Optimum Seating Arrangements and Tuscan Squares

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0. *Abstract*

It may be desired to seat n people along a row (as at a lunch counter), or $n+1$ people around a circular table, in n consecutive rounds of seating, so that each person x has every other person y on their right exactly once, and on their left exactly once, in one of the seatings. Alternatively, it may be desired to seat $2n$ people along a row, or $2n+1$ people around a circular table, in only n consecutive rounds, so that each person x is *adjacent* to every other person y (either on the right or the left) exactly once. We show that these problems are solved using the rows of Tuscan squares to specify the successive rounds of seatings.

1. *Introduction*

Suppose that King Arthur wishes to have dinner with $2n$ of his favorite knights on $2n$ consecutive evenings, around the Round Table, so that each of these $2n+1$ worthies will be seated to the immediate right of each of the $2n$ others, and to the immediate left of each of the $2n$ others, exactly once? This seating arrangement can be accomplished for every positive integer n, and every Tuscan square of order $2n$ provides a solution. If, instead, King Arthur wishes to have dinner with $2n$ of his favorite knights on only n consecutive evenings, around the Round Table, so that each of these $2n+1$ gentlemen will be seated *adjacent* to each of the $2n$ others (either to the right or to the left) exactly once, this can also be accomplished for every positive integer n. A zig-zag construction for a Tuscan square of order $2n$ can be used to provide the seating patterns.

Suppose that at lunch time, the same $2n$ knights, without King Arthur, are to be seated on one side of a very long lunch counter. If it is desired that on $2n$ consecutive days each of the $2n$ knights will be seated to the immediate right of each of the other $2n-1$, and to the left of each of the other $2n-1$, exactly once, the seating arrangements to accomplish this are again in one-to-one correspondence with the Tuscan squares of order $2n$. It will also happen that each of the $2n$ knights will be seated at the right end of the counter, and at the left end of the counter, exactly once each.

If these $2n$ knights are to be seated at the same lunch counter on only n consecutive days in such a way that each of these knights will be adjacent to each of the $2n-1$ others (either to the right or to the left) exactly once, the zig-zag constructed Tuscan square mentioned above (for the Round Table seating) can be used to provide the n seating arrangements, for every positive integer n. It will also happen that each of them will be seated at an end of the lunch counter (either at the right and/or the left end) exactly once.

2. *The Role of Tuscan Squares*

In [1], a *Tuscan square of order n* was defined as an $n \times n$ array of n symbols (say $1, 2, 3, \ldots, n$) in such a way that each row is a permutation of the n symbols, and such that each of the $n(n-1)$ ordered pairs of adjacent symbols (say (i, j), with $1 \leq i \neq j \leq n$) occurs exactly once, in some row of the array.

If the edges of the complete directed graph on n node, $\vec{K}_n$, can be partitioned into n disjoint Hamiltonian paths, the sequence of nodes on these paths provide the n rows of a Tuscan square of order n, and conversely.

A Tuscan square of order n need not be a Latin square, since its columns (unlike its rows) are not required to be permutations of the n symbols. However, the leftmost column, and the rightmost column, of every Tuscan square of order n will in fact be a permutation of the n symbols. As a result, if an $(n+1)$st column, consisting of a single $(n+1)$st symbol (say "0"), is adjoined, and the rows are now regarded cyclically as consisting of $n+1$ symbols, then each of the $n(n+1)$ ordered pairs of adjacent symbols (say (i, j), with $0 \leq i \neq j \leq n$) will occur exactly once in some row of the array.

Clearly, any Tuscan square of order n can have its rows viewed as n consecutive seating patterns for n individuals (e.g. numbered from 1 to n) so that each individual will have each of the other $n-1$ on his/her right, and on his/her left, exactly once (as at a lunch counter) in the course of the n seatings. If instead we use the $(n+1) \times n$ array where an additional column (say all 0's) has been adjoined to a Tuscan square of order n, and we now regard the n rows cyclically, we have a seating arrangement for $n+1$ people around a round table, where each of the $n+1$ people will have each of the other n to the right, and to the left, exactly once.

Thus, there is a precise correspondence between Tuscan squares of order n, and seating arrangements in n rounds, for n people along a row, and $n+1$ people around a circular table.

A Tuscan square of order n is said to be in *standard form* if the top row and the left column each have the numbers from 1 to n in consecutive order. In [1], it is shown that in standard form there are 736 Tuscan squares of order 6, and 466,144 Tuscan square of order 7. Some of the Tuscan squares of order 6, but none of order 7, are also Latin. Here are non-Latin examples for $n = 6$ and for $n = 7$.

1	2	3	4	5	6
2	1	4	6	3	5
3	1	6	5	2	4
4	2	6	1	5	3
5	4	1	3	6	2
6	4	3	2	5	1

1	2	3	4	5	6	7
2	7	5	3	6	4	1
3	5	1	7	6	2	4
4	3	1	6	5	7	2
5	2	6	1	4	7	3
6	3	7	4	2	1	5
7	1	3	2	5	4	6

3. *Constructions for Seatings in Only Half the Rounds*

Suppose that, in a Tuscan square of order $2n$, half the rows are the left-right reversals of the other half. Then we can use only n rows to specify the seating, in n rounds, of $2n$ people along a line (e.g. at a lunch counter) in such a way that each person x is seated adjacent (either on the left or the right) to each other person y, exactly once, in one of the rounds. Also, if we have adjoined an extra column (say all 0's) to such a Tuscan square of order $2n$, we can use only n of the circular rows of length $2n+1$ to seat $2n+1$ people around a circular table in such a way that each of the $2n+1$ people sits next to each of the other $2n$ people (either to the right or the left) exactly once.

If $p = 2n + 1$ is prime, we can use the multiplication table modulo p to obtain a Tuscan square where half the rows are the left-right reversals of the other half. Here are the multiplication tables for $p = 3, p = 5$, and $p = 7$.

(mod 3)

×	1	2
1	1	2
2	2	1

(mod 5)

×	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(mod 7)

×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

If we adjoin an all-0's column to each of these Tuscan squares, so that we now have $(2n) \times (2n+1)$ multiplication tables modulo p, we can use n of the $2n$ rows as seating arrangements in only n rounds, around a circular table, so that each of $2n+1$ people is seated adjacent (either left or right) to each of the other $2n$, exactly once.

(mod 3)

×	0	1	2
1	0	1	2
2	0	2	1

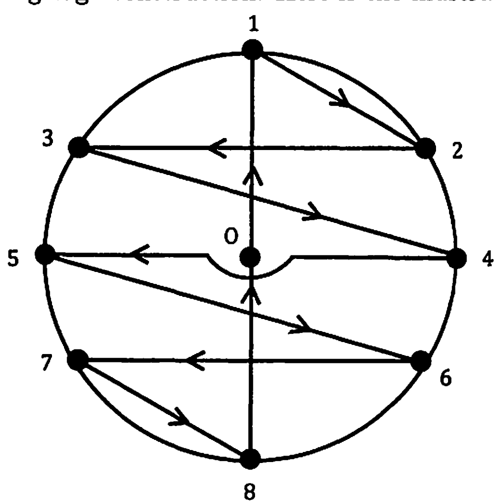
(mod 5)

×	0	1	2	3	4
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(mod 7)

×	0	1	2	3	4	5	6
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

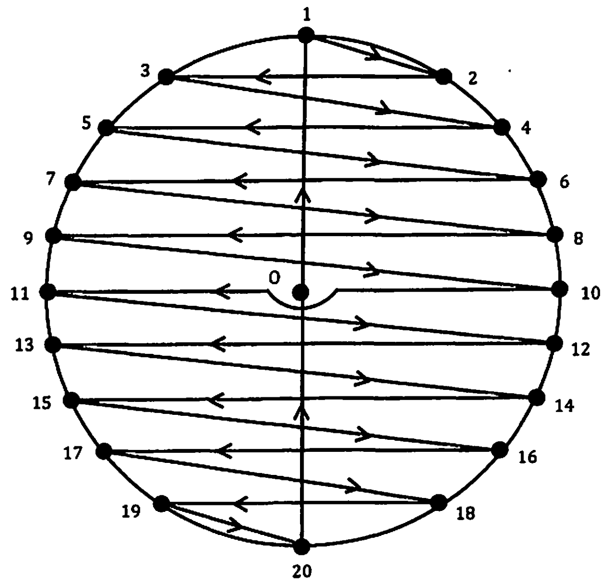
However, even if $2n+1$ is not prime, we can generate Tuscan squares of order $2n$ with the required property, for all positive integers n, using a "zig-zag" construction. Here is the illustration for $2n+1=9$.



0	1	2	3	4	5	6	7	8
0	2	4	1	6	3	8	5	7
0	3	1	5	2	7	4	8	6
0	4	6	2	8	1	7	3	5
0	5	3	7	1	8	2	6	4
0	6	8	4	7	2	5	1	3
0	7	5	8	3	6	1	4	2
0	8	7	6	5	4	3	2	1

The zig-zag pattern is rotated by $360°/2n = \pi/n$ to get the successive rows of the Tuscan square. (In this example, the successive rows will start, after the initial 0, with the values 1, 2, 4, 6, 8, 7, 5, 3, respectively, which have been reordered in the table to occur in numerical sequence.)

The next odd numbers which are not prime (beyond 9) are 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57, 63, ... To make the zig-zag construction completely clear, here is the illustration for $2n+1=21$.



Note that with $2n$ equally spaced nodes around a circle, with "1" at the top and "2n" at the bottom, the even numbers run consecutively down the right side, and the odd numbers down the left side. This pattern is then rotated consecutively by $360°/2n$ (in this case by $18°$) to generate the successive rows of the Tuscan square. The Tuscan squares so generated will also be Latin squares. (In the terminology of [1], a Tuscan square which is also a Latin square is called a *Roman square*.) When rotated $180°$, the edges on these circles look the same, except that all the arrows have their directions reversed. This is the property that guarantees that half the rows of the corresponding Tuscan square will be left-right reversals of the other half.

This zig-zag construction for Tuscan squares of order $2n$ is not the only way to generate Tuscan squares of order $2n$ where n of the rows are the left-right reversals of the other n rows.

4. *Conclusion*

The previous literature on Tuscan squares of order n ([1], [2], [3]) considered their relationship to the decomposition of the edges of the complete directed graphs on n nodes, $\vec{K}_n$, into n disjoint Hamiltonian paths, and their application to frequency hopping patterns in communications and radar and to statistical designs. In this paper, we show a direct correspondence between Tuscan squares of order n and seating arrangements of n, or of $n+1$, people in n rounds, to maximize the adjacency (both right and left) relationships. We further show that when n is even, certain

Tuscan squares exist to provide seating arrangements of n, or of $n+1$, people in $n/2$ rounds, where each person is adjacent to every other person, either on the right or left, exactly once. (The seatings with n people are along a row, while those with $n+1$ people are around a circular table.)

References.

[1] Golomb, Solomon W. and Herbert Taylor, "Tuscan Squares - A New Family of Combinatorial Designs", *Ars Combinatoria*, **20-B** (1985), pp. 115-132.

[2] Golomb, Solomon W., Herbert Taylor, and Tuvi Etzion, "Tuscan K-Squares", *Advances in Applied Mathematics*, **10** (1989), pp. 164-174.

[3] Golomb, Solomon W., Herbert Taylor, and Tuvi Etzion, "Polygonal Path Constructions for Tuscan K-Squares", *Ars Combinatoria*, **30** (1990), pp. 97-140.

[4] Chu, Wensong, Solomon W. Golomb, and Hong-Yeop Song, "Tuscan Squares", *The CRC Handbook of Combinatorial Designs*, 2^{nd} *Edition* (2006), Charles J. Colbourn and Jeffery H. Dinitz, eds., CRC Press.