Graph Decompositions of $K(v, \lambda)$ into Modified Triangles Using Langford and Skolem Sequences

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Abstract

We decompose the complete multigraph $K(v, \lambda)$ into copies of a graph H_i $(i = 1, 2, 3)$. Each H_i is a near triangle in that it is connected and has 3 vertices. In several case, the decompositions are completed using classical combinatorial sequences due to Langford and Skolem.

1 Introduction

A graph design is a decomposition of a graph $\mathcal{K}$ into copies of a graph H. For a complete and excellent survey of such decompositions see Adams, Bryant and Buchannan [1]. In this note we decompose $\mathcal{K} = K(v, \lambda)$, the complete multigraph on v vertices with λ edges between each pair of vertices, into copies of a graph H which is nearly a triangle. In Sections 2, 3, and 4, in fact, we consider three slightly different graphs H_1, H_2 and H_3, each of which is connected and

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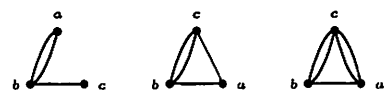
has 3 vertices and each has one or more double edges. We will refer to the corresponding designs as $H_i(v, \lambda)$ for $i = 1, 2, 3$. The three graphs H_i are shown in Figure 1. As is usual in design theory, we refer to each copy of H_i as a block of $H_i(v, \lambda)$.



H_1 H_2 H_3

Figure 1

In [5], the authors defined a triangle-like decomposition called a loop design, with blocks $\langle a, b, c, a \rangle$, a block on 3 vertices containing the edges $\{a, b\}$, $\{b, c\}$, $\{c, a\}$ and the loop $\{a, a\}$. In the present case, the H_i are different from graphs in [5], and they are different also from the small graphs listed in Chapter 24 of [4] and in [1]. The multigraphs H_i $(i = 1, 2, 3)$ which we consider here are the smallest multigraphs on three vertices which have not yet been considered in the literature and continue results from [5] and [4]. A Stanton graph of order k, S_k, is a graph on k vertices where, for each $i = 1, ..., k(k-1)/2$, there is exactly one edge of multiplicity i. There exists a Stanton graph of six edges on three vertices, and the graphs H_i considered here have 3, 4, or 5 edges; thus, the graph designs here are also different from those considered in [3]. Recently Saad El-Zanati [7] along with his co-authors proved that the necessary conditions are sufficient for the Stanton graph decomposition.

The work done in this paper is particularly interesting because of the new applications of certain well-known combinatorial sequences. Using these and other techniques, we have complete results for H_1 and H_2 decomposition in Sections 2 and 3 respectively. The decomposition problem for H_3, considered in Section 4, has several surprises and proved especially difficult and still not completely solved.

We first present several results which will be needed later beginning with Skolem and Langford sequences. A Skolem sequence of order n is a sequence $S = (s_1, s_2, ..., s_{2n})$ of $2n$ integers satisfying the conditions (1) for every $k \in \{1, 2, ..., n\}$ there exist exactly two elements s_i and s_j in S such that $s_i = s_j = k$, and (2) if $s_i = s_j = k$, with $i < j$, then $j - i = k$. Skolem sequences are also written as collections of ordered pairs $\{(a_i, b_i) : 1 \leq i \leq n,\ b_i - a_i = i\}$, with $\cup_{i=1}^{n}\{a_i, b_i\} = \{1, 2, ..., 2n\}$, and thus the differences $1, ..., n$ occur exactly once. See Chapter VI.53 of [4].

Somewhat more generally, a Langford sequence of order n and defect d is a sequence $L = \{y_1, y_2, ..., y_{2n}\}$ of $2n$ integers satisfying the conditions: (1) for every $k \in \{d, d+1, ..., d+n-1\}$ there exist exactly two elements $y_i, y_j \in L$, and (2) if $y_i = y_j = k$ with $i < j$, then $j - i = k$.

A sequence (Langford or Skolem) is called hooked if it contains $2n+1$ entries and the $2n$ entry is a place holder only.

Lemma 1 *(The S-L Lemma, p. 613 of [4]) (a) A Langford sequence of order* n *and defect* d *exists if and only if (1)* $n \geq 2d - 1$, *and (2)* $n \equiv 0, 3 \pmod 4)$ *for* *even d or $n \equiv 0, 1$ (mod 4) and d is odd. A hooked Langford sequence of order n and defect d exists if and only if (1) $n(n-2d+1)+2 \geq 0$ and (2) $n \equiv 2, 3$ (mod 4) and d is odd, or $n \equiv 1, 2$ (mod 4) and d is even. (b) A Skolem sequence of order t exists if and only if $t \equiv 0, 1$ (mod 4). A hooked Skolem sequence of order t exists if and only if $t \equiv 2, 3$ (mod 4).*

A pair-wise balanced design, a PBD(v, K), is a pair $(V, \mathcal{B})$ where V is a set of v points, $\mathcal{B}$ is a collection of blocks, and K is the set of block sizes. Each pair of points of a PBD meet in exactly one block. The following lemma is adapted from Table 3.23, p. 249, in [4].

Lemma 2 *(The PBD Lemma) (a) There exists a PBD(v, K) for $K = \{4, 5\}$ and any $v \equiv 0, 1$ (mod 4), except for $v = 8, 9, 12$. (b) There exists a PBD(v, K) for $K = \{3, 4, 6\}$ for all $v \equiv 0, 1$ (mod 3). (c) There exists a PBD(v, K) for $K = \{4, 6, 7, 9\}$ and any $v \equiv 0, 1$ (mod 3) except $v = 10, 12, 15, 18, 19, 24, 27$. (d) There exists a PBD$(v, K)$ for $K = \{8, 9\}$ for any $v \equiv 0, 1$ (mod 8), with several hundred possible exceptions less than 1680. (e) There exist PBD(v, K) for $K = \{3, 4, 5\}$ for all v except $v = 6, 8$.*

We will need a structure theorem of H. Agrawal.

Lemma 3 *[2] (Agrawal's Lemma) In every binary equi-replicate design of constant block size k (hence $bk = vr$ and $b = mv$), the treatments in each block can be rearranged such that in the k by b array, formed with blocks as columns, every treatment occurs in each row exactly m times.*

A path design, P(v, k, λ) is a decomposition of $K(v, \lambda)$ into paths of length $k-1$. The ordered path block $\langle a, b, c\rangle$ contains the edges $\{a, b\}$ amd $\{b, c\}$, and the path block $\langle a, b, c\rangle$ is identified with the block $\langle c, b, a\rangle$. It is well-known that there exists a P$(v, 3, 2)$ for every $v \geq 3$.

Lemma 4 *(Path Lemma) For every $v \geq 3$, there exists a P$(v, 3, 2)$.*

Proof: For $v = 3$, use blocks $\langle 1, 2, 3\rangle$, $\langle 2, 3, 1\rangle$, and $\langle 3, 1, 2\rangle$. Suppose $X =$ P$(v, 3, 2)$ exists for some $v \geq 3$. Then we create a P$(v+1, 3, 2)$ by adding the following blocks to those of X: $\langle i, v+1, i+1\rangle$, for $i = 1, 2, ..., v-1$, and the block $\langle v, v+1, 1\rangle$. The result follows by induction on v. □

2 H_1(v,λ) Designs

In this section we will decompose $K(v, \lambda)$ into copies of H_1 (see Figure 1) and we denote by $\langle a, b, c\rangle$ the *ordered* block with three vertices a, b, and c and the edges $\{a, b\}$, $\{a, b\}$ and $\{b, c\}$.

We seek necessary conditions first. Since each block contains 3 edges, we require $\lambda v(v-1) \equiv 0$ (mod 3). Since each pair will evidently occur at least twice, it is necessary that $\lambda \geq 2$.

2.1 The case $\lambda = 2$

The necessary condition, when $\lambda = 2$, is that $v = 3t$ or $3t + 1$. Whenever $\lambda = 2$, if a, b are the first two elements in a block, the pair of edges for a, b occur in that same block. Thus, if a, b occur together in some other block, they must later occur in the 1st and 3rd positions. Similarly, if b, c occur in the 2nd and 3rd positions in a block, they must occur together once more in the 2nd and 3rd positions in a second block so that their edges occur exactly twice.

Example 1 *The blocks* $\langle 1,2,3\rangle$, $\langle 2,3,1\rangle$, *and* $\langle 3,1,2\rangle$ *show that there exists an* $H_1(3,3)$. *The blocks* $\langle 1,2,3\rangle$ *and* $\langle 1,3,2\rangle$ *give an* $H_1(3,2)$.

For $v = 6t + 1$ or $6t + 3$ there exists a BIBD$(v, 3, 1)$. For any BIBD$(v, 3, 1)$, by identifying each block $\{p, q, r\}$ with the three blocks of the $H_1(3, 3)$, one may create an $H_1(v, 3)$, and similarly an $H_1(v, 2)$. Therefore, there exists an $H_1(v, \lambda)$ if $v = 6t + 1$ or $6t + 3$. for any $\lambda \geq 2$. This gives us a required $H_1(19, 2)$ design in order to apply Lemma 2(c).

Example 2 *The blocks* $\langle 3,2,1\rangle$, $\langle 4,1,2\rangle$, $\langle 2,4,3\rangle$, *and* $\langle 1,3,4\rangle$ *give an* $H_1(4,2)$. *It follows that there exists an* $H_1(v,2)$ *if* $v = 12t + 1$ *or* $12t + 4$ *since there exists BIBD*$(v,4,1)$ *for these parameters.*

Example 3 *The design* $H_1(6,2)$ *is cyclically generated mod* 5 *by the starter blocks* $\langle 1,3,\infty\rangle$ *and* $\langle 1,2,\infty\rangle$.

The examples and comments just above suffice to apply Lemma 2(b) to prove Theorem 1 below but we give further examples interesting in their own right which can be used to apply Lemma 2(c).

Example 4 *An* $H_1(7,2)$. *Use the starter blocks* $\langle 4,2,1\rangle$ *and* $\langle 4,1,2\rangle$ *and develop mod* 7.

Example 5 *An* $H_1(9,2)$. *There are* 72 *edges and* 24 *blocks. Since* 24 *is not a multiple of* 9, *the role of each vertex in the block cannot be uniform. Recall that the first vertex has degree* 2, *the second has degree* 3 *and the last has degree* 1. *The points are* $\{\infty, 0, 1, \ldots, 7\}$ *and the blocks are:*
(1) $\langle \infty, 1+i, 5+i\rangle$ *and* $\langle \infty, 5+i, 1+i\rangle$ *for* $i \in \{0,1,2,3\}$, *and*
(2) $\langle 4+i, 1+i, 2+i\rangle$ *and* $\langle 3+i, 1+i, 2+i\rangle$ *for* $i \in \{0,1,\ldots,7\}$, *with all sums modulo* 8.

Suppose now that $v > 27$ and $v = 3t$ or $3t + 1$. Let P denote a PBD on v points with block sizes from $K = \{4, 6, 7, 9\}$ and suppose b is any block of the PBD. Identify, arbitrarily, the points of block b with the points in the example just above with $|b|$ points. Form the blocks as in the above example using the points of b. Do this for each block of P. The set of blocks thus formed gives an $H_1(v, 2)$. This shows that all $H_1(v, 2)$ exist for $v = 3t$ or $3t+1$ with the possible exceptions from the list in the PBD Lemma. We now proceed to construct the other designs. We observe that, since $H_1(6t + 1, 2)$ and $H_1(6t + 3, 2)$ exist (see the comment after Lemma 1), we only need to construct designs for $v = 10, 12, 18$, and 24.

Example 6 *An* $H_1(10,2)$ *on points* $\{(i,j) : i = 0,1$ *and* $j = 0,1,2,3,4\}$.
Difference sets to be developed mod$(5,*)$*:*
$\{(0,1),(0,0),(1,0)\}$, $\{(1,1),(1,0),(0,0)\}$, $\{(0,3),(1,1),(0,0)\}$,
$\{(1,4),(0,0),(1,1)\}$, $\{(0,3),(0,0),(1,2)\}$, *and* $\{(1,0),(1,2),(0,0)\}$.

This is isomorphic to the following $H_1(10,2)$.

2	3	4	5	1
1	2	3	4	5
6	7	8	9	10

,

7	8	9	10	6
6	7	8	9	10
1	2	3	4	5

,

4	5	1	2	3
7	8	9	10	6
1	2	3	4	5

,

10	6	7	8	9
1	2	3	4	5
7	8	9	10	6

,

4	5	1	2	3
1	2	3	4	5
8	9	10	6	7

,

6	7	8	9	10
8	9	10	6	7
1	2	3	4	5

Example 7 *An* $H_1(12,2)$. *Use the starter blocks* $\langle 3,1,\infty\rangle$, $\langle 4,1,\infty\rangle$, $\langle 5,1,6\rangle$, $\langle 7,6,1\rangle$, *and develop mod* 11.

Example 8 *An* $H_1(18,2)$. *Use the starter blocks* $\langle 3,1,\infty\rangle$, $\langle 4,1,\infty\rangle$, $\langle 5,1,9\rangle$, $\langle 6,1,9\rangle$, $\langle 7,1,2\rangle$, $\langle 8,1,2\rangle$ *and develop mod* 17.

Example 9 *An* $H_1(24,2)$, *Use the starter blocks* $\langle 5,1,\infty\rangle$, $\langle 8,1,\infty\rangle$, $\langle 3,1,2\rangle$, $\langle 19,2,1\rangle$, $\langle 9,1,4\rangle$, $\langle 18,4,1\rangle$, $\langle 11,1,6\rangle$, $\langle 18,6,1\rangle$ *and develop mod* 23.

The following theorem now follows from Lemma 2.

Theorem 1 *The necessary conditions are sufficient for the existence of* $H_1(v,2)$.

A cyclic proof of Theorem 1, suggested by a reader of the paper, can be given succinctly as follows. We specify the needed difference sets developed modulo x by using $C(m) = \{\langle t+2i-1,0,i\rangle, \langle t+2i,0,i\rangle : 1 \le i \le m\}$ mod x.
$H_1(6t+1,2)$ on Z_{6t+1} : $C(t)$ mod $6t+1$;
$H_1(6t+3,2)$ on $Z_{6t+2}\bigcup\{\infty\}$: $C(t)$ and $\langle\infty,0,3t+1\rangle$ mod $6t+2$;
$H_1(6t+4,2)$ on Z_{6t+4} : $C(t)$ and $\langle 3t+1,0,3t+2\rangle$ mod $6t+4$;
$H_1(6t,2)$ on $Z_{6t-1}\bigcup\{\infty\}$: $C(t-1)$ and $\langle 3t-1,0,t\rangle$, $\langle\infty,0,t\rangle$ mod $6t-1$.

2.2 The case $\lambda = 3$

We now consider the case for $\lambda = 3$.

Theorem 2 *The necessary conditions are sufficient for the existence of* $H_1(v,3)$.

Proof: We will show that there exists an $H_1(v,3)$ for every $v \ge 3$. Suppose $\mathcal{B}$ is the set of blocks of a $\mathrm{P}(v,3,2)$. First, rewrite each block $\langle a,b,c\rangle$ of $\mathcal{B}$ as $\{\{a,b\},\{b,c\}\}$. If we regard the edges $\{a,b\}$ as the points of a new design, then as the new blocks of $\mathcal{B}$ have two points each and as each point appears twice in $\mathcal{B}$, the set of blocks is an equi-replicate design. Applying Agrawal's Lemma, put the blocks of $\mathcal{B}$ in an array, say A_1, with two rows so that each new point $\{a,b\}$

appears once in each row. Now create a new array, say A_2, whose first two rows are copies of the first row of A_1, and whose third row is a copy of the second row of A_1. The new array is now seen to be a decomposition of $\mathcal{K} = K(v, 3)$ into blocks of the form $\{\{p, q\}, \{p, q\}, \{q, r\}\}$ for some p, q, and r. That is, A_2 is the set of blocks of an $H_1(v, 3)$. □

We may observe that every $\lambda \geq 2$ may be written as $\lambda = 2s + 3t$ for some non-negative integers s and t. We have shown that, if $v = 3j$ or $3j + 1$, then the only necessary H_1-condition, $\lambda v(v-1) \equiv 0 \pmod 3$, is met, and that, for such v, there exist $H_1(v, 2s + 3t)$ by using the blocks of s-copies of the design in Theorem 1 and t-copies of the design from Theorem 2. If $v = 3j + 2$, then $H_1(v, 3t)$ exist by the previous theorem. This proves:

Theorem 3 *The necessary conditions are sufficient for the existence of* $H_1(v, \lambda)$.

3 $H_2(v, \lambda)$ Designs

In this section we decompose $\mathcal{K} = K(v, \lambda)$ into copies of H_2, and a single block $\langle a, b, c\rangle$ represents the four edges $\{a, b\}$, $\{b, c\}$, $\{b, c\}$, and $\{c, a\}$. Since there are 4 edges per block, the necessary condition is $\lambda v(v-1) \equiv 0 \pmod 8$.

3.1 The Case $\lambda = 2$

When $\lambda = 2$, it is necessary that $v \equiv 0, 1 \pmod 4$. We begin with examples which will allow us to apply the PBD Lemma.

Example 10 *There exists an* $H_2(4, 2)$ *design. The blocks are* $\langle a, b, c\rangle$, $\langle a, b, d\rangle$, *and* $\langle a, c, d\rangle$.

Example 11 *There exists an* $H_2(5, 2)$ *design. The blocks are*

1	2	3	4	5
4	3	5	2	1
5	4	2	1	3

Example 12 *An* $H_2(8, 2)$. *Use starter blocks* $\langle \infty, 1, 4\rangle$ *and* $\langle 0, 1, 6\rangle$ *and develop mod* 7.

Example 13 *There exists an* $H_2(9, 2)$. *Use starter blocks* $\langle 0, 3, 4\rangle$ *and* $\langle 0, 3, 5\rangle$ *and develop mod* 9.

Example 14 *An* $H_2(12, 2)$. *Use starter blocks* $\langle \infty, 0, 4\rangle$, $\langle 0, 3, 9\rangle$, $\langle 0, 3, 2\rangle$ *and develop mod* 11.

Theorem 4 *There exists an* $H_2(v, 2)$ *for* $v \equiv 0, 1 \pmod 4$.

Proof: Apply the PBD Lemma. In view of the examples, we may assume $v \neq 8, 9, 12$. Let P denote a PBD on v points, where $v = 4t$ or $4t+1$ but $v \neq 8, 9, 12$. For each block β of size 4 in P, identify arbitrarily the points of β with the points of the $H_2(4,2)$ above and create the 3 blocks as in the example. For each block of P of size 5, create five blocks as in the example for the $H_2(5,2)$. Since each pair of points met exactly once in the PBD, they meet twice in the $H_2(v,2)$. □

3.2 The Case $\lambda = 3$

Next we consider $H_2(v,3)$, and the necessary condition is $v(v-1) \equiv 0 \pmod 8$.

Theorem 5 *There exists* $H_2(8t+1,3)$ *for all* $t \geq 1$.

Proof: Develop cyclically the following starter blocks modulo v, for $v = 8t+1$: $\langle 4(t-r), 0, s+4r \rangle$ where $1 \leq s \leq 3$ and $0 \leq r \leq t-1$. □

Example 15 *If* $v = 8, 16, 24$, *then there is an* $H_2(v,3)$*:*

v	Develop cyclically modulo $v-1$
8	$\langle 4,\infty,5\rangle$, $\langle 3,0,2\rangle$, $\langle 2,0,3\rangle$
16	$\langle 8,\infty,9\rangle$, $\langle 6,0,7\rangle$, $\langle 7,0,5\rangle$, $\langle 3,0,4\rangle$, $\langle 5,0,3\rangle$, $\langle 4,0,6\rangle$
24	$\langle 13,\infty,12\rangle$, $\langle 10,0,11\rangle$, $\langle 9,0,10\rangle$ $\langle 3,0,5\rangle$, $\langle 6,0,8\rangle$, $\langle 8,0,6\rangle$ $\langle 7,0,3\rangle$, $\langle 5,0,9\rangle$, $\langle 11,0,7\rangle$

It is interesting that, for each $v = 8t$ in this last example, replace ∞ by 0 in the first block, and obtain a set of starter blocks for $8t+1$ (developed mod $8t+1$).

The results from this section and the PBD Lemma prove that:

Theorem 6 *The necessary conditions are sufficient for the existence of* $H_2(v,3)$, *except possibly for the* $v = 8t < 1680$ *listed in Table 3.23 in [4].*

Sarvate and Zhang [8] have recently completed the existence by resolving all these cases in affirmative.

3.3 The case $\lambda = 4$

This subsection is devoted to proving the following main theorem:

Theorem 7 *There exists an* $H_2(v,4)$ *for all* $v \geq 3$.

An $H_2(v,4)$ exists for $v \equiv 0, 1 \pmod 4$ by taking two copies of the design for index 2.

Theorem 8 *There exists an* $H_2(4t+3,4)$ *for every* $t \geq 0$.

Proof: Use a BIBD$(4t+3,3,3)$ which exists as v is odd. Write each BIBD block $\delta = \{a,b,c\}$ as a block of edges $\{\{a,b\},\{b,c\},\{c,a\}\}$. Apply Agrawal's Lemma to the newly written blocks of the BIBD. In the Agrawal array, for the block δ given, suppose $\{b,c\}$ occurs in row 3. Then choose $\{b,c\}$ to be the double edge in the H_2 block $\langle a,b,c\rangle$. Since each edge occurs once in row 3, every pair now occurs 4 times in H_2 blocks and the index of the design is 4. □

We now suppose $v = 4t+2$. In order to show that there is always a cyclic $H_2(4t+2,4)$, we divide $H_2(4t+2,4)$ into two cases, $v = 8m+2$ and $8m+6$.

Example 16 *An $H_2(6,4)$. Use the blocks (developed mod 5): $\langle\infty,1,2\rangle$, $\langle\infty,1,3\rangle$, $\langle 4,1,2\rangle$.*

Example 17 *An $H_2(14,4)$. Use a Langford sequence of order 4 and defect 2. (See array below.) The starter blocks (mod 13) from the sequence are $\langle 0,2,7\rangle$, $\langle 0,3,5\rangle$, $\langle 0,4,8\rangle$, $\langle 0,6,9\rangle$. The remaining blocks are $\langle\infty,0,1\rangle$, $\langle\infty,0,6\rangle$, and $\langle 0,10,11\rangle$. The Langford pairs determine starter block differences of 2, 3, 4, 5, which are associated with double edges. The vertices used for these blocks, $2,3,\ldots,9$ omit 1, 10, and 11 $(1,-2,-3$ mod $13)$. The remaining three blocks use differences of 1 (four times) and 6 (twice). Since 6 and 7 $(\equiv -6 \pmod{13})$ are used as vertices, all pairs with difference 6 occur 4 times in the blocks developed. The final block, with vertices 10, 11 covers the differences $-3,-2$ (mod 13). It follows that the index is 4.*

4	1	3	1	2	4	3	2
2	3	4	5	6	7	8	9

Example 18 *An $H_2(18,4)$. Use a Langford sequence of order 7 and defect 2 to create starter blocks (mod $v-1=17$). In the array below, the top row is a Langford sequence of order 7. In the second row are the vertices $\{2,3,4,\ldots,15\}$. First use the "initial" starter blocks $\langle\infty,0,1\rangle$ and $\langle\infty,1,2\rangle$. The other starter blocks are determined by the pairs of vertices which correspond to like elements in the sequence. They are $\langle 0,2,4\rangle$, $\langle 0,5,8\rangle$, $\langle 0,10,14\rangle$, $\langle 0,7,12\rangle$, $\langle 0,9,15\rangle$, $\langle 0,6,13\rangle$, $\langle 0,3,11\rangle$.*

1	7	1	2	6	4	2	5	3	7	4	6	3	5
2	3	4	5	6	7	8	9	10	11	12	13	14	15

Theorem 9 *There exists an $H_2(4t+2,4)$ for all $t \geq 1$.*

Proof: First suppose $v = 8m+2$. The cyclic solution begins with the (mod $v-1$) starter blocks $\langle\infty,0,1\rangle$ and $\langle\infty,1,2\rangle$. Next, use a Langford sequence of order $4m-1$ and defect 2 to determine pairs $\{a,b\}$, as in the example, with differences $2,3,\ldots,4m$ and form starter blocks $\langle 0,a,b\rangle$ for each pair. The blocks are developed mod $v-1$. Now suppose $v = 8m+6$. Use a Langford sequence of order $4m$ and defect 2. The first $4m$ starter blocks (mod $v-1$) are $\langle 0,a_i,b_i\rangle$ where $\{a_i,b_i\}$ are the pairs determined by the sequence as in the array. In these starter blocks, the differnces $|a_i-b_i|$ include the values $2,3,\ldots,4m+1$ which correspond to double edges. The vertices used in these blocks, $2, 3, ..., 8m + 1$, correspond to single edges.The last three starter blocks, which fill in the missing needed differences, are $\langle\infty, 0, 1\rangle$, $\langle\infty, 0, 4m + 2\rangle$ and $\langle 0, 8m + 2, 8m + 3\rangle$. □

We may observe that necessary conditions $v \geq 3$ and $v \equiv 0, 1 \pmod 8$ are sufficient for the existence of a $H_2(v, 3)$ for all $v's$ except for finitely many values and $H_2(v, 2)$ exists for all $v \geq 3$ and $v \equiv 0, 1 \pmod 4$. Hence given any odd λ, say, $2t + 3$, we can construct $H_2(v, 2t + 3)$ for all integers $t \geq 0$. As the necessary conditions when $\lambda \equiv 0 \pmod 4$ are same as the necessary and sufficient conditions for the existence of $H_2(v, \lambda = 4)$ and for $\lambda \equiv 2 \pmod 4$ are same as the necessary and sufficient conditions for the existence of $H_2(v, \lambda = 2)$, we have:

Theorem 10 *The necessary conditions are sufficient for the existence of $H_2(v, \lambda)$ except possibly for the $v = 8t < 1680$ listed in Table 3.23 in [4].*.

As mentioned earlier, Sarvate and Zhang [8] have resolved all these cases in affirmative.

4 $H_3(v, \lambda)$ Designs

In this section we decompose $\mathcal{K} = K(v, \lambda)$ into copies of H_3, and a single block $\langle a, b, c\rangle$ represents the five edges $\{a, b\}$, $\{b, c\}$, $\{b, c\}$, $\{c, a\}$, $\{c, a\}$. The third vertex listed in each block will always be the vertex with degree 4 and the other two will be the vertices with degree 3. The necessary condition is that $\lambda v(v-1) \equiv 0 \pmod{10}$.

4.1 The $\lambda = 2$ case for small v

For an $H_3(v, 2)$ to exist, it is necessary that $v = 5t$ or $5t + 1$ since there are 5 edges per block.

Example 19 *We develop the starter block $\langle 4, 1, 0\rangle$ cyclically modulo 6, to obtain an $H_3(6, 2)$.*

Theorem 11 *There does not exist an $H_3(5, 2)$.*

Proof: For $v = 5$, from the necessary conditions, there would be 20 edges and 4 blocks. Suppose $\langle 1, 2, 3\rangle$, corresponding to edges $\{1, 2\}$, $\{2, 3\}$, $\{2, 3\}$, $\{1, 3\}$, and $\{1, 3\}$, is a block. Now, since $\{1, 2\}$ must be a single edge in exactly one more block, we may assume WLOG it is $\langle 1, 2, 4\rangle$. Thus, $\{1, 2\}$ appears twice in H_3 blocks. However, in $K(5, 2)$, the degree of vertex 1 is eight, and in the first two blocks, vertex 1 has a degree sum of six. In a third and last block with edge $\{1, 2\}$, vertex 1 must have degree two. But there are no vertices of degree two in an H_3 block. □

Example 20 *There exists an* $H_3(5,4)$.

b	b	d	d	d	d	b	b
c	c	e	e	e	e	c	c
a	a	a	a	b	c	d	e

Example 21 *An* $H_3(10,2)$.

7	9	5	7	9	1	7	8	1	7	8	5	7	8	2	7	8	2
8	10	6	8	10	4	9	10	4	9	10	6	10	9	3	10	9	3
1	1	1	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6

Example 22 *In constructing an* $H_3(15,2)$, *if one vertex occurs only as a degree 4 vertex, experience suggests we refer to it as* ∞ *and consider a cycled block* $\langle a+i, b+i, \infty\rangle$. *However, here* $0 \le i \le 6$. *This suggests having six starter blocks which cycle seven times each for the needed 42 blocks. This further suggested dividing the vertices into two sets* $\{8,9,...,14\}$ *and* $\{1,2,...,7\}$ *which cycle independently. Starter blocks meeting the conditions are* $\langle 1,11,\infty\rangle$, $\langle 1,11,8\rangle$, $\langle 1,12,13\rangle$, $\langle 1,12,10\rangle$, $\langle 1,3,9\rangle$, *and* $\langle 1,3,7\rangle$. *For example note the development* $\langle 1,8,11\rangle$, $\langle 2,9,12\rangle$, $\langle 3,10,13\rangle$, $\langle 4,11,14\rangle$, $\langle 5,12,8\rangle$, $\langle 6,13,9\rangle$, $\langle 7,14,10\rangle$. *It follows that we have constructed an* $H_3(15,2)$.

The general case for $H_3(5t,2)$ remains open.

When $v = 5t+1$, however, counting [for any vertex x, $deg(x) = \lambda(v-1) = 2(5t) = 10t = 4(t) + 3(2t)$] shows that each vertex may occur t-times as a degree 4 vertex in some H_3 block and $2t$-times as a vertex of degree 3. This suggests a complete cyclic construction. We will actually show that there exists a cyclic $H_3(v,2)$ if $v = 5t+1$ for some $t \ge 1$. The proof will be completed in the next subsection in which we show how to create the starter blocks.

4.2 More applications of Skolem and Langford sequences

We apply Skolem sequences to $v = 10t+1$ and Langford sequences to $v = 10t+6$.

Example 23 *An* $H_3(41,2)$. *The "Skolem differences" are* $\{1,2,3,4\}$ *and the pairs* (a_i, b_i) *are determined from the array. The top row is a Skolem sequence of order* $t = 4$ *(length 8), and the same sequence is listed again. The vertices* (a_i, b_i) *are listed sequentially in the second row. The starter blocks determined are* $\langle 5,6,0\rangle$, $\langle 7,10,0\rangle$, $\langle 8,12,0\rangle$, $\langle 9,11,0\rangle$, $\langle 13,14,0\rangle$, $\langle 15,18,0\rangle$, $\langle 16,20,0\rangle$, $\langle 17,19,0\rangle$. *There are* $\lambda(v-1)/10 = 2t = 8$ *starter blocks. Because of the degree 4 vertex in each* H_3 *starter block is the vertex 0, it is important that the Skolem differences do not appear as vertices in the starter blocks.*

1	1	3	4	2	3	2	4	1	1	3	4	2	3	2	4
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Example 24 *An $H_3(31,2)$. Use a hooked Skolem sequence and two arrays. The Skolem sequence 31132*2 is repeated in reverse order for the second array. For the first pairs,* 9 *is not used, and in the second group* 10 *is not used. The starter blocks are* $\langle 4,7,0\rangle$, $\langle 5,6,0\rangle$, $\langle 8,10,0\rangle$, $\langle 9,11,0\rangle$, $\langle 12,15,0\rangle$, *and* $\langle 13,14,0\rangle$.

3	1	1	3	2	*	2
4	5	6	7	8	9	10

and

2	*	2	3	1	1	3
9	10	11	12	13	14	15

When $v = 10t + 6$, there are $2t + 1$ starter blocks needed. The method used requires an initial block $\langle 5t+2, 10t+5, 0\rangle$. In this way, the vertices with difference $v/2$ occur twice as single edges when this block is developed mod v. The remaining blocks are determined as above but using a Langford sequence of order t and defect 2.

Example 25 *An $H_3(46,2)$. For $v = 46 = (10)(4) + 6$, the parameter t is* 4. *Begin with the starter block* $\langle 22,45,0\rangle$. *When this block is developed mod* 46, *each pair of vertices with difference* 23 *will occur twice. The differences* 1 *(*$45 - 0$ *mod* 46*) and* 22 *correspond to double edges. The remaining differences needed for the starter blocks are* $2,3,...,21$. *The single edges will correspond to differences* $\{2,3,4,5\}$. *We illustrate with the arrays below. The top row is a Langford sequence of order $t = 4$ and defect* 2, *and it is repeated in the next array. Between the two* 4*'s are four terms. Between the two* 3*'s, there are three terms, and so on. Corresponding to the two* 4*'s in the first row, in the second row are the terms 6 and 11, whose difference is* $4 + 1$. *The four corresponding pairs in row* 2 *are the vertices in the starter blocks. These are:* $\langle 23,45,0\rangle$, $\langle 6,11,0\rangle$, $\langle 7,9,0\rangle$, $\langle 8,12,0\rangle$, $\langle 10,13,0\rangle$, *and* $\langle 14,19,0\rangle$, $\langle 15,17,0\rangle$, $\langle 16,20,0\rangle$, $\langle 18,21,0\rangle$.

4	1	3	1	2	4	3	2
6	7	8	9	10	11	12	13

and

4	1	3	1	2	4	3	2
14	15	16	17	18	19	20	21

Example 26 *An $H_3(56,2)$. With $t = 5$, the Langford differences are* $\{2,3,4,5,6\}$ *and a suitable set of pairs is determined by a hooked Langford sequence. The first* 11 *entries in the top row below give a hooked sequence, and the vertices are listed sequentially. The entries are reversed to finish the array. The pairs which determine the starter blocks are the matches in the second row to the corresponding entries in the first row. The starter blocks are* $\langle 27,55,0\rangle$, $\langle 7,11,0\rangle$, $\langle 8,13,0\rangle$, $\langle 9,15,0\rangle$, $\langle 10,12,0\rangle$, $\langle 14,17,0\rangle$, $\langle 16,19,0\rangle$, $\langle 18,24,0\rangle$, $\langle 20,25,0\rangle$, $\langle 21,23,0\rangle$, $\langle 22,26,0\rangle$.

3	4	5	1	3	1	4	2	5	*	2
7	8	9	10	11	12	13	14	15	16	17

2	*	5	2	4	1	3	1	5	4	3
16	17	18	19	20	21	22	23	24	25	26

Theorem 12 *There exists an $H_3(10t+6,2)$ for all $t \geq 0$ and an $H_3(10t+1,2)$ for all $t \geq 1$.*

Proof: For $v = 10t+6$, use the starter block $\langle 5t+2, 10t+5, 0\rangle$. Then, if $t \equiv 0, 3 \pmod 4$, use a Langford sequence of order t and defect 2 and if $t \equiv 1, 2 \pmod 4$ use a hooked Langford sequence. For $v = 10t+1$, if $t \equiv 0, 1 \pmod 4$, then a Skolem sequence of order t exists and the sequence, and its shift, generate suitable starter blocks. If $t \equiv 2, 3 \pmod 4$, then a hooked Skolem sequence suffices as in the examples. □

4.3 Higher Values of the Index.

Theorem 13 *There does not exist an $H_3(v, 3)$ for any v.*

Proof: Since the index is three, which is odd, each edge must appear in an H_3 block as a singleton edge at least once. There is one such edge per block, and, therefore, there are not enough such edges as at least $v(v-1)/2$ are needed but there are only $3v(v-1)/10$ blocks. □

Theorem 14 *If v is even, there does not exist an $H_3(v, 5)$.*

Proof: Suppose $v = 2n$ for some $n \geq 2$. Any $H_3(2n, 5)$ would have exactly $\binom{2n}{2}$ blocks. Since each pair of vertices occurs 5 times (an odd number), each pair must occur as a single edge in an H_3 block at least once. It follows that each edge must occur exactly once in this way since that accounts for each singleton edge. As every vertex has degree $5(v-1)$ and occurs in blocks $2n-1$ times as a vertex of degree three, every vertex must occur with degree 4 in y blocks, where $y = [5(2n-1) - 3(2n-1)]/4 = (2n-1)/2$. But this y is not an integer. □

Theorem 15 *If v is odd, then there exists an $H_3(v, 5)$.*

Proof: If v is odd, there exists a BIBD$(v, 3, 3)$ with block set $\mathcal{B}$. Suppose $\{a, b, c\}$ is a block in $\mathcal{B}$. Re-write $\{a, b, c\}$ as a block of three edges $\{\{a, b\}, \{b, c\}, \{c, a\}\}$, and do so similarly for each block in $\mathcal{B}$. With this new way to write the blocks, apply Agrawal's Lemma and form an array in which each column is a block and each new point (*i.e.*, edge) appears in each of the three rows exactly once. For each new block, say $\{\{p, q\}, \{q, r\}, \{r, p\}\}$, with $\{p, q\}$ in row one in the array, create the H_3 block $\langle p, q, r\rangle$ with five edges such that $\{p, q\}$ is the single edge. Since the index of the BIBD is 3, each pair appears in each row exactly once. The edge $\{p, q\}$ will appear in two more H_3 blocks, each time as a double edge. Thus, in the new H_3 blocks, each edge occurs five times. □

Corollary 1 *For every $v \geq 3$, there exists an $H_3(v, 10)$.*

Proof: Apply the argument in the previous theorem using the fact that a BIBD$(v, 3, 6)$ always exists. □

The argument against $H_3(2n, 5)$ does not apply to $H_3(2n, 5m)$ for m odd.

Example 27 *An* $H_3(4, 15)$.

4	4	4	4	4	2	4	4	4	4	4	3	4	4	4	4	4	1
2	2	2	3	3	3	3	3	3	1	1	1	1	1	1	2	2	2
1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3

Example 28 *There exists an* $H_3(v, 15)$ *for* $v = 6$ *and* $v = 8$. *For* $v = 6$, *develop the following starter blocks mod* 5*:* $\langle 0, \infty, 1\rangle$ *three times;* $\langle 0, \infty, 2\rangle$, *and* $\langle 0, 1, 3\rangle$ *twice each;* $\langle 0, 1, 2\rangle$, $\langle 0, 2, 1\rangle$. *For* $v = 8$ *develop the following mod* 7*:* $\langle 0, i, \infty\rangle$, *for* $i = 1, 2, 3$*;* $\langle 0, \infty, 1\rangle$, $\langle 0, 1, 3\rangle$ *three times;* $\langle 0, 3, 1\rangle$, $\langle 1, 3, 0\rangle$, $\langle 1, 0, 3\rangle$, $\langle 3, 1, 0\rangle$, $\langle 3, 0, 1\rangle$.

Theorem 16 *There exists an* $H_3(v, 15)$ *for all* $v \geq 3$.

Proof: There exists such designs for $v = 3, 5$ and index 15 since they exist for index 5. An $H_3(v, 15)$ exists for $v = 4, 6, 8$ by the examples above in this subsection. The remaining designs exist by applying the PBD Lemma. □

Corollary 2 *There exists an* $H_3(v, 5t)$ *for all* v *and all* $t \geq 2$.

5 Summarizing the Results

We have shown that the necessary conditions are sufficient for existence of graph designs $H_i(v, \lambda)$ for $i = 1, 2$, except the results are undecided for $H_2(8t, 3)$ for $24 < 8t \leq 1680$ whereas [8] have constructed all undecided H_2 decompositions. For H_3-decomposition the situation is much different, clearly $H_3(5t, 2)$ for $t > 3$ is undecided and probably the first family one needs to work on. Also, when $\lambda \equiv 0 \pmod 5$, there is no condition on v, for all other values of λ, $v \equiv 0, 1$ (mod 5). What makes H_3, harder and interesting is $H_3(v, 3)$ does not exist and for even v, $H_3(v, 5)$ does not exist.

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