On the f-chromatic class of a multi-wheel graph*

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Abstract An f-coloring of a graph G is an edge-coloring of G such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. A multi-wheel graph is a graph obtained from s cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_s}$ $(s \geq 1)$ by adding a new vertex, say w, and edges joining w to all the vertices of the s cycles. In this article, we solve a conjecture posed by Yu et al. in 2006 and prove that it is not always true. Furthermore, the classification problem of multi-wheel graphs on f-colorings is solved completely.

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1 Introduction

Throughout this paper, the term *graph* is used to denote a finite undirected simple graph. The reader is referred to [1] for the undefined terms.

An *edge-coloring* of G is an assignment of colors to all the edges of G. Let G be a graph and let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An *f-coloring* of G is an edge-coloring of G such that each vertex $v \in V(G)$ has at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the *f-chromatic index* of G and denoted by $\chi'_f(G)$. We denote the *degree* of vertex v by $d(v)$. Define

$$\Delta_f(G) = \max_{v \in V(G)} \{\lceil \frac{d(v)}{f(v)} \rceil\},$$

where $\lceil x \rceil$ is the smallest integer not smaller than x.

Hakimi and Kariv [2] firstly posed and studied f-colorings. One of their results will be used in the rest of this article as follows.

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Theorem 1 *[2] Let G be a graph. Then*

$$\Delta_f(G) \leq \chi'_f(G) \leq \max_{v\in V(G)}\{\lceil\frac{d(v)+1}{f(v)}\rceil\} \leq \Delta_f(G)+1.$$

G is called a graph of *f-class* 1 if $\chi'_f(G) = \Delta_f(G)$, and of *f-class* 2 otherwise. The problem of deciding whether a graph G is of f-class 1 or f-class 2 is called the *classification problem* on f-colorings.

If $f(v) = 1$ for all $v \in V(G)$, the f-coloring is reduced to the *proper edge-coloring*, and the f-chromatic index of G is reduced to the *chromatic index* of G and denoted by $\chi'(G)$. The well-known theorem of Vizing [4], i.e. $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$, can be deduced from Theorem 1.

Zhang and Liu [6]-[10], Zhang et al. [11, 12], Yu et al. [5] studied the classification problem of complete graphs, regular graphs and some other special classes of graphs on f-colorings. Liu et al. [3] studied some properties of f-critical graphs. (A graph G is called *f-critical* if G is of f-class 2 and $\chi'_f(G-e) < \chi'_f(G)$ for every edge $e \in E(G)$.)

Let

$$V_0^*(G) = \{v \in V(G) : d(v) = f(v)\Delta_f(G)\}.$$

The *f-core* of a graph G is the subgraph of G induced by the vertices of $V_0^*(G)$ and is denoted by G_{Δ_f}. The number $d(v)/f(v)$ is called the *f-ratio* of vertex v in G. We call a graph G *RP-removable*, if all the vertices of G can be iteratively removed using the following vertex removal operations:

(1) removal of a vertex v with degree at most $(f(v)-1)\Delta_f(G)+1$;

(2) removal of a vertex v, which has at most one remaining neighbor of f-ratio $\Delta_f(G)$.

Zhang et al. [11] got the following result.

Theorem 2 *[11] Let G be a graph. If G is RP-removable, then G is of f-class* 1.

Clearly, a graph without f-core is *RP*-removable. Furthermore, a graph whose f-core is a forest is also *RP*-removable because we can iteratively remove the remaining vertices of degree one in the f-core first. Hence, the following results can be deduced from Theorem 2.

Corollary 3 *[7] Let G be a graph. If $V_0^*(G) = \emptyset$, then G is of f-class* 1.

Corollary 4 *[7] Let G be a graph. If G_{Δ_f} is a forest, then G is of f-class* 1.

A *cycle* is a closed walk $v_1v_2\ldots v_pv_1$ in which $v_1, v_2, \ldots, v_p$ are distinct. A cycle with p vertices is denoted by C_p. By Corollary 4, it is easy to see that if G is of f-class 2, then G_{Δ_f} must contain cycles. When G_{Δ_f} contains cycles, the simplest non-trivial case is that G_{Δ_f} consists of vertex-disjoint cycles and paths. In this paper, we will give a class of graphs of f-class 2 whose f-core has maximum degree two.

A *multi-wheel graph* is a graph obtained from s cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_s}$ ($s \geq 1$) by adding a new vertex, say w, and edges joining w to all the vertices of the s cycles. The new vertex is called the *hub* of the multi-wheel graph. Let $n = \Sigma_{1 \leq i \leq s} n_i$. Clearly, for a multi-wheel graph G with order $n+1$, the hub w has $d(w) = n$ and each $v \in V(G) \backslash \{w\}$ has $d(v) = 3$. When $s = 1$, a multi-wheel graph with order $n+1$ is just a wheel graph, which is denoted by W_{n+1}. Yu et al. obtained the following result and posed a conjecture on wheel graphs [5].

Theorem 5 *[5] Let G be a wheel graph of order $n+1$ with the hub w and the cycle $C_n = v_1 v_2 \ldots v_n v_1$. If $d(w) \neq 3r + 2$ ($r \in Z^+$) when $\Delta_f(G) = 3$, then G is of f-class 1.*

Conjecture 1 *[5] Let G be a wheel graph of order $n+1$ with the hub w and the cycle $C_n = v_1 v_2 \ldots v_n v_1$. If $d(w) = 3r + 2$ ($r \in Z^+$) when $\Delta_f(G) = 3$, then G is of f-class 2.*

However, the following counterexample in Fig. 1 implies that the conjecture is not true.

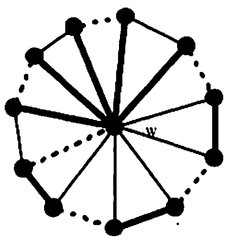

Fig 1. An f-colorings of W_{12} with $\Delta_f(W_{12}) = 3$ colors, where $f(w) = 5$ and $f(v) = 1$ for each $v \in V(W_{12}) \setminus \{w\}$.

In Section 2, we solve Conjecture 1 and show that case $f(w) = r + 1$ and $f(v) = 1$ for all $v \in V(G) \setminus \{w\}$ is the only one when conjecture works (see Theorem 7). Furthermore, we solve the classification problem of multi-wheel graphs on f-colorings completely. In Section 3, we give a problem for further research.

2 Main Results

Let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Suppose that G has been given an edge-coloring $\tilde{c}$ with colors in C. An edge colored with color $\alpha \in C$ is called an *α-edge*. We denote by $|\alpha(v)|$ the number of α-edges of G incident with the vertex $v \in V(G)$. Define $m(v, \alpha) = f(v) - |\alpha(v)|$ for each $v \in V(G)$ and each $\alpha \in C$. For two distinct colors $a, b \in C$, a trail $W = v_0 e_1 v_1 e_2 v_2 \ldots e_h v_h$ is called an *ab-alternating trail* if W satisfies the following conditions:

(a) the edges of W are colored alternately with a and b, and the first edge of W is colored with b;

(b) $m(v_0, a) \geq 2$ if $v_0 = v_h$ and h is odd;
$m(v_0, a) \geq 1$, $m(v_h, b) \geq 1$ if $v_0 \neq v_h$ and h is even;
$m(v_0, a) \geq 1$, $m(v_h, a) \geq 1$ if $v_0 \neq v_h$ and h is odd.

The operation, interchanging the colors a and b of the edges in an ab-alternating trail W, is called *switching* W. After W was switched, $m(v_i, a)$ and $m(v_i, b)$ remain as they were if $i \neq 0, h$, while $m(v_0, b) \geq 1$ and $m(v_0, a) \geq 0$ if W is not a closed trail, or $m(v_0, b) \geq 2$ and $m(v_0, a) \geq 0$ if W is a closed trail of odd length.

Theorem 6 *Let G be a multi-wheel graph of order $n+1$ with the hub w and the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_s}$, and $f(w) \geq (n+1)/3$. Then, when $f(w) = (n+1)/3$ and $f(v) = 1$ for all $v \in V(G) \setminus \{w\}$, G is of f-class 2; when $f(w) > (n+1)/3$ or $f(u) \geq 2$ for at least one vertex $u \in V(G) \setminus \{w\}$, G is of f-class 1.*

Proof. Since $f(w) \geq (n+1)/3$, $d(v) = 3$ and $f(v) \geq 1$ for each $v \in V(G) \setminus \{w\}$, there is $\Delta_f(G) \leq 3$. If $\Delta_f(G) = 1$, i.e. $d(v) \leq f(v)$ for all $v \in V(G)$, then $\chi'_f(G) = 1$. If $\Delta_f(G) = 2$, then either $V_0^*(G) = \emptyset$ or $V_0^*(G) = \{w\}$. In either case, G is of f-class 1 according to Corollary 3 and Corollary 4. Next, we discuss the cases with $\Delta_f(G) = 3$.

If $f(w) = (n+1)/3$ and $f(v) = 1$ for all $v \in V(G) \setminus \{w\}$, then $\frac{d(w)}{f(w)} = \frac{n}{f(w)} < 3$ and $\frac{d(v)}{f(v)} = 3$ for each $v \in V(G) \setminus \{w\}$. So this is a case with $\Delta_f(G) = 3$. In addition, $V_0^*(G) = V(G) \setminus \{w\}$. By contradiction, suppose that ζ is an f-coloring of G with 3 colors. Then one color appears exactly $f(w) - 1$ times and either of the other two appears $f(w)$ times at vertex w in ζ. So there will exist some color, say a, such that the number of all a-edges in ζ, i.e. $(f(w) \times 1 + 1 \times (3f(w) - 1))/2$, is not an integer. This is a contradiction. So G is of f-class 2 if $f(w) = (n+1)/3$ and $f(v) = 1$ for all $v \in V(G) \setminus \{w\}$.

If $f(w) > (n+1)/3$, then $\frac{d(w)}{f(w)} = \frac{n}{f(w)} < 3$. We consider two cases.

Case 1. *There exists a vertex $v \in V(G) \setminus \{w\}$ with $f(v) = 1$.*
Clearly, $\Delta_f(G) = 3$. Since $d(w) = n \leq 3f(w) - 2 = 3(f(w) - 1) + 1$, we can RP-remove w first. After that, the f-ratio of any remaining vertex is at most 2. That is to say that G is RP-removable. By Theorem 2, G is of f-class 1.

Case 2. *Each vertex $v \in V(G) \setminus \{w\}$ has $f(v) \geq 2$.*
$\Delta_f(G) = 3$ only if $\lceil \frac{d(w)}{f(w)} \rceil = 3$ in this case. This implies that $V_0^*(G) = \emptyset$. By Corollary 3, G is of f-class 1.

Now, the remaining case is that $f(w) = (n+1)/3$ and there exists a vertex $u \in V(G) \setminus \{w\}$ such that $f(u) \geq 2$. We suppose that $u \in V(C_{n_k}) (1 \leq k \leq s)$. Let $u' \in V(C_{n_k}) \cap N_G(u)$. Clearly, $w, u \notin V_0^*(G)$ by $\Delta_f(G) = 3$. This implies that u' has at most one neighbor of f-ratio $\Delta_f(G)$ in G. Thus we can RP-remove u' first. Then RP-remove w because $d_{G-u'}(w) = 3f(w) - 2 = 3(f(w) - 1) + 1$. Now, each remaining vertex has f-ratio at most 2. Therefore G is RP-removable. By Theorem 2, G is of f-class 1. ∎

Now we can solve Conjecture 1 by virtue of Theorem 6.

Theorem 7 *Let G be a wheel graph of order $n+1$ with the hub w and the cycle C_n. Suppose that $d(w) = 3r + 2$ $(r \in Z^+)$ and $\Delta_f(G) = 3$. G is of f-class 2 if and only if $f(w) = r + 1$ and $f(v) = 1$ for all $v \in V(G) \setminus \{w\}$.*

Proof. $\Delta_f(G) = 3$ and $3 \nmid d(w)$ implies that $\frac{d(w)}{f(w)} < 3$. Thus $f(w) \geq (d(w)+1)/3 = (n+1)/3$. By Theorem 6 for the case that $s = 1$, G is of f-class 2 if and only if $f(w) = (n+1)/3 = (d(w)+1)/3 = r+1$ and $f(v) = 1$ for all $v \in V(G) \setminus \{w\}$. ∎

In general, we have the following result to solve the classification problem of multi-wheel graphs on f-colorings.

Theorem 8 *Let G be a multi-wheel graph of order $n+1$ with the hub w and the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_s}$. Then G is of f-class 2 if and only if $f(w) = (n+1)/3$ and $f(v) = 1$ for all $v \in V(G) \setminus \{w\}$.*

Proof. By Theorem 6, the sufficiency is verified. Next, we show the necessity that G is of f-class 1 when $f(w) \neq (n+1)/3$ or $f(u) \geq 2$ for at least one vertex $u \in V(G) \setminus \{w\}$.

When $f(w) > (n+1)/3$, or $f(w) = (n+1)/3$ and $f(u) \geq 2$ for at least one vertex $u \in V(G) \setminus \{w\}$, G is of f-class 1 by Theorem 6. So the remaining cases to be considered are the ones with $f(w) < (n+1)/3$, i.e. $n \geq 3f(w)$.

Case 1. $n > 3f(w)$.

Then $\frac{d(w)}{f(w)} = \frac{n}{f(w)} > 3$. This means $\Delta_f(G) \geq 4$. Since $d(v) = 3$ for each $v \in V(G) \setminus \{w\}$, $V_0^*(G) = \emptyset$ or $V_0^*(G) = \{w\}$. By Corollary 3 or Corollary 4, G is of f-class 1.

Case 2. $n = 3f(w)$.

Since $\frac{d(w)}{f(w)} = 3$ and $\frac{d(v)}{f(v)} = \frac{3}{f(v)} \leq 3$, there is $\Delta_f(G) = 3$. We give an f-coloring of G with 3 colors $\alpha_1, \alpha_2, \alpha_3$ as follows. First, for each $i = 1, 2, \ldots, s$, we color the edges in cycle C_{n_i} orderly starting from an arbitrary edge. If $n_i \equiv 0 \pmod 3$, color the edges with colors $\alpha_1, \alpha_2, \alpha_3$ alternately. If $n_i \equiv 1 \pmod 3$, color the first $n_i - 1$ edges with colors $\alpha_1, \alpha_2, \alpha_3$ alternately and the last edge with color α_2. If $n_i \equiv 2 \pmod 3$, color the first $n_i - 2$ edges with colors $\alpha_1, \alpha_2, \alpha_3$ alternately, the $(n_i - 1)$th edge with α_1 and the last edge with α_2. Second, for each $v \in V(G) \setminus \{w\}$, we color the edge wv with the color which is one of $\{\alpha_1, \alpha_2, \alpha_3\}$ and does not appear at v so far. Now, we obtain an edge-coloring ζ in which each of $\{\alpha_1, \alpha_2, \alpha_3\}$ appears at each vertex $v \in V(G) \setminus \{w\}$ exactly one time. Let $q = \max_{1 \leq i < j \leq 3}\{||\alpha_i(w)| - |\alpha_j(w)||\}$. If $q = 0$, i.e. $|\alpha_i(w)| = f(w)$ for each $i = 1, 2, 3$, then ζ is an f-coloring of G with 3 colors. Otherwise, we claim that $q \geq 4$. Suppose that a and b are two colors in $\{\alpha_1, \alpha_2, \alpha_3\}$ satisfying that $|a(w)| - |b(w)| = q$. Clearly, $|a(w)| \geq f(w) + 1$ and $|b(w)| \leq f(w) - 1$ since $d(w) = 3f(w)$. Thus $q \geq 2$. If $q = 2$, then the number of a-edges in ζ, i.e. $((f(w) + 1) + 3f(w))/2$, is not an integer. If $q = 3$, then the number of either a-edges or b-edges in ζ, i.e. one of $\{(|a(w)| + 3f(w))/2, (|b(w)| + 3f(w))/2\}$, is not an integer. When $q \geq 4$, we can find a closed ba-alternating trail P with odd length starting at w. Switching P makes $|a(w)|$ decrease by two and $|b(w)|$ increase by two. Thus $|a(w)| - |b(w)|$ decreases by four. Repeat this operation until $q = 0$. Therefore, G is of f-class 1 in this case. ∎

3 Further research problem

For a multi-wheel graph G of f-class 2, G_{Δ_f} is a union of disjoint cycles. We can show another example on a graph of f-class 2, which is not a multi-wheel graph and whose f-core is a cycle, as below.

Example. Let H be the join graph of C_{14} and $\overline{K_3}$, where $f(v) = 1$ for each $v \in V(C_{14})$ and $f(u) = 3$ for each $u \in V(\overline{K_3})$. Then H is of f-class 2.

Proof. Clearly, $d(v) = 5$ for each $v \in V(C_{14})$ and $d(u) = 14$ for each $u \in V(\overline{K_3})$ in H. So $\Delta_f(H) = 5$.

By contradiction, suppose that H is of f-class 1. Then H has an f-coloring with 5 colors $c_1, c_2, \ldots, c_5$. For whatever distribution of 5 colors in f-colorings of H, each of 5 colors appears exactly 1 time at v when vertex $v \in V(C_{14})$ and some color appears 2 times and each of the others appears 3 times at u when vertex $u \in V(\overline{K_3})$. Since there are 5 colors and $|V(\overline{K_3})| = 3$, there must exist at least 2 colors, each of which appears 1 time at each $v \in V(C_{14})$ and 3 times at each $u \in V(\overline{K_3})$. This means that, for an arbitrary f-coloring of H with 5 colors, there exists one color c_i, $1 \leq i \leq 5$, in such a way that the number of the all c_i-edges is $(1 \times 14 + 3 \times 3)/2$. This is not an integer, a contradiction. ∎

Based on the discussion above, the following problem is interesting for further research.

Problem. *What kinds of graphs G are of f-class 2 when $\Delta(G_{\Delta_f}) = 2$?*

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