Construction quasi-regular semilattices with singular linear spaces

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Abstract Let $\mathbb{F}_q^{n+l}$ denote the $(n+l)$-dimensional singular linear space over a finite field $\mathbb{F}_q$. For a fixed integer $m \leq \min\{n, l\}$, denote by $\mathcal{L}_o^m(\mathbb{F}_q^{n+l})$ the set of all subspaces of type (t, t_1), where $t_1 \leq t \leq m$. Partially ordered by ordinary inclusion, one family of quasi-regular semilattices is obtained. Moreover, we compute its all parameters.

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1 Introduction

It is well known that lattice is an important part of poset's theory. Its theory play an important role in many branches of mathematics, such as computer logical design and procedure theory. In recent times there has been great interest in constructing more kinds of practical lattices and semilattices. For example, Guo, Gao and Wang [2] constructed lattices based on d-bounded distance-regular graphs. Wang, Guo, Li [4, 5, 6] constructed lattices in singular linear space, totally isotropic flats and classical spaces, respectively. Wang and Li [7] constructed lattices in vector space over a finite field. In [3], Guo, Li and Wang constructed semilattices in

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*Corresponding author. E-mail address: lizengti@126.com(Li Zengti) symplectic spaces, we construct a new quasi-regular semilattice in singular linear spaces and compute its parameters.

The rest of this paper is organized as follows. In section 2, we introduce some definitions and terminologies about lattice, regular semilattice and quasi-regular semilattice. In section 3, we construct a family of quasi-regular semilattices, and then compute its parameters.

2 Preliminaries

In this section, we first recall some definitions and terminologies about lattice and regular semilattice. The reader is referred to [1] for details. And then introduce quasi-regular semilattice in singular linear spaces.

Let $(P, \leq)$ be a poset. We write $a < b$ whenever $a \leq b$ and $a \neq b$. If P has the minimum (respectively maximum) element, then we denote it by 0 (respectively 1), and say that P is a poset with 0 (respectively 1). A poset P is said to be a *semilattice* if $a \wedge b := \inf\{\mathrm{a}, \mathrm{b}\}$ exists for any two elements $a, b \in P$. Let P be a finite poset with 0. If there is a function r from P to set of all the nonnegative integers such that

(1) r(0)=0,

(2) $r(b) = r(a) + 1$, if $a < b$.

Then r is said to be the *rank function* on P. Note that the rank function on P is unique if it exists.

Let P be a semilattice, and let $P = X_0 \cup X_1 \cup \cdots \cup X_m$, where $X_i = \{x \in P | r(x) = i\}, i = 0, 1, \cdots, m$. The semilattice $(P, \leq)$ is called *regular* if the following three properties hold:

(i) Given $y \in X_m, z \in X_r$ with $z \leq y$, the number of points $u \in X_s$ such that $z \leq u \leq y$ is a constant $\mu(r, s)$.

(ii) Given $u \in X_s$, the number of points $z \in X_r$, such that $z \leq u$ is a constant $\nu(r, s)$.

(iii) Given $a \in X_r, y \in X_m$, with $a \wedge y \in X_j$, the number of pairs $(b, z) \in X_s \times X_m$ such that $b \leq z, b \leq y, a \leq z$ is a constant $\pi(j, r, s)$.

In this paper, we define the concept of quasi-regular semilattice.

Let P be a semilattice, and let $P = X_0 \cup X_1 \cup \cdots \cup X_m$, where $X_i = \{x \in P | r(x) = i\} = X_i^0 \cup X_i^1 \cup \cdots \cup X_i^i, i = 0, 1, \cdots, m$, and $X_i^j \cap X_i^k = \emptyset$ for $j \neq k$. The semilattice $(P, \leq)$ is called *quasi-regular* if the following three properties hold:

(i) Given $y \in X_m^{m'}, z \in X_r^{r'}$ with $z \leq y$, the number of points $u \in X_s^{s'}$ such that $z \leq u \leq y$ is a constant $\mu(r(r'), s(s'); m')$.

(ii) Given $u \in X_s^{s'}$, the number of points $z \in X_r^{r'}$, such that $z \leq u$ is a constant $\nu(r(r'), s(s'))$.

(iii) Given $a \in X_r^{r'}, y \in X_m^{m_1}$, with $a \wedge y \in X_j^{j'}$, the number of pairs $(b, z) \in X_s^{s'} \times X_m^{m'}$ such that $b \leq z, b \leq y, a \leq z$ is a constant $\pi(j(j'), r(r'), s(s'); m')$.

Pick $P = X_0^0 \cup X_1^1 \cup \cdots \cup X_m^m$, Then the quasi-regular semilattice $(P, \leq)$ is a regular semilattice.

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a prime power. For two nonnegative integers n and l, $\mathbb{F}_q^{n+l}$ denotes the $(n+l)$-dimensional row vector space over $\mathbb{F}_q$. The set of all $(n+l) \times (n+l)$ nonsingular matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are nonsingular $n \times n$ and $l \times l$ matrices, respectively, forms a group under matrix multiplication, called the singular general linear group of degree $n + l$ over $\mathbb{F}_q$ and denoted by $GL_{n+l,n}(\mathbb{F}_q)$.

Let P be an m-dimensional subspace of $\mathbb{F}_q^{n+l}$, denote also by P a $m \times (n+l)$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P . There is an action of $GL_{n+l,n}(\mathbb{F}_q)$ on $\mathbb{F}_q^{n+l}$ defined as follows

$$\begin{array}{lcl} \mathbb{F}_q^{n+l} \times GL_{n+l,n}(\mathbb{F}_q) & \longrightarrow & \mathbb{F}_q^{n+l} \\ ((x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+l}), T) & \longmapsto & (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+l})T. \end{array}$$

The above action induces an action on the set of subspaces of $\mathbb{F}_q^{n+l}$; i.e., a subspace P is carried by $T \in GL_{n+l,n}(\mathbb{F}_q)$ to the subspace PT. The vector space $\mathbb{F}_q^{n+l}$ together with the above group action, is called the $(n+l)$-dimensional singular linear space over $\mathbb{F}_q$.

For $1 \leq i \leq n+l$, let e_i be the row vector in $\mathbb{F}_q^{n+l}$ whose i-th coordinate is 1 and all other coordinates are 0. Denote by E the l-dimensional subspace of $\mathbb{F}_q^{n+l}$ generated by $e_{n+1}, e_{n+2}, \cdots, e_{n+l}$. A m-dimensional subspace P of $\mathbb{F}_q^{n+l}$ is called a subspace of type (m,k) if $\dim(P \cap E) = k$.

For a fixed integer $m \leq \min\{n,l\}$, denote by $\mathcal{L}^m(\mathbb{F}_q^{n+l})$ the set of all subspaces of type (t, t_1), where $t_1 \leq t \leq m$. If we partially order $\mathcal{L}^m(\mathbb{F}_q^{n+l})$ by the ordinary inclusion, then $\mathcal{L}^m(\mathbb{F}_q^{n+l})$ is a semilattice, denoted by $\mathcal{L}_o^m(\mathbb{F}_q^{n+l})$. For any $A \in \mathcal{L}_o^m(\mathbb{F}_q^{n+l})$, the rank function of $\mathcal{L}_o(\mathbb{F}_q^{n+l})$ is defined as follows

$$r(A) = \dim(A).$$

Let

$$X_i = \{B \in \mathcal{L}_o^m(\mathbb{F}_q^{n+l}) | r(B) = i\},$$

and

$$X_i^j = \{B \in X_i | \dim(B \cap E) = j\}, j = 0, 1, \cdots, i,$$

where $E = \langle e_{n+1}, e_{n+2}, \cdots, e_{n+l} \rangle \in \mathbb{F}_q^{n+l}$.

In this paper we obtain the following result.

Theorem 2.1 *Semilattice $\mathcal{L}_0^m(\mathbb{F}_q^{n+l})$ is a quasi-regular semilattice. Its parameters are given by the formulas*

$$\mu(r(r_1), s(s_1); m_1) = q^{(s-s_1-r+r_1)(m_1-s_1)} \begin{bmatrix} m-r+r_1-m_1 \\ s-s_1-r+r_1 \end{bmatrix}_q \begin{bmatrix} m_1-r_1 \\ s_1-r_1 \end{bmatrix}_q,$$

$$\nu(r(r_1), s(s_1)) = q^{(r-r_1)(s_1-r_1)} \begin{bmatrix} s-s_1 \\ r-r_1 \end{bmatrix}_q \begin{bmatrix} s_1 \\ r_1 \end{bmatrix}_q,$$

and

$$\begin{aligned} & \pi(j(j_1), r(r_1), s(s_1); m_1) \\ = & \sum_{0 \leq i \leq j, 0 \leq i_1 \leq i} q^{(s-s_1-i+i_1)(j-j_1-i+i_1+m_1-s_1)+(s_1-i_1)(j_1-i_1)} \\ & \times \begin{bmatrix} (m-m_1)-(j-j_1) \\ (s-s_1)-(i-i_1) \end{bmatrix}_q \begin{bmatrix} m_1-j_1 \\ s_1-i_1 \end{bmatrix}_q N'(r+s-i, r_1+s_1-i_1; m, m_1; n+l, n). \end{aligned}$$

Here $N'(r+s-i, r_1+s_1-i_1; m, m_1; n+l, n)$ is given in [4, Lemma 2.3].

3 Proof of Theorem 2.1

Lemma 3.1 *Let $0 \leq k_1 \leq k_2 \leq k \leq l, 0 \leq m_1-k_1 \leq m_2-k_2 \leq m-k \leq n$, and let U_1, U_2 and U are subspaces of type $(m_1, k_1), (m_2, k_2)$ and (m, k) in $\mathbb{F}_q^{n+l}$, with $U_1 \subseteq U_2 \subseteq U$, respectively. If U_3 is a subspace of type (m_3, k_3) in $\mathbb{F}_q^{n+l}$ with $U_3 \subseteq U$ and $U_3 \cap U_2 = U_1$, then the number of U_3 is*

$$q^{(\delta_3-\delta_1)(\delta_2-\delta_1+k-k_3)+(k_3-k_1)(k_2-k_1)} \begin{bmatrix} \delta-\delta_2 \\ \delta_3-\delta_1 \end{bmatrix}_q \begin{bmatrix} k-k_2 \\ k_3-k_1 \end{bmatrix}_q ,$$

where $\delta = m-k, \delta_i = m_i - k_i, i = 1, 2, 3$.

Proof. Let $\delta_1 = m_1 - k_1, \delta_2 = m_2 - k_2, \delta_3 = m_3 - k_3, \delta = m - k$. By the transitivity of $GL_{n+l,n}(\mathbb{F}_q)$ on the set of subspaces of the same type, we may assume that

$$\begin{array}{ccc} & \begin{array}{cccc} \delta_1 & n-\delta_1 & k_1 & l-k_1 \end{array} & \\ U_1 = & \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} & \begin{array}{c} \delta_1 \\ k_1 \end{array} \end{array} ,$$

$$\begin{array}{ccc} & \begin{array}{cccccc} \delta_1 & \delta_2-\delta_1 & n-\delta_2 & k_1 & k_2-k_1 & l-k_2 \end{array} & \\ U_2 = & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{pmatrix} & \begin{array}{c} \delta_1 \\ k_1 \\ \delta_2-\delta_1 \\ k_2-k_1 \end{array} \end{array} ,$$

and

$$\begin{array}{ccc} & \begin{array}{cccccccc} \delta_1 & \delta_2-\delta_1 & \delta-\delta_2 & n-\delta & k_1 & k_2-k_1 & k-k_2 & l-k \end{array} & \\ U = & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{pmatrix} & \begin{array}{c} \delta_1 \\ k_1 \\ \delta_2-\delta_1 \\ k_2-k_1 \\ \delta-\delta_2 \\ k-k_2 \end{array} \end{array} .$$

Since $U_3 \cap U_2 = U_1$ and $U_3 \subset U$, we have

$$\begin{array}{ccc} & \begin{array}{cccccccc} \delta_1 & \delta_2-\delta_1 & \delta-\delta_2 & n-\delta & k_1 & k_2-k_1 & k-k_2 & l-k \end{array} & \\ U_3 = & \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & u_{32} & u_{33} & 0 & 0 & u_{36} & u_{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & u_{46} & u_{47} & 0 \end{pmatrix} & \begin{array}{c} \delta_1 \\ k_1 \\ \delta_3-\delta_1 \\ k_3-k_1 \end{array} \end{array} ,$$

where rank $u_{47} = k_3 - k_1$ and rank $u_{33} = \delta_3 - \delta_1$. Note that there are $\begin{bmatrix} k-k_2 \\ k_3-k_1 \end{bmatrix}_q$ choices for u_{47} and there are $\begin{bmatrix} \delta-\delta_2 \\ \delta_3-\delta_1 \end{bmatrix}_q$ choices for u_{33}. By the transitivity of $GL_{n+l,n}(\mathbb{F}_q)$ on the set of subspaces of the same type, the number of U_3's does not depend on the particular choice of u_{47} and u_{33}. Pick $u_{47} = (I^{(k_3-k_1)}, 0)$ and $u_{33} = (I^{(\delta_3-\delta_1)}, 0)$. Then U_3 has a matrix representation

$$\begin{array}{c} \begin{array}{cccccccccc} \delta_1 & \delta_2-\delta_1 & \gamma & \delta' & n-\delta & k_1 & k_2-k_1 & \lambda & k' & l-k \end{array} \\ \left(\begin{array}{cccccccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & u_{32} & I & 0 & 0 & 0 & u_{360} & 0 & u_{372} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{46} & I & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ \delta_1 \\ k_1 \\ \gamma \\ \lambda \end{array},$$

where $\delta' = \delta - \delta_2 - \delta_3 + \delta_1, k' = k - k_2 - k_3 + k_1, \gamma = \delta_3 - \delta_1, \lambda = k_3 - k_1$. Therefore the number of U_3 is equal to

$$q^{(\delta_3-\delta_1)(\delta_2-\delta_1+k-k_3)+(k_3-k_1)(k_2-k_1)} \begin{bmatrix} \delta-\delta_2 \\ \delta_3-\delta_1 \end{bmatrix}_q \begin{bmatrix} k-k_2 \\ k_3-k_1 \end{bmatrix}_q.$$

□

Lemma 3.2 *Let $A \in X_r^{r_1}$, $C \in X_m^{m_1}$ and $A \le C$. Then the number of $B \in X_s^{s_1}$ such that $A \le B \le C$ is equal to*

$$\mu(r(r_1), s(s_1); m_1) = q^{(s-s_1-r+r_1)(m_1-s_1)} \begin{bmatrix} m-r+r_1-m_1 \\ s-s_1-r+r_1 \end{bmatrix}_q \begin{bmatrix} m_1-r_1 \\ s_1-r_1 \end{bmatrix}_q.$$

Proof. By the transitivity of $GL_{n+l}(\mathbb{F}_q)$ on the set of subspaces of the same type, we may assume that

$$A = \begin{array}{c} \begin{array}{cccc} r-r_1 & n-r+r_1 & r_1 & l-r_1 \end{array} \\ \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right) \end{array} \begin{array}{c} \\ r-r_1 \\ r_1 \end{array},$$

and

$$C = \begin{array}{c} \begin{array}{cccccc} r-r_1 & m' & n' & r_1 & m_1-r_1 & l-m_1 \end{array} \\ \left(\begin{array}{cccccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{array}\right) \end{array} \begin{array}{c} \\ r-r_1 \\ r_1 \\ m' \\ m_1-r_1 \end{array},$$

where $m' = m - m_1 - r + r_1, n' = n - m + m_1$. Since

$$A \leq B \leq C,$$

we have

$$B = \begin{array}{c} \begin{array}{cccccc} r-r_1 & m' & n' & r_1 & m_1-r_1 & l-m_1 \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & u_{32} & 0 & 0 & u_{35} & 0 \\ 0 & 0 & 0 & 0 & u_{45} & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ r-r_1 \\ r_1 \\ s' \\ s_1-r_1 \end{array},$$

where $m' = m - m_1 - r + r_1, n' = n - m + m_1, s' = s - s_1 - r + r_1$, u_{45} denotes the (s_1-r_1)-subspace in $\mathbb{F}_q^{m_1-r_1}$ and u_{32} denotes the $(s-s_1-r+r_1)$-subspace in $\mathbb{F}_q^{m-m_1-r+r_1}$. Note that there are $\begin{bmatrix} m_1-r_1 \\ s_1-r_1 \end{bmatrix}_q$ choices for u_{45} and there are $\begin{bmatrix} m-m_1-r+r_1 \\ s-s_1-r+r_1 \end{bmatrix}_q$ choices for u_{32}. By the transitivity of $GL_{n+l}(\mathbb{F}_q)$ on the set of subspaces of the same type, the number of B's does not depend on the particular choice of u_{45} and u_{32}. Pick $u_{45} = (I^{(s_1-r_1)}, 0)$ and $u_{32} = (I^{(s-s_1-r+r_1)}, 0)$. Then B has a matrix representation

$$\begin{array}{c} \begin{array}{cccccccc} \delta_2 & \delta_3-\delta_2 & \delta_1-\delta_3 & n-\delta_1 & r_1 & s_1-r_1 & m_1-s_1 & l-m_1 \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & u_{352} & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \delta_2 \\ r_1 \\ \delta_3-\delta_2 \\ s_1-r_1 \end{array}.$$

where $\delta_1 = m - m_1, \delta_2 = r - r_1, \delta_3 = s - s_1$. Therefore the number of B is equal to

$$q^{(s-s_1-r+r_1)(m_1-s_1)} \begin{bmatrix} m-r+r_1-m_1 \\ s-s_1-r+r_1 \end{bmatrix}_q \begin{bmatrix} m_1-r_1 \\ s_1-r_1 \end{bmatrix}_q.$$

□

Lemma 3.3 *If $B \in X_s^{s_1}$, then the number of $A \in X_r^{r_1}$ such that $A \leq B$ is equal to*

$$\nu(r(r_1), s(s_1)) = q^{(r-r_1)(s_1-r_1)} \begin{bmatrix} s-s_1 \\ r-r_1 \end{bmatrix}_q \begin{bmatrix} s_1 \\ r_1 \end{bmatrix}_q.$$

Proof. By the transitivity of $G_{n+l,l}(\mathbb{F}_q)$ on the set of subspaces of the same type, we may assume that

$$B = \begin{array}{cc} \begin{array}{cccc} s-s_1 & n-s+s_1 & s_1 & l-s_1 \end{array} & \\ \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array} \right) & \begin{array}{c} s-s_1 \\ s_1 \end{array} \end{array} .$$

Since

$$A \leq B,$$

we have

$$A = \begin{array}{cc} \begin{array}{cccc} s-s_1 & n-s+s_1 & s_1 & l-s_1 \end{array} & \\ \left(\begin{array}{cccc} u_{11} & 0 & u_{13} & 0 \\ 0 & 0 & u_{23} & 0 \end{array} \right) & \begin{array}{c} r-r_1 \\ r_1 \end{array} \end{array} ,$$

where rank $u_{11} = r - r_1$ and rank $u_{23} = r_1$. Note that there are $\begin{bmatrix} s_1 \\ r_1 \end{bmatrix}_q$ choices for u_{23}. By the transitivity of $GL_{n+l}(\mathbb{F}_q)$ on the set of subspaces of the same type, the number of A's does not depend on the particular choice of u_{23}. Pick $u_{23} = (I^{(r_1)}\ 0)$. Then A has a matrix representation

$$\begin{array}{cc} \begin{array}{ccccc} s-s_1 & n-s+s_1 & r_1 & s_1-r_1 & l-s_1 \end{array} & \\ \left(\begin{array}{ccccc} u_{11} & 0 & 0 & u_{132} & 0 \\ 0 & 0 & I & 0 & 0 \end{array} \right) & \begin{array}{c} r-r_1 \\ r_1 \end{array} \end{array} .$$

Therefore the number of subspace A is equal to

$$\nu(r(r_1), s(s_1)) = q^{(r-r_1)(s_1-r_1)} \begin{bmatrix} s-s_1 \\ r-r_1 \end{bmatrix}_q \begin{bmatrix} s_1 \\ r_1 \end{bmatrix}_q .$$

□

Lemma 3.4 *Let $A \in X_r^{r_1}$ and $B \in X_m^{m_2}$. Assume that $A \wedge B \in X_j^{j_1}$, and $(C, D) \in X_s^{s_1} \times X_m^{m_1}$. If $C \leq D, C \leq B, A \leq D$, then the number of (C, D) is equal to*

$$\begin{aligned} & \pi(j(j_1), r(r_1), s(s_1); m_1) \\ = & \sum_{0 \leq i \leq j, 0 \leq i_1 \leq \min\{i, j_1\}} q^{(s-s_1-i+i_1)(j-j_1-i+i_1+m_1-s_1)+(s_1-i_1)(j_1-i_1)} \\ & \times \begin{bmatrix} (m-m_1)-(j-j_1) \\ (s-s_1)-(i-i_1) \end{bmatrix}_q \begin{bmatrix} m_1-j_1 \\ s_1-i_1 \end{bmatrix}_q N'(r+s-i, r_1+s_1-i_1; m, m_1; n+l, n). \end{aligned}$$

Here $N'(r+s-i, r_1+s_1-i_1; m, m_1; n+l, n)$ is given in [4, Lemma 2.3].

Proof. Since $A \leq D, C \leq B$, we have

$$A \wedge C \leq A \wedge B \leq D.$$

Since

$$C \leq D,$$

and

$$C \wedge (A \wedge B) = A \wedge (B \wedge C) = A \wedge C \in X_i^{i_1}, 0 \leq i \leq j, 0 \leq i_1 \leq \min\{i, j_1\}.$$

For $i \in [0, j]$ and $i_1 \in [0, \min\{i, j_1\}]$, by Lemma 3.1, the number of C is equal to

$$q^{(s-s_1-i+i_1)(j-j_1-i+i_1+m_1-s_1)+(s_1-i_1)(j_1-i_1)} \begin{bmatrix} (m-m_1)-(j-j_1) \\ (s-s_1)-(i-i_1) \end{bmatrix}_q \begin{bmatrix} m_1-j_1 \\ s_1-i_1 \end{bmatrix}_q.$$

Since $C \leq D, A \leq D$, $C + A \leq D$. It follows from $A \wedge C \in X_i^{i_1}$ that $A + C$ is a subspace of type $(r + s - i, r_1 + s_1 - i_1)$. Therefore, the number of D is equal to $N'(r + s - i, r_1 + s_1 - i_1; m, m_1; n + l, n)$. Hence the desired result follows. □

Combining Lemma 3.2, Lemma 3.3 and Lemma 3.4, we complete the proof of Theorem 2.1.

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