ON THE SUPER EDGE-MAGIC DEFICIENCY OF 2-REGULAR GRAPHS WITH TWO COMPONENTS

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To our dear friend and colleague Ramón Manuel Figueroa-Centeno

ABSTRACT. A graph G is called super edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$ and $f(u) + f(v) + f(uv)$ is a constant for each $uv \in E(G)$. The super edge-magic deficiency, $\mu_s(G)$, of a graph G is defined as the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. In this paper, the super edge-magic deficiency of certain 2-regular graphs with two components is computed, which leads us to a conjecture on the super edge-magic deficiency of graphs in this class.

1. INTRODUCTION

We generally follow the notation and terminology pertaining to graphs of [3]. All graphs that we consider in this paper are finite, simple and undirected. We will denote the set of vertices and edges of a graph G by $V(G)$ and $E(G)$, respectively. For two graphs G_1 and G_2 with disjoint vertex sets, the *union* $G \cong G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If a graph G consists of m disjoint copies of a graph H, then we write $G \cong mH$.

For two integers a and b with $a \leq b$, we will denote the set $\{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$ by simply writing $[a, b]$, where $\mathbb{Z}$ denotes the set of all integers.

In 1970, Kotzig and Rosa [17] introduced the notion of edge-magic labelings. These labelings were called magic valuations by them. These were rediscovered in 1996 by Ringel and Lladó [19] who coined one of the now popular terms for them: edge-magic labelings. More recently, they have also been referred to as edge-magic total labelings by Wallis [21]. For a graph G of order p and size q, a bijective function $f : V(G) \cup E(G) \to [1, p+q]$ is called an *edge-magic labeling* if $f(u) + f(v) + f(uv)$ is a constant (called the *valence* of f) for each $uv \in E(G)$. If such a labeling exists, then G is called an *edge-magic graph*. In 1998, Enomoto et al. [4] defined a slightly restricted version of edge-magic labeling f of a graph G by requiring that $f(V(G)) = [1, p]$. They called such a labeling *super edge-magic*. Thus, a *super edge-magic graph* is a graph that admits a super edge-magic labeling. Lately, super edge-magic labelings and super edge-magic graphs are referred to by Wallis [21] as strong edge-magic total labelings and strongly edge-magic graphs, respectively. Furthermore, according to the latest version of the survey on graph

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labelings by Gallian [10] available to the authors, Hegde and Shetty [13] showed that the properties of being super edge-magic and strongly k-indexable are equivalent.

For every graph G, Kotzig and Rosa [17] proved that there exists an edge-magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This motivated them to define the edge-magic deficiency of a graph. The *edge-magic deficiency*, $\mu(G)$, of a graph G is the smallest nonnegative integer n for which the graph $G \cup nK_1$ is edge-magic. Inspired by Kotzig and Rosa's notion, Figueroa-Centeno et al. [7] analogously defined the concept of *super edge-magic deficiency*, $\mu_s(G)$, of a graph G to be either the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n. Thus, the super edge-magic deficiency of a graph G is a measure of how "close" G is to being super edge-magic.

In two separate papers [6, 7], Figueroa-Centeno et al. computed the super edge-magic deficiencies of the 2-regular graphs mC_n when $m = 1, 2, 3$ and $n \geq 3$. In [7], they also determined the exact values of $\mu_s(4C_{4n})$ and $\mu_s(4C_6)$. According to the parity of m, Ngurah et al. [18] discovered an upper bound for $\mu_s(mC_4)$. Afterwards, the authors determined the exact value of $\mu_s(mC_4)$ in [15]. Figueroa-Centeno et al. [8] studied the super edge-magic properties of 2-regular graphs with two components and proved several classes of such graphs are super edge-magic.

Note that for 2-regular graphs, a super edge-magic labeling is a strong vertex-magic total labeling (see [12, 14] for the definition and results on strong vertex-magic total labelings of 2-regular graphs) and vice versa. In this paper, we will only deal with 2-regular graphs. Thus, we will refer to strong vertex-magic total labelings as super edge-magic labelings.

To present the new results contained in this paper, we will frequently use the following lemma taken from [5].

Lemma 1. *A graph G of order p and size q is super edge-magic if and only if there exists a bijective function $f : V(G) \to [1,p]$ such that the set*

$$S = \{f(u) + f(v) \,|uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = \min(S)$ and

$$S = [k - (p+q), k - (p+1)].$$

Due to Lemma 1, it is sufficient to exhibit the vertex labeling of a super edge-magic graph; however, we will provide the valences to increase the clarity of our results.

Figueroa-Centeno et al. [5] established the following necessary condition for an r-regular graph to be super edge-magic.

Lemma 2. *If G is a super edge-magic r-regular graph of order p and size q, where $r \geq 1$, then q is odd and the valence of any super edge-magic labeling of G is $(q + 4p + 3)/2$.*

Kotzig and Rosa [17] found an upper bound for the edge-magic deficiency of a graph of order p, namely, $\mu(G) \leq F_{p+2} - 2 - \binom{p+1}{2}$, where F_p is the p-th term of the Fibonacci sequence. This implies that every graph has finite edge-magic deficiency. However, not all graphs have finite super edge-magic deficiency as the following lemma indicates (see [7]).

Lemma 3. *If G is a graph of size q such that the degrees of all vertices are even and $q \equiv 2 \pmod 4$, then $\mu_s(G) = +\infty$.*

The type of graph labelings that have received the most attention over the years was introduced by Rosa [20] in 1967, who called them *β-valuations.* They were later studied by Golomb [11] who called them *graceful labelings*, which is the popular term in the current literature of graph labelings. For a graph of size q, an injective function $f : V(G) \to [0, q]$ is called a graceful labeling if each $uv \in E(G)$ is labeled $|f(u) - f(v)|$ and the resulting edge labels are distinct. In [20], Rosa also introduced the concept of α-valuations (a particular type of graceful labeling) as a tool for decomposing the complete graph into isomorphic subgraphs. A graceful labeling f is called an *α-valuation* if there exists an integer λ (called the *critical value* of f) so that $\min\{f(u), f(v)\} \leq \lambda < \max\{f(u), f(v)\}$ for each $uv \in E(G)$. Moreover, he pointed out that a graph that admits an *α-valuation* is necessarily bipartite and therefore can not contain a cycle of odd length.

The following result presented in [15] provides us an upper bound for the super edge-magic deficiency of a graph without isolated vertices that has an α-valuation in terms of its order and size.

Theorem 1. *If G is a graph of order p and size q without isolated vertices that has an α-valuation, then*

$$\mu_s(G) \leq q - p + 1.$$

The bound presented in Theorem 1 is sharp in the sense that there are infinitely many graphs G for which $\mu_s(G) = |E(G)| - |V(G)| + 1$. Indeed, all complete bipartite graphs (see [9]) and some 2-regular bipartite graphs (see [6, 7, 15]) attain the bound.

In the following section, we present some results concerning the super edge-magic deficiency of 2-regular graphs with two components. Notice that the bipartite graphs in this class achieve the bound provided in Theorem 1 if the number of edges in these bipartite graphs is a multiple of 4 (see Theorems 2 and 4, and Corollaries 1 and 2).

For a thorough study of graph labeling problems, see the survey by Gallian [10]. For more information on super edge-magic graphs and related topics, see the books by Bača and Miller [2], and Wallis [21].

2. New Results

Our first result concerns the super edge-magic deficiency of 2-regular bipartite graphs of even order with two components.

Theorem 2. *For even integers m and n with $m \geq 4$ and $n \geq 4$,*

$$\mu_s(C_m \cup C_n) = \begin{cases} 1, & \text{if } m + n \equiv 0 \pmod 4; \\ +\infty, & \text{if } m + n \equiv 2 \pmod 4. \end{cases}$$

Proof. Throughout this proof, assume that m and n are even with $m \geq 4$ and $n \geq 4$. Suppose that $m + n \equiv 0 \pmod 4$. By Lemma 2, the 2-regular graph $C_m \cup C_n$ is not super edge-magic, implying that $\mu_s(C_m \cup C_n) \geq 1$. Also, Abrham and Kotzig [1] have proved that $C_m \cup C_n$ has an α-valuation if and only if both m and n are even and $m + n \equiv 0 \pmod 4$. This together with Theorem 1 provides that $\mu_s(C_m \cup C_n) \leq 1$. Thus, $\mu_s(C_m \cup C_n) = 1$ when $m + n \equiv 0 \pmod 4$. The other case is an immediate consequence of Lemma 3. □

Figueroa-Centeno et al. [8], and Gray and MacDougall [12] independently found a necessary and sufficient condition for the 2-regular graph $C_3 \cup C_n$ to be super edge-magic. This result is now extended in the following theorem.

Theorem 3. *For every integer $n \geq 3$,*

$$\mu_s\left(C_3 \cup C_n\right) = \begin{cases} 0, & \text{if } n \geq 6 \text{ and } n \text{ is even;} \\ 1, & \text{if } n \equiv 1 \pmod 4; \\ 2, & \text{if } n = 4; \\ +\infty, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. One can verify, by an exhaustive computer search, that $C_3 \cup C_4 \cup K_1$ is not super edge-magic. However, $C_3 \cup C_4 \cup 2K_1$ is super edge-magic by labeling the vertices in its cycles with $4-5-8-4$ and $1-6-2-9-1$, and its isolated vertices with 3 and 7 to obtain a valence of 23. This implies that $\mu_s\left(C_3 \cup C_4\right) = 2$. The 2-regular graph $C_3 \cup C_n$ has shown to be super edge-magic if and only if $n \geq 6$ and n is even (see [8, 12]). Thus, $\mu_s\left(C_3 \cup C_n\right) = 0$ if $n \geq 6$ and n is even, whereas $\mu_s\left(C_3 \cup C_n\right) \geq 1$ if $n \equiv 1 \pmod 4$.

To establish that $\mu_s\left(C_3 \cup C_n\right) \leq 1$ when $n \equiv 1 \pmod 4$, let $n = 4k+1$, where k is a positive integer, and define the graph $G \cong C_3 \cup C_{4k+1} \cup K_1$ with

$$V(G) = \{x_i \,|\, i \in [1,3]\} \cup \{y_i \,|\, i \in [1,4k+1]\} \cup \{z\}$$

and

$$E(G) = \{x_i x_{i+1} \,|\, i \in [1,2]\} \cup \{x_1 x_3\} \cup \{y_i y_{i+1} \,|\, i \in [1,4k]\} \cup \{y_1 y_{4k+1}\}.$$

Then the vertex labeling $f : V(G) \to [1, 4k+5]$ such that$f(x_1) = 1$; $f(x_2) = 2k+4$; $f(x_3) = 2k+5$;

$$f(y_l) = \begin{cases} i+1, & \text{if } l = 2i-1 \text{ and } i \in [1,k]; \\ k+3, & \text{if } l = 2i-1 \text{ and } i = k+1; \\ 2k+i+4, & \text{if } l = 2i-1 \text{ and } i \in [k+2, 2k+1]; \\ 2k+i+5, & \text{if } l = 2i \text{ and } i \in [1,k]; \\ i+3, & \text{if } l = 2i \text{ and } i \in [k+1, 2k]; \end{cases}$$

and $f(z) = k+2$ induces a super edge-magic labeling of G with valence $10k+14$, which leads to conclude that $\mu_s\left(C_3 \cup C_n\right) = 1$ when $n \equiv 1 \pmod 4$.

Finally, the remaining case immediately follows from Lemma 3. □

The super edge-magic characterization of the 2-regular graph $C_4 \cup C_n$ was independently given by Figueroa-Centeno et al. [8], and Gray and MacDougall [12]. This result is now extended in the following theorem.

Theorem 4. *For every integer $n \geq 3$,*

$$\mu_s\left(C_4 \cup C_n\right) = \begin{cases} 0, & \text{if } n \geq 5 \text{ and } n \text{ is odd;} \\ 1, & \text{if } n \equiv 0 \pmod 4; \\ 2, & \text{if } n = 3; \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. The 2-regular graph $C_4 \cup C_n$ has shown to be super edge-magic if and only if $n \geq 5$ and n is odd (see [8, 12]). This implies that if $n \geq 5$ and n is odd, then $\mu_s\left(C_4 \cup C_n\right) = 0$. The case where $n \equiv 0 \pmod 4$ immediately follows from Theorem 2. The cases where $n = 3$ and $n \equiv 2 \pmod 4$ easily follows from Theorem 3 and Lemma 3, respectively. This completes the proof of the theorem. □

With the aid of the super edge-magic characterization of the 2-regular graph $C_5 \cup C_n$ found in [8], we are now able to provide the following result.

Theorem 5. *For every integer $n \geq 3$,*

$$\mu_s(C_5 \cup C_n) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \equiv 3 \pmod 4; \\ +\infty, & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Proof. The 2-regular graph $C_5 \cup C_n$ has proven to be super edge-magic if and only if $n \geq 4$ and n is even (see [8]). This implies that $\mu_s(C_5 \cup C_n) = 0$ if $n \geq 4$ and n is even, whereas $\mu_s(C_5 \cup C_n) \geq 1$ if $n \equiv 3 \pmod 4$. Also, it follows from Theorem 3 that $\mu_s(C_5 \cup C_3) = 1$. Thus, it suffices to show that $\mu_s(C_5 \cup C_n) \leq 1$ when $n \geq 7$ and $n \equiv 3 \pmod 4$. To do this, let $G \cong C_5 \cup C_n \cup K_1$ be the graph with

$$V(G) = \{x_i \,|\, i \in [1,5]\} \cup \{y_i \,|\, i \in [1,n]\} \cup \{z\}$$

and

$$E(G) = \{x_i x_{i+1} \,|\, i \in [1,4]\} \cup \{x_1 x_5\} \cup \{y_i y_{i+1} \,|\, i \in [1,n-1]\} \cup \{y_1 y_n\},$$

and consider three cases.

(1) [label=Case 0:]

(2) For $n = 12k - 5$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 12k+1]$ such that

$$f(x_1) = 3k+2; \quad f(x_2) = 9k;$$
$$f(x_3) = 9k+1; \quad f(x_4) = 3k+3;$$
$$f(x_5) = 9k+3;$$

$$f(y_1) = 1;\ f(y_{2i-1}) = i+1, \text{ if } i \in [2, 3k-1];$$
$$f(y_{2i}) = 6k+i+1, \text{ if } i \in [1, 3k-2];$$

$$f(y_l) = \begin{cases} 3k+3i+1, & \text{if } l = 6k+6i-8 \text{ and } i \in [1,k]; \\ 9k+3i+3, & \text{if } l = 6k+6i-7 \text{ and } i \in [1,k-1]; \\ 3k+3i+3, & \text{if } l = 6k+6i-6 \text{ and } i \in [1,k-1]; \\ 9k+3i-1, & \text{if } l = 6k+6i-5 \text{ and } i \in [1,k-1]; \\ 3k+3i+2, & \text{if } l = 6k+6i-4 \text{ and } i \in [1,k-1]; \\ 9k+3i+1, & \text{if } l = 6k+6i-3 \text{ and } i \in [1,k-1]; \end{cases}$$

$$f(y_{12k-7}) = 12k+1; \quad f(y_{12k-6}) = 2;$$
$$f(y_{12k-5}) = 12k-1;$$

and $f(z) = 3k+1$.

(3) For $n = 12k - 1$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 12k+5]$ such that

$$f(x_1) = 3k+3; \quad f(x_2) = 9k+8;$$
$$f(x_3) = 3k+4; \quad f(x_4) = 9k+3;$$
$$f(x_5) = 9k+5;$$

$$f(y_1) = 1;\ f(y_{2i-1}) = i+1, \text{ if } i \in [2, 3k];$$

$$f(y_{2i}) = \begin{cases} 6k+i+3, & \text{if } i \in [1, 3k-1]; \\ i+5, & \text{if } i \in [3k, 6k-2]; \end{cases}$$

$$f(y_l) = \begin{cases} 9k+3i+1, & \text{if } l = 6k+6i-5 \text{ and } i \in [1,k]; \\ 9k+3i+8, & \text{if } l = 6k+6i-3 \text{ and } i \in [1,k-1]; \\ 9k+3i+3, & \text{if } l = 6k+6i-1 \text{ and } i \in [1,k]; \end{cases}$$

$$f(y_{12k-3}) = 12k+4; \quad f(y_{12k-2}) = 2;$$

and $f(z) = 3k+2$.

(4) For $n = 12k+3$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 12k+9]$ such that

$$\begin{aligned} f(x_1) &= 3k+4; & f(x_2) &= 9k+9; \\ f(x_3) &= 3k+5; & f(x_4) &= 9k+7; \\ f(x_5) &= 9k+6; & & \end{aligned}$$

$$\begin{aligned} f(y_1) &= 1;\ f(y_{2i-1}) = i+1, \text{ if } i \in [2, 3k+1]; \\ f(y_{6k+5}) &= 9k+8; \end{aligned}$$

$$f(y_{2i}) = \begin{cases} 6k+i+5, & \text{if } i \in [1, 3k]; \\ i+5, & \text{if } i \in [3k+1, 3k+3]; \end{cases}$$

$$f(y_l) = \begin{cases} 9k+3i+8, & \text{if } l = 6k+6i-3 \text{ and } i \in [1,k]; \\ 9k+3i+9, & \text{if } l = 6k+6i+1 \text{ and } i \in [1,k]; \\ 3k+3i+7, & \text{if } l = 6k+6i+2 \text{ and } i \in [1,k-1]; \\ 3k+3i+6, & \text{if } l = 6k+6i+4 \text{ and } i \in [1,k-1]; \\ 9k+3i+7, & \text{if } l = 6k+6i+5 \text{ and } i \in [1,k-1]; \\ 3k+3i+8, & \text{if } l = 6k+6i+6 \text{ and } i \in [1,k-1]; \end{cases}$$

$$f(y_{12k+2}) = 2; \quad f(y_{12k+3}) = 12k+7;$$

and $f(z) = 3k+3$.

Therefore, by Lemma 1, f extends to a super edge-magic labeling of G with valence $(5n+33)/2$, which implies that $\mu_s(C_5 \cup C_n) = 1$ when $n \geq 7$ and $n \equiv 3 \pmod 4$.

Finally, the remaining case immediately follows from Lemma 3. □

We now explore the super edge-magic deficiency of the 2-regular graph $C_7 \cup C_n$. In this case, the authors have only been able to provide a partial solution to this question, which is contained in the following result and Table 1.

Theorem 6. *For every integer $n \geq 3$,*

$$\mu_s(C_7 \cup C_n) = \begin{cases} 0, & \text{if } n \equiv 0 \text{ or } 8 \pmod{12}; \\ 1, & \text{if } n \equiv 1 \pmod 4; \\ +\infty, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. First, assume that $n \equiv 0$ or $8 \pmod{12}$, and let $G \cong C_7 \cup C_n$ be the 2-regular graph with

$$V(G) = \{x_i \,|\, i \in [1,7]\} \cup \{y_i \,|\, i \in [1,n]\}$$

and

$$E(G) = \{x_i x_{i+1} \,|\, i \in [1,6]\} \cup \{x_1 x_7\} \cup \{y_i y_{i+1} \,|\, i \in [1, n-1]\} \cup \{y_1 y_n\}.$$

Now, consider three cases.

(1) [label= Case 1.0:,, leftmargin=*]

(2) For $n = 8$, define the vertex labeling $f : V(G) \to [1, 15]$ such that

$$(f(x_i))_{i=1}^{7} = (1, 10, 2, 11, 12, 7, 8)$$

and

$$(f(y_i))_{i=1}^{8} = (3, 13, 9, 5, 15, 6, 4, 14).$$

(3) For $n = 12k$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 12k+7]$ such that

$$\begin{aligned} f(x_1) &= 2; & f(x_2) &= 6k+3; \\ f(x_3) &= 3; & f(x_4) &= 6k+6; \\ f(x_5) &= 1; & f(x_6) &= 6k+7; \\ f(x_7) &= 6k+8; & & \\ f(y_1) &= 8; & f(y_3) &= 4; \\ f(y_5) &= 5; & & \end{aligned}$$

$$f(y_{2i}) = 6k+i+8, \text{ if } i \in [1,3k]\,; \; f(y_{6k+1}) = 3k+8;$$

$$f(y_l) = \begin{cases} 3i+8, & \text{if } l = 6i+1 \text{ and } i \in [1,k-1]\,; \\ 3i+3, & \text{if } l = 6i+3 \text{ and } i \in [1,k-1]\,; \\ 3i+4, & \text{if } l = 6i+5 \text{ and } i \in [1,k-1]\,; \\ 3k+3i, & \text{if } l = 6k+6i-4 \text{ and } i \in [1,k]\,; \\ 3k+3i+1, & \text{if } l = 6k+6i-2 \text{ and } i \in [1,k]\,; \\ 3k+3i+8, & \text{if } l = 6k+6i \text{ and } i \in [1,k-1]\,; \end{cases}$$

$$f(y_{2i-1}) = 6k+i+7, \text{ if } i \in [3k+2, 6k]\,; \; f(y_{12k}) = 6k+4.$$

(4) For $n = 12k+8$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 12k+15]$ such that

$$\begin{aligned} f(x_1) &= 1; & f(x_2) &= 6k+10; \\ f(x_3) &= 2; & f(x_4) &= 6k+11; \\ f(x_5) &= 6k+12; & f(x_6) &= 5; \\ f(x_7) &= 6k+9; & & \\ f(y_1) &= 3; & f(y_3) &= 7; \\ f(y_5) &= 4; & f(y_7) &= 9; \\ f(y_9) &= 6; & & \end{aligned}$$

$$f(y_{2i}) = 6k+i+12, \text{ if } i \in [1,5]\,;$$

$$f(y_l) = \begin{cases} 3i+7, & \text{if } l = 6i+5 \text{ and } i \in [1,k-1]\,; \\ 6k+3i+16, & \text{if } l = 6i+6 \text{ and } i \in [1,k-1]\,; \\ 3i+9, & \text{if } l = 6i+7 \text{ and } i \in [1,k-1]\,; \\ 6k+3i+15, & \text{if } l = 6i+8 \text{ and } i \in [1,k-1]\,; \\ 3i+5, & \text{if } l = 6i+9 \text{ and } i \in [1,k-1]\,; \\ 6k+3i+17, & \text{if } l = 6i+10 \text{ and } i \in [1,k-1]\,; \end{cases}$$

$$\begin{aligned} f(y_{6k+5}) &= 3k+7; & f(y_{6k+6}) &= 3k+8; \\ f(y_{6k+8}) &= 3k+10; & f(y_{6k+10}) &= 3k+5; \end{aligned}$$

$$f(y_l) = \begin{cases} 9k+3i+13, & \text{if } l = 6k+6i+1 \text{ and } i \in [1,k]\,; \\ 9k+3i+12, & \text{if } l = 6k+6i+3 \text{ and } i \in [1,k]\,; \\ 9k+3i+14, & \text{if } l = 6k+6i+5 \text{ and } i \in [1,k]\,; \\ 3k+3i+10, & \text{if } l = 6k+6i+6 \text{ and } i \in [1,k-1]\,; \\ 3k+3i+6, & \text{if } l = 6k+6i+8 \text{ and } i \in [1,k-1]\,; \\ 3k+3i+8, & \text{if } l = 6k+6i+10 \text{ and } i \in [1,k-1]\,; \end{cases}$$

$$\begin{aligned} f(y_{12k+6}) &= 6k+8; & f(y_{12k+7}) &= 12k+15; \\ f(y_{12k+8}) &= 6k+6; & & \end{aligned}$$

Thus, by Lemma 1, f extends to a super edge-magic labeling of G with valence $5n/2+19$, which implies that $\mu_s(C_7 \cup C_n) = 0$ when $n \equiv 0$ or $8 \pmod{12}$.

Next, assume that $n \equiv 1 \pmod 4$. By Lemma 2, the 2-regular graph $C_7 \cup C_n$ is not super edge-magic. This implies that $\mu_s(C_7 \cup C_n) \geq 1$. Also, it follows from Theorem 5 that $\mu_s(C_7 \cup C_5) = 1$. Thus, it suffices to show that $\mu_s(C_7 \cup C_n) \leq 1$ when $n \geq 9$ and $n \equiv 1 \pmod 4$. To do this, let $H \cong G \cup K_1$ be the graph with

$$V(H) = V(G) \cup \{z\} \text{ and } E(H) = E(G),$$

where $G \cong C_7 \cup C_n$ as defined above, and consider three cases.

Case 1: [label= Case 2.0:,, leftmargin=*]

Case 2: For $n = 12k-3$, where k is a positive integer, define the vertex labeling $f : V(H) \to [1, 12k+5]$ such that

$$\begin{array}{rclrcl} f(x_1) &=& 1; & f(x_2) &=& 6k+5; \\ f(x_3) &=& 2; & f(x_4) &=& 6k+6; \\ f(x_5) &=& 6k+7; & f(x_6) &=& 6; \\ f(x_7) &=& 6k+4; & & & \end{array}$$

$$f(y_{2i-1}) = 6k+i+7, \text{ if } i \in [1, 3k-1]; \ f(y_2) = 3;$$

$$f(y_l) = \begin{cases} 3i+6, & \text{if } l = 6i-2 \text{ and } i \in [1, k-1]; \\ 3i+1, & \text{if } l = 6i \text{ and } i \in [1, k-1]; \\ 3i+2, & \text{if } l = 6i+2 \text{ and } i \in [1, k-1]; \end{cases}$$

$$\begin{array}{rclrcl} f(y_{6k-2}) &=& 3k+4; & f(y_{6k-1}) &=& 3k+5; \\ f(y_{6k+1}) &=& 3k+1; & & & \end{array}$$

$$f(y_l) = \begin{cases} 6k+i+7, & \text{if } l = 2i \text{ and } i \in [3k, 6k-2]; \\ i+4, & \text{if } l = 2i-1 \text{ and } i \in [3k+2, 6k-1]; \end{cases}$$

and $f(z) = 3k+2$.

Case 3: For $n = 12k+1$, where k is a positive integer, define the vertex labeling $f : V(H) \to [1, 12k+9]$ such that

$$\begin{array}{rclrcl} f(x_1) &=& 1; & f(x_2) &=& 6k+6; \\ f(x_3) &=& 2; & f(x_4) &=& 6k+7; \\ f(x_5) &=& 5; & f(x_6) &=& 6k+8; \\ f(x_7) &=& 6k+9; & & & \end{array}$$

$$f(y_{2i-1}) = 6k+i+9, \text{ if } i \in [1, 3k]; \ f(y_2) = 4;$$

$$f(y_l) = \begin{cases} 3i+4, & \text{if } l = 6i-2 \text{ and } i \in [1, k]; \\ 3i+5, & \text{if } l = 6i \text{ and } i \in [1, k-1]; \\ 3i, & \text{if } l = 6i+2 \text{ and } i \in [1, k-1]; \end{cases}$$

$$\begin{array}{rclrcl} f(y_{6k}) &=& 3k+5; & f(y_{6k+1}) &=& 3k+6; \\ f(y_{6k+3}) &=& 3k; & & & \end{array}$$

$$f(y_l) = \begin{cases} 6k+i+9, & \text{if } l = 2i \text{ and } i \in [3k+1, 6k]; \\ i+4, & \text{if } l = 2i-1 \text{ and } i \in [3k+3, 6k+1]; \end{cases}$$

and $f(z) = 3k+3$.

Case 4: For $n = 12k + 5$, where k is a positive integer, define the vertex labeling $f : V(G) \to [1, 12k + 13]$ such that

$$\begin{aligned} f(x_1) &= 1; & f(x_2) &= 6k+9; \\ f(x_3) &= 2; & f(x_4) &= 6k+10; \\ f(x_5) &= 6k+11; & f(x_6) &= 6; \\ f(x_7) &= 6k+8; \end{aligned}$$

$$f(y_{2i-1}) = 6k + i + 11, \text{ if } i \in [1, 3k+1];$$

$$f(y_2) = 3; \qquad f(y_4) = 5;$$

$$f(y_l) = \begin{cases} 3i+5, & \text{if } l = 6i \text{ and } i \in [1,k]; \\ 3i+6, & \text{if } l = 6i+2 \text{ and } i \in [1,k-1]; \\ 3i+1, & \text{if } l = 6i+4 \text{ and } i \in [1,k-1]; \end{cases}$$

$$\begin{aligned} f(y_{6k+2}) &= 3k+6; & f(y_{6k+3}) &= 3k+7; \\ f(y_{6k+5}) &= 3k+1; \end{aligned}$$

$$f(y_l) = \begin{cases} 6k+i+11, & \text{if } l = 2i \text{ and } i \in [3k+2, 6k+2]; \\ i+4, & \text{if } l = 2i-1 \text{ and } i \in [3k+4, 6k+3]; \end{cases}$$

and $f(z) = 3k + 4$.

Thus, by Lemma 1, f extends to a super edge-magic labeling of H with valence $(5n + 43)/2$, which implies that $\mu_s(C_7 \cup C_n) = 1$ when $n \geq 9$ and $n \equiv 1 \pmod 4$.

Finally, the remaining case immediately follows from Lemma 3. □

In light of Table 1, it seems plausible to have a more general result than the one just presented.

TABLE 1. Super Edge-Magic Labelings of $C_m \cup C_n$ for small m and n

m	n	C_m	C_n	k
7	10	(1, 9, 2, 13, 12, 5, 11, 1)	(7, 6, 14, 8, 15, 3, 16, 10, 4, 17, 7)	44
	14	(1, 12, 2, 13, 3, 9, 16, 1)	(7, 14, 4, 15, 5, 17, 6, 18, 8, 19, 10, 20, 11, 21, 7)	54
	16	(2, 13, 5, 9, 4, 15, 14, 2)	(1, 16, 6, 17, 3, 18, 7, 19, 8, 20, 10, 21, 11, 22, 12, 23, 1)	59
	18	(1, 15, 5, 13, 2, 17, 16, 1)	(6, 8, 18, 4, 19, 10, 20, 7, 21, 3, 22, 12, 9, 23, 14, 24, 11, 25, 6)	64
	22	(1, 17, 4, 16, 6, 19, 18, 1)	(9, 8, 20, 12, 21, 2, 22, 7, 23, 3, 24, 10, 25, 11, 5, 26, 13, 27, 14, 28, 15, 29)	74
	26	(1, 19, 5, 17, 2, 21, 20, 1)	(6, 12, 22, 4, 23, 7, 24, 8, 25, 3, 26, 9, 27, 10, 28, 14, 11, 29, 16, 30, 13, 31, 18, 32, 15, 33, 6)	84

Figueroa-Centeno et al. [8] have proved that if m is even with $m \geq 4$ and n is odd satisfying $n \geq m/2 + 2$, then the 2-regular graph $C_m \cup C_n$ is super edge-magic. Combining this with Lemma 3, the above results in this section and Table 1, we obtain the following two corollaries.

Corollary 1. *For every two integers m and n with $1 \le m \le 3$ and $n \ge 3$,*

$$\mu_s(C_{4m} \cup C_n) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \equiv 0 \pmod 4; \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Corollary 2. *For every two integers m and n with $1 \le m \le 3$ and $n \ge 3$,*

$$\mu_s(C_{4m+2} \cup C_n) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \equiv 2 \pmod 4; \\ +\infty, & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

3. Conclusions

In the preceding section, we computed $\mu_s(C_m \cup C_n)$ for some positive integers m and n. In particular, we determine the exact value of $\mu_s(C_3 \cup C_4)$ to be 2 (see Theorem 3). Also, we presented in Theorem 1 that $\mu_s(C_m \cup C_n) = 1$ if $m + n \equiv 0 \pmod 4$, whereas $\mu_s(C_m \cup C_n) = +\infty$ if $m + n \equiv 2 \pmod 4$. On the other hand, Figueroa-Centeno et al. [8] conjectured that the 2-regular graph $C_m \cup C_n$ is super edge-magic if and only if $m + n \ge 9$ and $m + n$ is odd. All of these lead us to the following conjecture.

Conjecture 1. *For every two integers $m \ge 3$ and $n \ge 3$,*

$$\mu_s(C_m \cup C_n) = \begin{cases} 0, & \text{if } m+n \ge 9 \text{ and } m+n \text{ is odd;} \\ 1, & \text{if } m+n \equiv 0 \pmod 4; \\ 2, & \text{if } m = 3 \text{ and } n = 4; \\ +\infty, & \text{if } m+n \equiv 2 \pmod 4. \end{cases}$$

Holden et al. [14] have made a stronger conjecture than Conjecture 1 that with the exception of $C_3 \cup C_4$, $3C_3 \cup C_4$ and $2C_3 \cup C_5$, all 2-regular graphs of odd order possess super edge-magic labelings. Thus, our results in this paper adds credence to their conjecture.

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