A computer search of maximal partial spreads in PG$(3,q)$

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Abstract

From a computer search new minimum sizes for the maximal partial spreads in PG$(3,q)$ have been obtained for $q = 8,9,16$ and for every q such that $25 \leq q \leq 101$. Furthermore, density results in the cases $q = 8,9,16,19,23,25,27$ have been obtained. Finally, the already known exceptional size 45 for $q = 7$ has been founded again.

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1 Introduction

A spread in PG$(3,q)$, the projective space of three dimensions over the field GF(q), is a set of mutually skew lines covering the space. A partial spread is a set of mutually skew lines which is not a spread. A partial spread is said to be maximal if it is neither properly contained in a spread nor in a partial spread.

Maximal partial spreads have been investigated by several authors, but a complete knowledge of them is still far away.

This work is the natural continuation of the paper "A new method to construct maximal partial spreads in PG$(3,q)$" [20], where we found new minimums for the sizes of maximal partial spreads in PG$(3,q)$, with $q = 11,13,17,19,23$.

Moreover in [20], for $q = 11,13,17$, we constructed maximal partial spreads (in the following Mps) having all the cardinalities between our minimums and those of the density results found by O. Heden (see Section 2). In the cases $q = 19$ and $q = 23$ we did not fill the previous gap, but we do it here.

In this paper we found new minimums for the sizes of Mps in PG$(3,q)$, with $q = 8,9,16$ and for every q such that $25 \leq q \leq 101$.

Afterwards, we found the necessary cardinalities to fill the gaps between our minimums and the size $q^2 - q + 2$, and do it for $q = 8,9,19,23,25,27$. We also obtained density results in the case $q = 16$.

During the research, we found many known values, such as the exceptional cardinality 45 for $q = 7$.

To construct the Mps, we used several programs, written in C language, and let them run on a notebook with processor Intel Core i5-430M, 2.26 GHz, 3 MB L3 cache and 4 GB RAM.

The first program, identified as the "max-intersecting program", is a much more efficient version than the one used in [20], and it works in the following way: first of all, the program eliminates all the lines meeting some lines of an initial partial spread; then it calculates the number of the remaining lines meeting each remaining line, and adds to the initial partial spread the remaining line meeting the maximum number of remaining lines. The program proceeds in this way until a Mps is obtained.

For all the values of q studied in this paper we found minimums less than

$$(\lceil 2\log_2 q\rceil + 1)q + 1 - 3q,$$

where $(\lceil 2\log_2 q\rceil + 1)q + 1$ is the known minimum for q odd and $q \geq 25$, while for q even the known minimum is much higher than that.

Furthermore, we used two other versions of the program which again calculate the number of the lines meeting a fixed line, but select it when its value is the minimum or the closest to the average. Such programs will be identified as the "min-intersecting program" and the "middle-intersecting program", respectively. We use these versions to get unknown cardinalities greater than the found minimums.

Furthermore, we used programs that construct several Mps at the same time.

For simplicity reasons all the previous programs use the line of Plücker coordinates $(0,0,0,0,0,1)$ as initial line.

Afterwards, we wrote a program which constructs Mps in the following way. The program, that we call "linear program", chooses the first line in the order of construction, that is the order in which our algorithm constructs the Plücker coordinates of the lines (see Subsection 3.1), and eliminates all the lines meeting it. Next, the program chooses the first of the remaining lines and proceeds similarly until to construct a Mps. Then the program chooses the second line, in the order of construction, as first line and constructs the second Mps, and so on. So the program constructs $\theta_3\theta_2/\theta_1$ Mps, where $\theta_r = q^r + q^{r-1} + \ldots + 1$.

The linear program, besides giving many unknown cardinalities, finds Mps of sizes greater than those obtained by the max-intersecting program, but lower than the previous known minimums.

2 Known results about maximal partial spreads

The first lower bound for maximal partial spreads follows A. A. Bruen, who in 1971 (see [4]) proved the following result.

Theorem 2.1. *If $\mathcal{S}$ is a maximal partial spread in* PG$(3,q)$, *then*

$$q+\sqrt{q}+1 \le |\mathcal{S}| \le q^2-\sqrt{q}.$$

The upper bound was given by D. M. Mesner in 1967 (see [22]) and later by A. A. Bruen, by using blocking sets theory. In 1976, A. A. Bruen and J. A. Thas (see [6]) improved the previous result, ruling out the equal sign on the left.

The next lower bound is due to D. G. Glynn (see [9]), who in 1982 proved the following result.

Theorem 2.2. *If $\mathcal{S}$ is a maximal partial spread in* PG$(3,q)$, *then*

$$|\mathcal{S}| \ge 2q.$$

The best upper bound for maximal partial spreads in PG$(3,q)$, q a prime, is now given by A. Blokhuis. It follows from his results about blocking sets [3] through which we get that

$$|\mathcal{S}| < q^2+1-\frac{q+1}{2}$$

for a Mps $\mathcal{S}$ in PG$(3,q)$.

In [4], A. A. Bruen proved the existence of a Mps $\mathcal{S}$ in PG$(3,q)$, with

$$|\mathcal{S}| = q^2-q+1, \quad q>2, \tag{1}$$

and of a Mps $\mathcal{S}$ with

$$|\mathcal{S}| = q^2-q+2, \quad q \text{ odd}, q>3. \tag{2}$$

In [6], A. A. Bruen and J. A. Thas constructed a Mps $\mathcal{S}$ with

$$|\mathcal{S}| = q^2-q+2, \quad q=2^{2h+1},\ h\ge 1. \tag{3}$$

In [7], J. W. Freeman constructed a Mps $\mathcal{S}$, with

$$|\mathcal{S}| = q^2-q+2, \quad q=2^{2h},\ h\ge 1. \tag{4}$$

For q odd, the best density result known by us is due to O. Heden, who in [16] proved that for any integer n in the interval

$$\frac{q^2+1}{2}+6 \le n \le q^2-q+2 \tag{5}$$

there is a maximal partial spread of size n in PG$(3,q)$, $q \ge 7$. In [13] O. Heden found the following density result:

$$\frac{q^2+1}{2}+3 \leq n \leq q^2-q+2, \qquad \text{when } q+1 \equiv 8 \text{ or } 16 \bmod 24.$$

O. Heden also found

$$\frac{q^2+1}{2}+1, \qquad \text{when } \gcd(q+1,24)=2 \text{ or } 4, \tag{6}$$

$$\frac{q^2+1}{2}+2, \qquad \text{when } \gcd(q+1,24)=4, \tag{7}$$

in [12] and

$$\frac{q^2+1}{2}+n, \qquad \text{for } n=3,4,5 \qquad \text{if } q+1 \equiv \pm 2 \bmod 6, \tag{8}$$

$$\frac{q^2+1}{2}+n, \qquad \text{for } n=4 \qquad \text{if } q+1 \equiv 0 \bmod 6 \text{ and } q \geq 17, \tag{9}$$

$$\frac{q^2+1}{2}+n, \qquad \text{for } n=1,2,3,4,5 \quad \text{if } q+1 \equiv \pm 2 \bmod 12, \tag{10}$$

$$\frac{q^2+1}{2}+n, \qquad \text{for } n=3,4,5 \qquad \text{if } q+1 \equiv \pm 4 \bmod 12, \tag{11}$$

in [16].

In [12], [11] and [14], O. Heden found the following other density results: $13-18$ for $q=5$, $23-30$ for $q=7$ and $58-66$ for $q=11$.

In [10], A. Gács and T. Szőnyi constructed maximal partial spreads in $PG(3,q)$ of size

$$cq+1,$$

where c is an integer satisfying the condition

$$6 \ln q + 1 \leq c \leq q. \tag{12}$$

For q odd, the previous condition becomes

$$2 \log_2 q + 1 \leq c \leq q. \tag{13}$$

It follows that for q odd, $q \geq 25$, the smallest known maximal partial spreads have size $(\lceil 2\log_2 q \rceil + 1)q+1$.

We remark that in [10] and in [20] the formula (13) has been mistakenly reported with the symbol ln instead of $\log_2$ [26].

Moreover in [20], by a computer search, we found minimum sizes for $q = 11,13,17,19,23$ and, for $q=11,13,17$, we also found all the values between our minimums and the previous minimums of the density results. The cases $q = 19$ and $q = 23$ were not yet completed. More precisely, in the case $q = 19$, thirty-one sizes were missing, and in the case $q = 23$ only one. In this paper, as already said, we complete the previous density results.

Other results about Mps are the following.

In [2], A. Beutelspacher showed that in PG$(3,q)$ there is a Mps $\mathscr{S}$, with

$$|\mathscr{S}| = q^2 + 1 - nq, \qquad 0 \le n \le \frac{1}{2}q - 1, \ n \in \mathbb{N}. \tag{14}$$

In [23], S. Rajola, together with M. S. Tallini, showed that in PG$(3,q)$, q even and $q \ge 8$, there are Mps $\mathscr{S}$, with

$$|\mathscr{S}| = q^2 - 2nq + 2n + 1, \qquad n < \min\left\{\frac{q-1}{4}, \frac{1+\sqrt{2q-1}}{2}\right\}, \ n \in \mathbb{N}. \tag{15}$$

In [21], D. Jungnickel and L. Storme proved the existence of a Mps $\mathscr{S}$ in PG$(3,q)$, q even and $q \ge 4$, such that

$$|\mathscr{S}| = q^2 - q. \tag{16}$$

In [5], A. A. Bruen and J. W. P. Hirschfeld showed that in PG$(3,q)$, with $(q+1,3) = 1$, there is a Mps $\mathscr{S}$, with

$$|\mathscr{S}| = \frac{q^2+q+2}{2}. \tag{17}$$

Moreover, in [17], O. Heden proved that there are no maximal partial spreads of size 115 in PG$(3,11)$ and in [18] O. Heden, S. Marcugini, F. Pambianco and L. Storme proved the non-existence of a Mps of size 76 in PG$(3,9)$.

Finally, in [1], J. Bárat, A. Del Fra, S. Innamorati and L. Storme proved that 58 is the largest size for a maximal partial spread in PG$(3,8)$.

In the Table 1 we give all the known cardinalities for $q \le 101$ (we recall that, in the case $q = 2$, there is only the spread).

Moreover, in the Table 1 the notation $\forall k \le n$ means $k = 1, 2, \ldots, n$.

Table 1: The known sizes of maximal partial spreads in PG$(3,q)$, $q \le 101$

q	Min.	Ref.	Others	Ref.
3	7	[25]		
4	11	[25]	$12-14$	[25]
5	13	[14]	$14-22$	[14]
7	23	[11]	$24-25$; $26-30$; $31-44$; 45	[11]; [12]; (5); [15]

Table 1: The known sizes of maximal partial spreads in PG$(3,q)$, $q \leq 101$

q	Min.	Ref.	Others	Ref.
8	41	(14)	49; 51; 56; 57; 58	(14); (15); (16); (1); (3)
9	46	(17)	$47-74$	(5)
11	48	[20]	$49-57$; $58-66$; $67-112$	[20]; [12]; (5)
13	62	[20]	$63-85$; $86-158$	[20]; (5)
16	145	(14)	$145+k\cdot 16, \forall k \leq 5$; 193, 197, 227; 240; 241; 242	(14); (15); (16); (1); (4)
17	95	[20]	$96-148$; 149; 150; $151-274$	[20]; (9); [20]; (5)
19	114	[20]	$115-146$; 150; 156; 158; 163; $182-344$	[20]; (5)
23	148	[20]	$150-253$; 254; $255-270$; $271-508$	[20]; (13); [20]; (5)
25	276	(13)	301; $314-602$	(13); (5)
27	298	(13)	325,352; $368-704$	(13); (5)
29	320	(13)	349,378,407; $427-814$	(13); (5)
31	342	(13)	373,404,435,466; $484-932$	(13); (5)
32	545	(14)	$545+k\cdot 32, \forall k \leq 13$; $705+k\cdot 32, \forall k \leq 6$; 777,839,901,963; 992; 993; 994	(14); (12); (15); (16); (1); (4)
37	445	(13)	$445+k\cdot 37, \forall k \leq 6$; $686-1334$	(13); (10)
41	493	(13)	$493+k\cdot 41, \forall k \leq 8$; $847-1642$	(13); (5)
43	517	(13)	$517+k\cdot 43, \forall k \leq 9$; $928-930$; $931-1808$	(13); (8); (5)
47	612	(13)	$612+k\cdot 47, \forall k \leq 10$; $1111-2164$	(13); (5)
49	638	(13)	$638+k\cdot 49, \forall k \leq 11$; $1202-1206$; $1207-2354$	(13); (10); (5)

Table 1: The known sizes of maximal partial spreads in PG$(3,q)$, $q \leq 101$

q	Min.	Ref.	Others	Ref.
53	690	(13)	$690+k\cdot 53, \forall k \leq 13$; $1411-2758$	(13); (5)
59	768	(13)	$768+k\cdot 59, \forall k \leq 16$; $1747-3424$	(13); (5)
61	794	(13)	$794+k\cdot 61, \forall k \leq 17$; $1862-1866$; $1867-3662$	(13); (10); (5)
64	1665	(12)	$1665+k\cdot 64, \forall k \leq 36$; $2113+k\cdot 64, \forall k \leq 29$; 3341, 3467, 3593, 3719, 3845; 3971; 4032; 4033; 4034	(12); (14); (15); (15); (16); (1); (4)
67	939	(13)	$939+k\cdot 67, \forall k \leq 19$; $2248-2250$; $2251-4424$	(13); (8); (5);
71	995	(13)	$995+k\cdot 71, \forall k \leq 21$; $2527-4972$	(13); (5)
73	1023	(13)	$1023+k\cdot 73, \forall k \leq 22$; $2666-2670$; $2671-5258$	(13); (10); (5)
79	1107	(13)	$1107+k\cdot 79, \forall k \leq 25$; $3124-3126$; $3127-6164$	(13); (8); (5)
81	1135	(13)	$1135+k\cdot 81, \forall k \leq 26$; $3287-6482$	(13); (5)
83	1163	(13)	$1163+k\cdot 83, \forall k \leq 27$; $3451-6808$	(13); (5)
89	1247	(13)	$1247+k\cdot 89, \forall k \leq 30$; $3967-7834$	(13); (5)
97	1456	(13)	$1456+k\cdot 97, \forall k \leq 33$; $4706-4710$; $4711-9314$	(13); (10); (5)
101	1516	(13)	$1516+k\cdot 101, \forall k \leq 35$; $5107-10102$	(13); (5)

3 Algorithms description

In this section we give the details of the algorithms used in our search for maximal partial spreads.

We show the construction of the Plücker coordinates, the initial partial spreads and the way to construct maximal partial spreads.

3.1 Construction of Plücker coordinates

We recall that the Plücker coordinates represent a line in three dimensional space using six coordinates $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$, different from $(0,0,0,0,0,0)$, which are determined up to proportionality and such that

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0. \tag{18}$$

In order to construct the Plücker coordinates of the lines we proceed in the following way.

We fix the first coordinate equal to 1 and vary the others in all the possible ways, but taking into account the condition (18).

After this we fix the first coordinate equal to 0 and the second equal to 1 and vary the others in all the possible ways, taking again into account the previous relation. We proceed in this way up to the line $(0,0,0,0,0,1)$.

In the table below we describe the six different groups of lines obtained as above described. We write in bold font the coordinates depending on other coordinates, since (18) holds. In the last column we report the number of lines for each group of lines. The symbols $I,\dots,V$ denote the line groups.

p_{01}	p_{02}	p_{03}	p_{12}	p_{13}	p_{23}	$\sharp$
1	p^I_{02}	p^I_{03}	p^I_{12}	p^I_{13}	$\mathbf{p}^I_{23}$	q^4
0	1	p^{II}_{03}	p^{II}_{12}	$\mathbf{p}^{II}_{13}$	p^{II}_{23}	q^3
0	0	1	$\mathbf{p}^{III}_{12}$	p^{III}_{13}	p^{III}_{23}	q^2
0	0	0	1	p^{IV}_{13}	p^{IV}_{23}	q^2
0	0	0	0	1	p^V_{23}	q
0	0	0	0	0	1	1

We remark that, by (18), we get:

$$p^I_{23} = p^I_{02}p^I_{13} - p^I_{03}p^I_{12},$$
$$p^{II}_{13} = p^{II}_{03}p^{II}_{12},$$
$$p^{III}_{12} = 0.$$

As an example, we show the construction of the lines of the first group, that is

$$(1, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}).$$

We use the following algorithm. The Plücker coordinates of the $(i+1)$-th line, with $i = 0, 1, \ldots, q^4 - 1$, are:

$$\begin{aligned}
p_{01} &= 1, \\
p_{02} &= i \bmod q; \\
p_{03} &= \lfloor i/q \rfloor \bmod q, \\
p_{12} &= \lfloor i/q^2 \rfloor \bmod q, \\
p_{13} &= \lfloor i/q^3 \rfloor \bmod q, \\
p_{23} &= p_{02}p_{13} - p_{03}p_{12},
\end{aligned}$$

where $\lfloor X \rfloor$ denotes the integer part of X. We remark that the calculus of p_{23} is done in $\mathrm{GF}(q)$.

As easily checked, the obtained sextuples are all distinct, and two of them are never proportional, since the first element is 1 for all of them. Moreover all these sextuples represent lines of $\mathrm{PG}(3,q)$, since each of them satisfies the condition (18) and is not formed by six zeros.

For example, in the case $q = 2$, the above algorithm gives the following sextuples, where p_{23} depends on the line and is given by (18):

$$\begin{array}{llllll}
(1, & 0, & 0, & 0, & 0, & p_{23}) \\
(1, & 1, & 0, & 0, & 0, & p_{23}) \\
(1, & 0, & 1, & 0, & 0, & p_{23}) \\
(1, & 1, & 1, & 0, & 0, & p_{23}) \\
(1, & 0, & 0, & 1, & 0, & p_{23}) \\
(1, & 1, & 0, & 1, & 0, & p_{23}) \\
(1, & 0, & 1, & 1, & 0, & p_{23}) \\
(1, & 1, & 1, & 1, & 0, & p_{23}) \\
(1, & 0, & 0, & 0, & 1, & p_{23}) \\
(1, & 1, & 0, & 0, & 1, & p_{23}) \\
(1, & 0, & 1, & 0, & 1, & p_{23}) \\
(1, & 1, & 1, & 0, & 1, & p_{23}) \\
(1, & 0, & 0, & 1, & 1, & p_{23}) \\
(1, & 1, & 0, & 1, & 1, & p_{23}) \\
(1, & 0, & 1, & 1, & 1, & p_{23}) \\
(1, & 1, & 1, & 1, & 1, & p_{23})
\end{array}$$

We remark that p_{02} varies after $1 = q^0$ lines, p_{03} varies after q^1 lines, p_{12} varies after q^2 and finally p_{13} varies after q^3 lines.

3.2 Construction of the initial partial spread

First of all, we recall that the line $(0,0,0,0,0,1)$ belongs to all the different initial partial spreads that we use.

As first initial partial spread we choose some lines of the spread in $PG(3,q)$ obtained by A. A. Bruen and J. W. P. Hirschfeld and formed by tangent lines, imaginary chords and imaginary axes of a twisted cubic, with $\gcd(q+1,3)=3$ (see [5]).

The Plücker coordinates $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$ of the tangent lines different from $(1,0,0,0,0,0)$ are

$$\left(t^4, 2t^3, 3t^2, t^2, 2t, 1\right), \tag{19}$$

for every $t \in GF(q)$. The algorithm constructs from 0 to $q-1$ tangent lines, through (19), which gives the possibility to construct the tangent line $(1,0,0,0,0,0)$ or not. The $(q+1)$-th tangent line is the line $(0,0,0,0,0,1)$ which, as above said, is always chosen as first line.

The Plücker coordinates of the imaginary axes and the imaginary chords are respectively

$$\left(a^4\left(b^2+3\right)^2, 2a^3b\left(b^2+3\right), 3a^2\left(b^2-1\right), a^2\left(b^2+3\right), 2ab, 1\right),$$

$$\left(a^4\left(b^2+3\right)^2, 2a^3b\left(b^2+3\right), 3a^2\left(b^2+3\right), a^2\left(b^2-1\right), 2ab, 1\right),$$

where a varies in $GF(q)\backslash\{0\}$ and b in $GF(q)$. It is easy to check that the pairs (a,b) and $(-a,-b)$ give the same coordinates. So we make a vary in $\{1,\ldots,(q-1)/2\}$ and b in $\{0,\ldots,q-1\}$.

For every choice $(\bar{a},\bar{b})$ the algorithm constructs all the Plücker coordinates associated with the pairs (a,b), with $a=1,2,\ldots,\bar{a}$ and $b=0,1,2,\ldots,\bar{b}$ and so the algorithm constructs $\bar{a}\left(\bar{b}+1\right)$ lines (by considering the elements $\bar{a}$ and $\bar{b}$ of $\mathbb{Z}_p$ as integer numbers).

The algorithm constructs from 0 to $q(q-1)/2$ axes and from 0 to $q(q-1)/2$ chords.

If a program uses this spread we add the number 1 to its name.

As a second initial partial spread, we use, either entirely or partially, the following partial spread:

$$\left(a^2, -k, a, -ka, 0, k\right), \qquad \left(-k', b^2, b, k'b, k', 0\right),$$

where k, k' are fixed elements of $\mathbb{Z}_p\backslash\{0\}$, with $k \neq k'$, while a and b vary in $\mathbb{Z}_p\backslash\{0\}$. Such a partial spread has been found by using a representation of the space $PG(3,q)$ in the geometry of $AG(2,q)$. Such a representation is shown in [24]. For every choice of k and k', the algorithm allows us to take from 0 to $q-1$ lines of the first set and from 0 to $q-1$ lines of the second set.

If a program uses this spread we add the number 2 to its name.

We also use the following spread (see [19], 17.3.3).

Let $q = p^h$, with $h > 1$ and let $x^{p+1} + bx - c$ be a polynomial without roots in $\mathrm{F} = \mathrm{GF}(q)$. Then the set

$$\{\langle(1,0,0,0),(0,1,0,0)\rangle\} \cup \{\langle(z,y,1,0),(cy^p, z^p + by^p, 0, 1)\rangle \mid\mid (y,z) \in \mathrm{F}^2\}$$

is an aregular spread of $\mathrm{PG}(3,q)$.

If a program uses this spread we add the number 3 to its name.

3.3 The algorithms

First of all, the programs identified as the max-intersecting program, the min-intersecting program and the middle-intersecting program work with the set $\mathscr{L}$ formed by the lines not meeting $(0,0,0,0,0,1)$.

We first explain the max-intersecting program.

The algorithm firstly constructs the array of the Plücker coordinates of the lines of the initial partial spread $\mathscr{F}$ and cancels out all the lines meeting a line of $\mathscr{F}$.

Denoting by $\mathscr{L}_0$ the set of the remaining lines. For each line $l_i \in \mathscr{L}_0$, $i = 0,\ldots,|\mathscr{L}_0| - 1$, the algorithm calculates the number $n(l_i)$ of the lines of $\mathscr{L}_0$ meeting l_i. The program selects the first line $l_{\bar{i}}$ having the value $n(l_{\bar{i}}) = \max n(l_i)$, that is the line of $\mathscr{L}_0$ meeting the maximum number of lines of $\mathscr{L}_0$, or the first one, following our order of construction, in the case there are at least two lines of $\mathscr{L}_0$ meeting the same (maximum) number of lines of $\mathscr{L}_0$. Therefore the selected line $l_{\bar{i}}$ is marked, as a line of our line set, and all the lines meeting $l_{\bar{i}}$ are ruled out. Denote by $\mathscr{L}_1$ the new set of the remaining lines. At the second step the program calculates the number $n(l_i)$, $l_i \in \mathscr{L}_1$, $i = 0,\ldots,|\mathscr{L}_1| - 1$, and again determines the maximum of $n(l_i)$. The program stops at the n-th step, when $\mathscr{L}_n = \emptyset$.

As concern the other two versions of the program, they again calculate the number $n(l_i)$ of the lines meeting a fixed line l_i, but the program selects it when its value is the minimum of $n(l_i)$ or the closest to the average of all the numbers $n(l_i)$.

Again, as mentioned in the introduction, we wrote a program which constructs Mps in the following way. The program, that we call linear program, chooses the first line in our order of construction of the lines and the program eliminates all the lines meeting it. Next, the program chooses the first of the remaining lines and proceeds similarly until a Mps is achieved. Then the program chooses the second line, in the order of construction, as first line and it constructs the second Mps, and so on. So the program constructs $\theta_3\theta_2/\theta_1$ Mps, where $\theta_r = q^r + q^{r-1} + \ldots + 1$.

4 Results

We found some new minimums and new density results for the sizes of Mps of $\mathrm{PG}(3,q)$. In particular, we found new minimums for $q = 8, 9, 16$ and for every q

such that $25 \leq q \leq 101$.

Moreover, we found new density results for $q = 8, 9, 16, 19, 25, 27$. We also found a Mps of the size 149 for $q = 23$, which is the missing value between the minimum found in [20] and the minimum of the density result found in the same article.

We also found many well known results, such as the size 45 for $q = 7$.

Totally, we constructed about one million and a half Mps or spreads.

The density results found here and which appear in the Table 2 include also some known values. The knowledge of such values is not specified for brevity reasons.

Table 2: New sizes of maximal partial spreads in PG$(3, q)$

q	Min.	Previous min.	Density results	Previous density results
8	30	41	31–55	56–58
9	36	46	37–45	46–74
16	87	145	88–221, 225–231	240–242
19		114	147–181	115–146; 182–344
23		148	149	150–508;
25	173	276	174–313	314–602
27	193	298	194–367	368–704
29	210	320		
31	231	342		
32	238	545		
37	306	445		
41	345	493		
43	372	517		
47	417	612		
49	474	638		
53	488	690		
59	569	768		
61	600	794		
64	623	1665		
67	672	939		
71	732	995		
73	761	1023		
79	848	1107		

Table 2: New sizes of maximal partial spreads in PG$(3,q)$

q	Min.	Previous min.	Density results	Previous density results
81	873	1135		
83	903	1163		
89	968	1247		
97	1102	1456		
101	1160	1516		

From the already known results and from our results, we get the following theorem.

Theorem 4.1. *In* PG$(3,q)$*, for every q such that* $5 \leq q \leq 101$*, there are maximal partial spreads of size less than*

$$(\lceil 2\log_2 q \rceil + 1)q + 1 - 3q.$$

Concerning density results, in the case $q = 16$ we have not found all the unknown cardinalities included between the minimum value we found and the biggest unknown cardinality, in spite of numerous attempts. This is really unexpected, because in the other cases we have found all the unknown cardinalities in a very easy way.

In addition, from the already known results and from our results, we get the following theorem.

Theorem 4.2. *In* PG$(3,q)$*, for every q such that* $5 \leq q \leq 27$ *and* $q \neq 16$*, there is a maximal partial spread of size n for any integer n in the interval*

$$(\lceil 2\log_2 q \rceil + 1)q + 1 - 3q \leq n \leq q^2 - q + 2.$$

For every new example of Mps, we specify the program through which we have obtained it. Obviously, we have obtained several results using different programs.

In the following scheme we do not report the already known values, even if we remark that in many cases we have obtained them.

- $q = 8$. Sizes 30, 31, 32 by the max-intersecting program 3. Sizes 33-40, 42-48, 50 and 52-55 by the linear program.
- $q = 9$. Sizes 36-40 by the max-intersecting program 3. Sizes 41-45 by the linear program.
- $q = 16$. Sizes 87-99, 121-140, 142-144, 146-160, 162, 164, 166-168, 170-173, 175, 176, 178, 181, 182, 184-186, 188, 190, 194, 196, 198, 200, 202,

204, 212-214 and 226 by the max-intersecting program 3. Sizes 100-120 by the linear program. Sizes 141, 163, 165, 169, 174, 179, 183, 187, 189, 191, 192, 195, 199, 201, 203, 205-208, 210, 211, 215-221, 228-231 by the middle-intersecting program 3. Size 180 by the min-intersecting program 3.

- $q = 19$. Sizes 147-149, 151-155, 157, 159-162 and 164-181 by the linear program.
- $q = 23$. Size 149 by the max-intersecting program 1.
- $q = 25$. Sizes 173-193, 286, 289, 290, 302 and 304 by the max-intersecting program 3. Sizes 194-275, 277-285, 287-288, 291-300, 305-313 by the linear program. Size 303 by the middle-intersecting program 3.
- $q = 27$. Sizes 193-215 by the max-intersecting program 3. Sizes 216-297, 299-324, 326-351 and 353-366 by the linear program. Size 367 by the middle-intersecting program 3.
- $29 \le q \le 101$. All the minimum values have been found by the three versions of the max-intersecting program.

We give some examples about the execution time of the programs.

For $q = 7$ the linear program finds all the sizes between 27 and 45, and does it in 1,37 seconds.

For $q = 8$ the linear program constructs, in 5,95 seconds, 4096 Mps or spreads having all the cardinalities between 33 and 52, and the cardinalities 54, 56, 57 and 65.

For $q = 9$ the linear program constructs, in 16,89 seconds, 7462 Mps or spreads having all the cardinalities between 41 and 69, and the cardinalities 71, 72 and 82.

The max-intersecting program 3 gives, for $q = 8$, the cardinality 30 in 0,46 seconds; for $q = 9$ the cardinality 36 in 0,87 seconds; for $q = 16$ the cardinality 87 in 19,80 seconds and, for $q = 32$, the cardinality 238 in 648,09 seconds.

For $q = 71$ the max-intersecting program 1 gives the cardinality 732 in 2571,34 seconds, the cardinality 785 in 119,78 seconds and the cardinality 983 (that is lower than the previous known minimum) in 42,71 seconds.

5 Some new examples of maximal partial spreads

In this section we describe the Plücker coordinates of the lines of some Mps that we found.

For every reported Mps, we first write the Plücker coordinates of the lines of the initial partial spread, and then the order numbers i of the added lines, whose

Plücker coordinates can be determined through the formulas:

$$\begin{aligned}
p_{01} &= 1, \\
p_{02} &= i \bmod q, \\
p_{03} &= \lfloor i/q \rfloor \bmod q, \\
p_{12} &= \lfloor i/q^2 \rfloor \bmod q, \\
p_{13} &= \lfloor i/q^3 \rfloor \bmod q, \\
p_{23} &= p_{02}p_{13} - p_{03}p_{12},
\end{aligned}$$

which appear in Subsection 3.1.

As a first example we describe the maximal partial spread of size 30 for $q = 8$. In order to construct this Mps, we have chosen seven lines from the spread reported at the end of Subsection 3.2.

Initial lines:
(0, 0, 0, 0, 0, 1), (1, 4, 1, 0, 6, 5), (1, 0, 0, 1, 6, 0), (1, 1, 2, 2, 6, 2), (1, 1, 3, 3, 6, 3), (1, 1, 4, 4, 6, 0), (1, 1, 5, 5, 6, 1), (1, 1, 6, 6, 6, 4).

Added lines:
24, 2367, 231, 3708, 455, 2394, 3784, 1165, 180, 3971, 2134, 2589, 1893, 1808, 631, 3883, 382, 1462, 2063, 708, 810, 1537.

As a second example we describe the maximal partial spread of size 210 for $q = 29$. In order to construct this Mps, we have chosen sixty-one lines from the Bruen-Hirschfeld's spread.

Initial lines:
(0, 0, 0, 0, 0, 1), (9, 0, 9, -1, 0, 1), (16, 8, 12, 0, 2, 1), (20, 28, 21, 3, 4, 1), (28, 14, 7, 8, 6, 1), (13, 7, 28, 15, 8, 1), (28, 0, 7, -4, 0, 1), (24, 6, 19, 0, 4, 1), (1, 21, 26, 12, 8, 1), (13, 25, 28, 3, 12, 1), (5, 27, 25, 2, 16, 1), (4, 0, 23, -9, 0, 1), (20, 13, 21, 0, 6, 1), (25, 2, 15, 27, 12, 1), (6, 1, 5, 14, 18, 1), (9, 15, 20, 19, 24, 1), (9, 0, -3, 3, 0, 1),(16, 8, 0, 4, 2, 1), (20, 28, 9, 7, 4, 1), (28, 14, 24, 12, 6, 1), (13, 7, 16, 19, 8, 1), (28, 0, -12, 12, 0, 1), (24, 6, 0, 16, 4, 1), (1, 21, 7, 28, 8, 1), (13, 25, 9, 19, 12, 1), (5, 27, 6, 18, 16, 1), (4, 0, -27, 27, 0, 1), (20, 13, 0, 7, 6, 1), (25, 2, 23, 5, 12, 1), (6, 1, 13, 21, 18, 1), (9, 15, 28, 26, 24, 1), (1, 2, 3, 1, 2, 1), (16, 16, 12, 4, 4, 1), (23, 25, 27, 9, 6, 1), (24, 12, 19, 16, 8, 1), (16, 18, 17, 25, 10, 1), (20, 26, 21, 7, 12, 1), (23, 19, 2, 20, 14, 1), (7, 9, 18, 6, 16, 1), (7, 8, 11, 23, 18, 1), (24, 28, 10, 13, 20, 1), (25, 23, 15, 5, 22, 1), (1, 5, 26, 28, 24, 1), (25, 15, 14, 24, 26, 1), (20, 7, 8, 22, 28, 1), (20, 22, 8, 22, 1, 1), (25, 14, 14, 24, 3, 1), (1, 24, 26, 28, 5, 1), (25, 6, 15, 5, 7, 1), (24, 1, 10, 13, 9, 1), (7, 21, 11, 23, 11, 1), (7, 20, 18, 6, 13, 1), (23, 10, 2, 20, 15, 1), (20, 3, 21, 7, 17, 1), (16, 11, 17, 25, 19, 1), (24, 17, 19, 16, 21, 1), (23, 4, 27, 9, 23, 1), (16, 13, 12, 4, 25, 1), (1, 27, 3, 1, 27, 1).

Added lines:
677253, 504585, 521560, 449301, 625597, 347945, 489072, 36563, 240119, 509323, 616226, 330155, 82544, 121871, 174971, 187236, 138497, 157222, 346096, 108275, 124884, 147268, 601567, 391027, 429148, 109152, 145311, 432435, 550591, 697973, 25022, 191857, 173609, 589158, 459617, 129059, 206486, 596160, 367651, 56530, 337034, 658419, 317597, 55325, 603163, 52495, 107491, 451648, 683065, 148086, 285155, 116416, 302602, 337486, 150168, 477206, 196604, 506753, 274083, 561501, 33049, 42382, 458736, 70067, 569409, 441523, 416479, 80220, 243346, 537506, 516647, 89547, 328090, 212003, 98520, 109483, 234264, 215347, 245551, 503476, 528854, 21953, 385516, 271778, 527360, 189087, 423060, 232916, 38771, 286659, 112330, 669444, 296968, 277363, 182475, 583897, 482186, 160767, 110259, 38321, 642746, 341987, 105983, 122833, 49273, 531042, 304797, 519957, 115948, 644653, 594328, 395375, 650790, 492067, 662581, 113012, 494299, 7416, 498804, 103763, 90926, 167220, 272780, 58186, 47265, 200268, 372443, 421720, 605597, 76597, 464438, 706631, 437079, 453946, 510541, 27980, 70865, 152474, 344471, 410036, 27349, 111830, 197156, 197918, 202937, 241613, 254354, 370916, 379262, 397943.

6 Validity of the results

In this section we explain the proves on which we base the validity of the used programs.

First of all, max-intersecting program, min-intersecting program and middle-intersecting program give the same results of the similar programs used in [20].

Furthermore, the validity of these programs is based on the following facts:

1. We check the validity of the Plücker coordinates construction and the writing of incidence condition for two lines. Moreover, we submitted the programs to the following preliminary test: we slightly changed the programs to calculate the number $n(l)$ (for all the lines of the space for several values of q and for many lines for the others) always obtaining the value $n(l) = (q^2+q)(q+1)+1$ for every line l. The number $n(l)$ has been calculated more than one million of times.

2. We constructed the Bruen-Hirschfeld's spread for the maximum value of q that we have studied and we checked that it is a set of q^2+1 mutually skew lines and so a spread, by using a macro of Microsoft Excel (see [20]). Afterwards, by starting from a number of its lines less than q^2+1, but close to it, the program constructs again the Bruen-Hirschfeld's spread.

3. The program never gives results against the theory, in particular for the cases PG(3,2), PG(3,3) and PG(3,4), for which there is a complete characteri-

zation of maximal partial spreads (see [25]) and for which we obtain all and only the known cardinalities, in addition to several spreads.

4. We obtain a spread when we expect it.
5. For q a prime and $q \leq 13$, we constructed maximal partial spreads by Microsoft Excel using a macro similar to the macro used in [20], and we have obtained the same results which we get through the C language program.
6. For q a prime and $q \leq 13$, we tested some constructed line sets through a macro in Microsoft Excel and verify that they are Mps.
7. We tested, also for high values of q, q a prime, some constructed line sets by Microsoft Excel using the macro described in [20], and than we verified that they are sets of mutually skew line.
8. We wrote a (very) simple C language program which we have tested the obtained results and that program always confirmed that they are Mps.
9. We tested also the test programs. For example, to checked the program that verify that the obtained line sets are Mps, proceeding in the following way. For instance, we tested the Bruen-Hirschfeld's spread and the test program confirms that it really consists of a spread. After this we tested the previous spread deprived of a line and the program replies that it is not maximal. Furthermore, we tested the line set obtained by adding a line to the previous spread and the program answers that it is not a set of mutually skew lines.

Concerning the programs that construct Mps in $\mathrm{PG}(3,p^h)$, with p a prime and $h > 1$, the points 1, 3, 4, 5, 8, 9 and 10 hold and we did also the following tests.

Firstly we verified that the used tables of sum and product, obtained through a specific software [8] but written as an array for the C language by a macro of Microsoft Excel, are really those of a Galois field, by using a program which checks the axioms of a finite field.

Secondly, for $\mathrm{PG}(3,2)$ and $\mathrm{PG}(3,3)$ we constructed Mps either by using operations $\bmod p$ or the tables of sum and product, and we obtained the same results.

Concerning the linear program, the points 1, 3, 7 and 9 still hold. Furthermore, we remark that the program finds the sizes q^2-q+2 and q^2+1, but not sizes between them, for all the values that we studied. In the case $q=7$, the program finds the size q^2-q+3 and this is the only case in which the existence of such cardinality is known.

7 Conclusion

This work has the aim not only of finding new minimum sizes for the maximal partial spreads in $\mathrm{PG}(3,q)$, but also of giving, as an obvious consequence, a theoretical indication and therefore a new impulse to the research.

In fact, the results we improve back to the year 2003, when A. Gács and T. Szőnyi managed to lower the previous minimums remarkably. However, the gaps between the Glynn's lower bound and the known minimums still appeared much too large. Here, for the values of q that we study, we succeed in getting a reduction up to 70% of the previous gaps, as happens in the case $q = 64$.

Moreover, we have noted not only that the new minimums are quite lower than the previous one, but also that an essential difference between the cases q even and q odd does not appear. Only the case $q = 16$ has been different from the others, but only for the density results.

Nevertheless, it is possible to develop the computer search, too. We are developing new programs from which we expect to be able to achieve new results for maximal partial spreads in $\mathrm{PG}(3,q)$ for values of q that are larger than those studied in this paper.

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