let $C : v_1, v_2, ..., v_{r+2}, v_1$ be the cycle of order $r+2$ and $W = K_1 + C_{d+2}$ be the wheel with $V(C_{d+2}) = \{u_1, u_2, ..., u_{d+2}\}$. Let H be the graph obtained from C and W by identifying v_1 of C and the central vertex K_1 of W.

Subcase 1. Both r and d are even. Add $n-3$ new vertices $w_1, w_2, ..., w_{n-3}$ to the graph H and join each $w_i (1 \leq i \leq n-3)$ to the vertex v_1 and obtain the graph G of Figure 2.2. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r, e_m(u_1) = d$. Then $rad_m G = r$ and $diam_m G = d$. Let $S = \{w_1, w_2, ..., w_{n-3}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every detour monophonic set of G contains S. It is clear that S is not a detour monophonic set of G. Also, for any $x, y \in V(H)$, $S \cup \{x\}$ and $S \cup \{x, y\}$ are not detour monophonic sets of G. Let $T = S \bigcup \{u_1, u_{\frac{d+4}{2}}, v_{\frac{r+4}{2}}\}$. It is easily verified that T is a minimum detour monophonic set of G and so $dm(G) = n$.

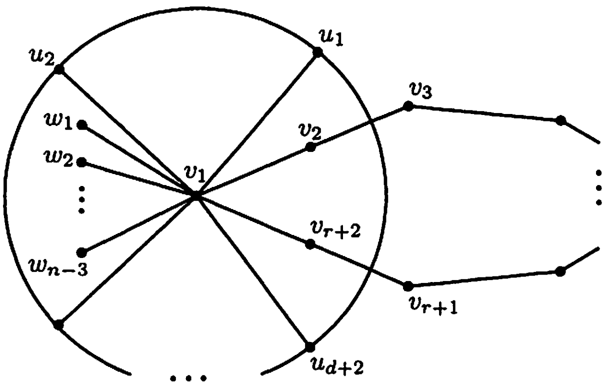

Figure 2.2: G

Subcase 2. Both r and d are odd. Add $n-3$ new vertices $w_1, w_2, ..., w_{n-4}, z$ to the graph H and join each $w_i (1 \leq i \leq n-4)$ to the vertex v_1; and join z to both $u_{\lceil \frac{r+2}{2} \rceil}$ and $u_{\lceil \frac{r+2}{2} \rceil + 1}$, and obtain the graph G of Figure 2.3. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$, $e_m(u_1) = d$. Then $rad_m G = r$ and $diam_m G = d$. Let $S = \{w_1, w_2, ..., w_{n-4}, z\}$ be the set of all extreme vertices of G. By Theorem 1.1, every detour monophonic set of G contains S. It is clear that S is not a detour monophonic set of G. Also, for any $x, y \in V(H)$, $S \cup \{x\}$ and $S \cup \{x, y\}$ are not detour monophonic sets of G. Let $T = S \cup \{u_1, u_2, u_3\}$. It is easily verified that T is a minimum detour monophonic set of G and so $dm(G) = n$.

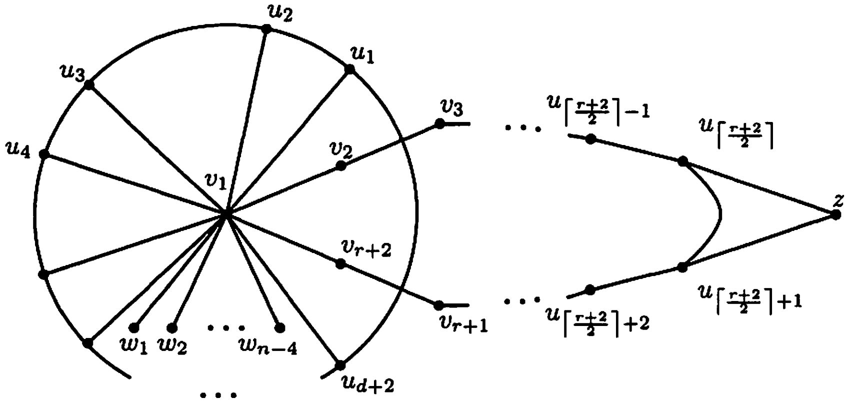

Figure 2.3: G

Subcase 3. r is odd and d is even. Add $n-2$ new vertices $w_1, w_2, ..., w_{n-3}, z$ to the graph H and join each $w_i (1 \leq i \leq n-3)$ to the vertex v_1 and join z to both $u_{\lceil \frac{r+2}{2} \rceil}$ and $u_{\lceil \frac{r+2}{2} \rceil +1}$, and obtain the graph G of Figure 2.4. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$ and $e_m(u_1) = d$. Then $rad_m G = r$ and $diam_m G = d$. Let $S = \{w_1, w_2, ..., w_{n-3}, z\}$ be the set of all extreme vertices of G. By Theorem 1.1, every detour monophonic set of G contains S. It is clear that S is not a detour monophonic set of G. Also, for any $x \in V(H)$, $S \bigcup \{x\}$ is not a detour monophonic set of G. Let $T = S \bigcup \{u_1, u_{\frac{d+4}{2}}\}$. It is easily verified that T is a minimum detour monophonic set of G and so $dm(G) = n$.

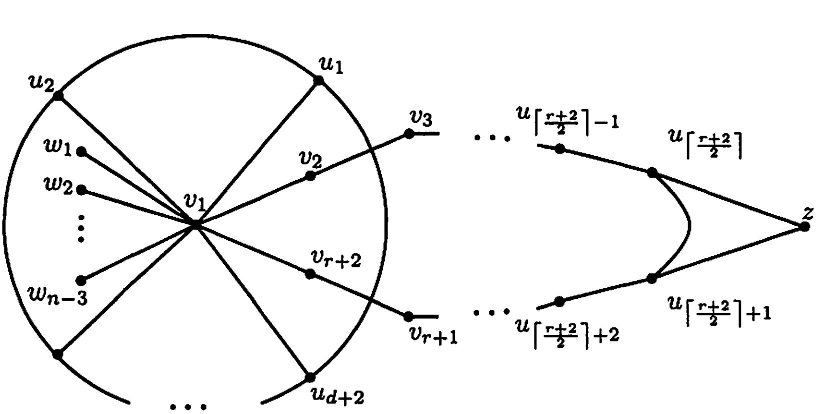

Figure 2.4: G

Subcase 4. r is even and d is odd. Add $n-4$ new vertices $w_1, w_2, ..., w_{n-4}$ to the graph H and join each $w_i (1 \leq i \leq n-4)$ to the vertex v_1, and obtain the graph G of Figure 2.5. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$, $e_m(u_1) = d$. Then $rad_m G = r$ and $diam_m G = d$. Let $S=\{w_1, w_2, ..., w_{n-4}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every detour monophonic set of G contains S. It is clear that S is not a detour monophonic set of G. Also, for any $x, y, z \in V(H)$, $S \cup \{x\}$; $S \cup \{x, y\}$; and $S \cup \{x, y, z\}$ are not detour monophonic sets of G. Let $T = S \cup \{u_1, u_2, u_3, v_{\frac{r+4}{2}}\}$. It is easily verified that T is a minimum detour monophonic set of G and so $dm(G) = n$. □

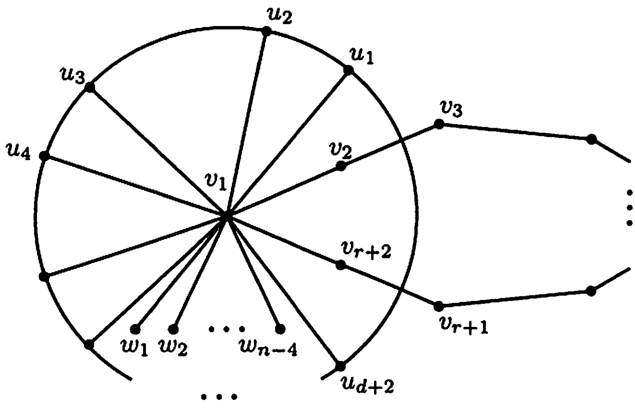

Figure 2.5: G

Problem 2.4. For any three positive integers r, d and $n \geq 4$ with $r = d$, does there exist a connected graph G with $rad_m G = r$, $diam_m G = d$ and $dm(G) = n$?

Theorem 2.5. *If p, d, n are integers with $2 \leq n \leq p - d + 1$ and $d \geq 3$, there is a connected graph G of order p, monophonic diameter d and detour monophonic number n.*

Proof. Let $P_{d+1} : u_0, u_1, u_2, ..., u_d$ be a path of length d. Let G be the graph obtained from the path P_{d+1} by (i) adding $n-2$ new vertices $v_1, v_2, \ldots, v_{n-2}$ and joining each vertex $v_i (1 \leq i \leq n-2)$ to u_1; and (ii) adding $p-d-n+1$ new vertices $w_1, w_2, \ldots, w_{p-d-n+1}$ and joining each vertex $w_i (1 \leq i \leq p-d-n+1)$ to both u_0 and u_2. The graph G has order p and monophonic diameter d. Let $S = \{u_d, v_1, v_2, ..., v_{n-2}\}$ is the set of all extreme vertices of G. Then by Theorem 1.1, every detour monophonic set of G contains S. Clearly S is not a detour monophonic set of G and so $dm(G) > n - 1$. Let $T = S \cup \{u_0\}$. It is easily verified that T is a detour monophonic set of G and so $dm(G) = n$. □

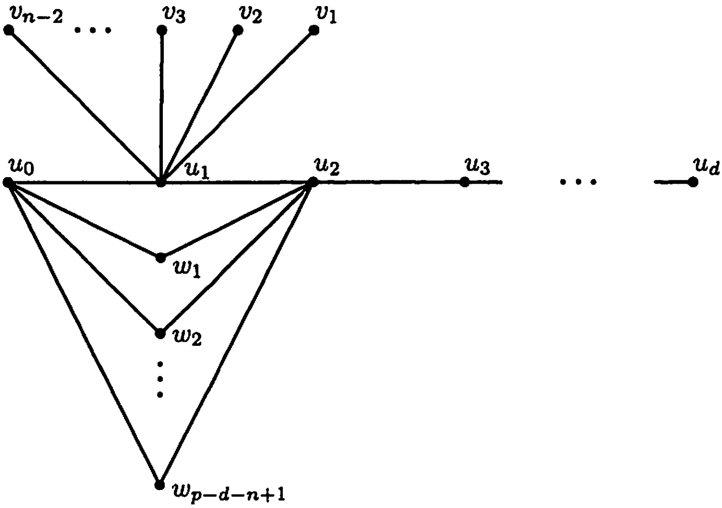

Figure 2.6: G

3 Detour monophonic number of a graph by adding some pendant edges

Theorem 3.1. *If G' is a graph obtained by adding k pendant edges to a connected graph G, then $max\{k, dm(G)\} \leq dm(G') \leq dm(G) + k$.*

Proof. Let G' be the connected graph obtained from G by adding k pendant edges $u_iv_i(1 \leq i \leq k)$, where each $u_i(1 \leq i \leq k)$ is a vertex of G and each $v_i(1 \leq i \leq k)$ is not a vertex of G. Let S be a minimum detour monophonic set of G. Then $S \bigcup \{v_1, v_2, ..., v_k\}$ is a detour monophonic set of G' and so $dm(G') \leq dm(G) + k$.

Now, we claim that $dm(G) \leq dm(G')$. Suppose that $dm(G) > dm(G')$. Then let S' be a detour monophonic set of G' with $|S'| < dm(G)$. Since each $v_i(1 \leq i \leq k)$ is an extreme vertex of G', it follows from Theorem 1.1 that $\{v_1, v_2, ..., v_k\} \subseteq S'$. Let $S = (S' - \{v_1, v_2, ..., v_k\}) \cup \{u_1, u_2, ..., u_k\}$. Then S is a subset of $V(G)$ and $|S| = |S'| < dm(G)$. Now, we show that S is a detour monophonic set of G. Let $w \in V(G) - S$. Since S' is a detour monophonic set of G', w lies on an $x - y$ detour monophoic path, say P, in G' for some vertices $x, y \in S'$. If neither x nor y is $v_i(1 \leq i \leq k)$, then $x, y \in S$. If exactly one of x, y is $v_i(1 \leq i \leq k)$, say $x = v_i$. Then w lies on the $u_i - y$ detour monophonic path in G obtained from P by removing v_i. If both $x, y \in \{v_1, v_2, ..., v_k\}$, then let $x = v_i$ and $y = v_j$ where $i \neq j$. Hence w lies on the $u_i - u_j$ detour monophonic path in G obtained from P by removing v_i and v_j. Thus S is a detour monophonic set of G. Hence $dm(G) \leq |S| < dm(G)$, which is a contradiction. Also, since G' contains k pendant vertices, by Theorem 1.1, $dm(G') \geq k$. Thus $dm(G') \geq max\{k, dm(G)\}$. □

Remark 3.2. The bounds for $dm(G')$ in Theorem 3.1 are sharp. Consider a tree T with number of end vetices $n \geq 3$. Let $S = \{v_1, v_2, ..., v_n\}$ be the set of all end vertices of T. Then by Theorem 1.3, S is the unique minimum detour monophonic set of T. If we add a pendant edge to an end vertex of T, then we obtain another tree T' with n end verties. Hence $dm(T) = dm(T')$. On the otherhand, if we add k pendant edges to a cut vertex of T, then we obtain another tree T'' with $n + k$ end vertices. Then by Theorem 1.3, $dm(T') = dm(T) + k$.

Now, we proceed to characterize graphs G for which $dm(G) = dm(G')$, where G' is obtained from G by adding k pendant edges.

Theorem 3.3. *Let G' be a graph obtained from a connected graph G by adding k pendant edges $u_iv_i (1 \leq i \leq k)$, where $u_i \in V(G)$ and $v_i \notin V(G)$. Then $dm(G) = dm(G')$ if and only if $\{u_1, u_2, ..., u_k\}$ is a subset of some minimum detour monophonic set of G.*

Proof. Let $\{u_1, u_2, ..., u_k\}$ is a subset of some minimum detour monophonic set S of G. Let $S' = (S - \{u_1, u_2, ..., u_k\}) \bigcup \{v_1, v_2, ..., v_k\}$. Then $|S'| = |S|$. Claim that S' is a detour monophonic set of G'. Let $z \in V(G') - S'$. If $z = u_i (1 \leq i \leq k)$, then z lies on every $v_i - w$ detour monophonic path in G', where $w \in S'$, since u_i is the only vertex adjacent to v_i. So we may assume that $z \neq u_i (1 \leq i \leq k)$. Since z is a vertex of G and S is a detour monophonic set of G, it follows that z lies on some $x - y$ detour monophonic path P in G for some $x, y \in S$. Then by an argument similar to the one used in the proof of Theorem 3.1, we can show that S' is a detour monophonic set of G'. Hence $dm(G') \leq |S'| = |S| = dm(G)$. Now, the result follows from Theorem 3.1.

Conversely, let $dm(G) = dm(G')$. Let S' be a minimum detour monophonic set of G'. Since each $u_i (1 \leq i \leq k)$ is a cut vertex of G', it follows from Theorem 1.2 that $u_i \notin S'$ for $1 \leq i \leq k$. Since $v_i (1 \leq i \leq k)$ is an end vertex of G', it follows from Theorem 1.1 that $v_i \in S'$ for $1 \leq i \leq k$. Let $S = (S' - \{v_1, v_2, ..., v_k\}) \bigcup \{u_1, u_2, ..., u_k\}$. Then $S \subseteq V(G)$ and $|S| = |S'|$. Then, as in the proof of Theorem 3.1, S is a detour monophonic set of G. Since $|S| = |S'| = dm(G') = dm(G)$, it follows that S is a minimum detour monophonic set of G that contains $\{u_1, u_2, ..., u_k\}$. □

Acknowledgements

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