D-GRAPHS FOR GRAPHS WITH CYCLOMATIC NUMBER 1

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Abstract

We identify a graph without proper cycles, which is comatching with a cycle. The result is then extended to certain general families of graphs with cyclomatic number 1, formed by attaching trees to cycles.

1. Introduction

The graphs considered here are finite and contain no loops. However, they may contain multiple edges. We refer to Harary [4], for the basic definitions in Graph Theory (however, we use "nodes" and "edges" instead of "points" and "lines" respectively) .

It is well known, that the matching polynomial is not a characterizing polynomial for graphs (See Farrell [1]). Given an arbitrary graph, it is of interest, to identify graphs which have the same matching polynomial. Some general constructions of such graphs are given in Farrell and Wahid [2]. It is also of interest, to identify graphs whose matching polynomial is the determinant of an associated matrix. Such graphs have been discussed in Farrell and Wahid [3].

Definition

Let G be a simple graph with p nodes, q edges and k components. Then, the ***cyclomatic number*** (or cycle rank) of G is q-p+k.

In this paper, we identify families of graphs, without proper cycles (i.e. cycles with at least three edges), which have the same matching polynomial as certain families graph formed by attaching trees to cycles; i.e. families of graphs with cyclomatic number 1.We conclude the paper with a summary of the results in his area and a discussion on some open problems.

2. The Matching Polynomial of a Graph

First of all, we give some definitions and results, which are relevant to the material which follows.

Let G be a graph. A ***matching*** in G is a spanning subgraph, whose components are nodes and edges only. Let p be the number of nodes in G; and a_k -the number of matchings in G, with k edges. Then, the ***matching polynomial*** of G is

$$M(G;\underline{w}) = \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} a_k w_1^{p-2k} w_2^k ,$$

where $\underline{w} = (w_1, w_2)$. The indeterminates w_1 and w_2 (over the complex numbers) are the weights associated with each node and edge respectively, in G. This polynomial was introduced in Farrell [1]. For brevity, we shall write **M(G)** for $M(G;\underline{w})$, when it would lead to no confusion. Graphs which have the same matching polynomial, are called ***comatching***.

The smallest pair of non-trivial comatching graphs is the following:

Figure 1

The following are the smallest connected comatching graphs.

Figure 2

The following result is taken from [1]. It provides the basic algorithm (called ***The Reduction Process***) for finding matching polynomials of arbitrary graphs.

Theorem 1(The Fundamental Edge Theorem)

Let G be a graph containing an edge uv (joining nodes u and v). Let G' be the graph obtained by deleting the edge uv; and G^* -the graph obtained from G by removing nodes u and v. Then

$$M(G;\underline{w}) = M(G';\underline{w}) + w_2 M(G^*;\underline{w}).$$

When this theorem is applied, the graph G' is referred to; as "the reduced graph" and the graph G^*; as "the incorporated graph"

Let G be a graph with p nodes. The ***matching matrix*** of G is the pxp matrix $A(G) = [a_{ij}]$, over the complex numbers, where

$$a_{ij} = \begin{cases} \sqrt{nw_2}, & \text{if nodes i and j are joined by n edges; and } i < j \\ -\sqrt{nw_2}, & \text{if nodes i and j are joined by n edges; and } i > j \\ w_1, & \text{if } i = j \\ 0, & \text{if nodes i and j are not adjacent.} \end{cases}$$

This matrix was introduced in [3].

Let G be a graph, such that its matching polynomial is the determinant of its matching matrix; i.e. $M(G) = |A(G)|$. Then G is called a ***D-graph***. If H is any D-graph, such that H is comatching with G, then H is called a **D-graph for G**; and we write **H = D(G)**.

The following lemma gives a sufficient condition for a graph to be a D-graph. This result is taken from [3].

Lemma 1

Let G be a graph without proper even cycles. Then

$$M(G) = |A(G)|.$$

3. D-graphs for Cycles

We will denote by $\mathbf{C_n}$, the cycle with n nodes. From Lemma 1, it is clear that an odd cycle is a D-graph. In the case of an even cycle, we will find a comatching graph, without proper cycles.

A path is a tree, with nodes of valencies 1 and 2 only. The path with n nodes will be denoted by $\mathbf{P_n}$. We will call P_n a ***chain***, when it is a component of a graph. The two nodes of valency 1 are called
end-nodes of P_n.

The ***tadpole graph*** (or ***tadpole***, for brevity) $\mathbf{T_n}$, is the multigraph with n edges, formed from P_n, by adding a new edge, joining an end node, to its adjacent node. The resulting node of valency 2 is called the ***head node*** of T_n.

The following theorem identifies a graph which is comatching with a cycle.

Theorem 2

Let C_n be the cycle with n nodes. Then tadpole T_n is a D-graph for C_n.

Proof

Let us apply The Reduction Process to C_n, by deleting an edge. Then the reduced graph G' will be the chain P_n. The incorporated graph G^* will be the chain P_{n-2}. Therefore

$$M(C_n) = M(P_n) + w_2M(P_{n-2}).$$

Apply the Reduction Process to T_n, by deleting one of its double edges. Then the reduced graph will be P_n. The incorporated graph will be P_{n-2}. Therefore

$$M(T_n) = M(P_n) + w_2M(P_{n-2})$$
$$= M(C_n).$$

Since T_n has no proper cycles, it is a D-graph. Hence $T_n = D(C_n)$. □

4. D-graphs for Cycles With Chains Attached

Let G and H be graphs, rooted at nodes u and v respectively. We ***attach*** G to H (or H to G) by identifying nodes u and v. The resulting node is called the ***node of attachment***. Throughout this paper, we will use an ***end node*** (a node of valency one) of a chain as its root. The ***connecting edge*** of an attached chain, is the edge incident to its node of attachment.

We now consider the graph $\mathbf{T_{n,m}}$ with cyclomatic number 1, formed by attaching the chain P_m to the cycle C_n.

Theorem 3

Let H be the graph formed by attaching the chain P_m to the head node of the tadpole T_n. Then $H = D(T_{n,m})$.

Proof

Let xy be the connecting edge of the chain P_m. Apply the Reduction Process to $T_{n,m}$, by deleting xy. The reduced graph will consist of two components; the cycle

C_n and the chain P_{m-1}. The incorporated graph will be the graph obtained from $T_{n,m}$, by removing nodes x and y. It will therefore consist of two components chains; P_{n-1} and P_{m-2}. Hence Theorem 1 yields,

$$M(T_{n,m}) = M(C_n)M(P_{m-1}) + w_2M(P_{n-1})\, M(P_{m-2}).$$

Now, apply the Reduction Process to H; by deleting the connecting edge xy of P_m . The reduced graph will consist of two components; the tadpole T_n and the chain P_{m-1}. The incorporated graph will be the graph obtained from H, by removing nodes x and y. It will therefore consist of the two component chains; P_{n-1} and P_{m-2}. Hence Theorem1 yields

$$M(H) = M(T_n)M(P_{m-1}) + w_2M(P_{n-1})\, M(P_{m-2}).$$
$$= M(C_n)M(P_{m-1}) + w_2M(P_{n-1})\, M(P_{m-2}) \text{ (using Theorem 2)}$$
$$= M(T_{n,m}).$$

Since H has no proper cycles, it is a D-graph. Hence the result follows. □

5. D-graphs for Cycles With Trees Attached

Let C_n be a cycle with n nodes. Let R_1, R_2, .. and R_n be trees, such that for $1 < i \leq n$, $R_i \cong R_{n-i+2}$. Let us attach these n trees to C_n , in the following manner. Firstly, attach R_1 to an arbitrary node of C_n. Then, attach the remaining trees R_2, R_3,..and R_n in sequential order (ie. for $1 < i \leq n-1$, the node of attachment of R_i is adjacent to the node of attachment of R_{i+1}) either clockwise or anticlockwise, to the remaining n-1 nodes of C_n.Then the n trees have been attached ***symmetrically*** to C_n .

We will now consider graphs formed by attaching trees (including trees with one node) symmetrically to cycles; with the restriction that two of the attached trees are isomorphic and are separated by a path of length 2, on the cycle. The node separating these trees is called a ***symmetric node*** of the cycle. Clearly then, R_1 is always attached to a symmetric node. The order in which the other trees are attached, relative to a symmetric node x, will be referred to, as the order "starting at x". We attach trees "in order" to a tadpole, starting with the head node as the first node of attachment. Therefore, for attachment purposes, the head node of a tadpole , is equivalent to a symmetric node of a cycle.

The following theorem identifies a D-graph for a graph consisting of a cycle with trees symmetrically attached, and containing a symmetric node.

Theorem 4

Let G be the graph with a symmetric node, formed by attaching trees symmetrically, to the nodes of the cycle C_n. Let H be the graph formed by attaching isomorphs of the trees in order, to the n nodes of the tadpole T_n. Then H is a D-graph for G.

Proof

Let x_1 be a symmetric node in G. Apply the Reduction Process to G, by deleting the edge x_1x_n of C_n, incident to x_1. The reduced graph G' will be the chain P_n, with the n trees attached in order, starting at x_1. In general, the incorporated graph G^*, will contain three subgraphs;

(i) R_1-$\{x_1\}$,
(ii) R_n-$\{x_n\}$,

[N.B. if R_1 and R_n are nodes, then R_1-$\{x_1\}$ and R_n-$\{x_n\}$ will be empty graphs. Also, if the roots of R_1 and R_n have valency1, then R_1-$\{x_1\}$ and R_n-$\{x_n\}$ will be connected graphs.]

and (iii) the graph $P_{n-2}(2,3,\ldots,n-1)$, consisting of the chain P_{n-2}, with the n-2 trees $R_2, R_3, R_4, \ldots R_{n-1}$ attached in order, starting at node x_2.

Clearly then,

$$M(G^{*};\underline{w}) = M(R_1\text{-}\{x_1\})\, M(R_n\text{-}\{x_n\})\, M(P_{n-2}(2,3,\ldots,n-1)). \quad \ldots(1)$$

From Theorem 1, we have

$$M(G;\underline{w}) = M(G';\underline{w}) + w_2M(G^{*};\underline{w}).$$
$$= M(G';\underline{w}) + w_2M(R_1\text{-}\{x_1\})M(R_n\text{-}\{x_n\})M(P_{n-2}(2,3,\ldots,n-1)). \quad \ldots(2)$$

Let head node of T_n be z, and its adjacent node- y, Apply the Reduction Process to H, by deleting one of the double edges yz. Then the reduced graph H' will be the chain P_n, with the n trees attached in order, starting with the tree R_1 at z. It will therefore be isomorphic to G'. Therefore

$$M(H';\underline{w}) = M(G';\underline{w}). \quad \ldots(3)$$

The incorporated graph H^{*}, will be the graph obtained from H, by removing nodes y and z. It will therefore contain three subgraphs;

(i) R_1-$\{z\}$(since R_1 is attached to the head node z),

(ii) R_2-$\{y\}$(since R_2 is attached to node y),

and (iii) the graph $P_{n-2}(3,4,\ldots,n)$, consisting of the chain P_{n-2}, with the n-2 trees $R_3, R_4, \ldots R_n$ attached in order, starting at node y.

Therefore

$$M(H^{*};\underline{w}) = M(R_1\text{-}\{z\})\, M(R_2\text{-}\{y\})\, M(P_{n-2}(3,4,\ldots,n)). \quad \ldots(4)$$

Clearly R_1-$\{x_1\}$ and R_1-$\{z\}$ are isomorphic graphs. Also, $R_2$and R_n are isomorphic (by data); and therefore R_n-$\{x_n\}$and R_2-$\{y\}$ are isomorphic.

Now, the graph $P_{n-2}(3,4,\ldots,n)$ can also be denoted by $P_{n-2}(n,n-1,\ldots,4,3)$, by viewing the chain from the opposite end. Since the trees are symmetrically attached to the nodes of the cycle C_n, then, for $1< i \leq n$, we have $R_i \cong R_{n-i+2}$.

$\Rightarrow P_{n-2}(2,3,\ldots,n-1)$ and $P_{n-2}(n,n-1,\ldots,4,3)$ are isomorphic.

$\Rightarrow P_{n-2}(2,3,\ldots,n-1)$ and $P_{n-2}(3,4,\ldots,n)$ are isomorphic.

Therefore, from Equation (4), we get

$$M(H^{*};\underline{w}) = M(R_1\text{-}\{x_1\}\, M(R_n\text{-}\{x_n\})M(P_{n-2}(2,3,\ldots,n-1)).$$

From Theorem 1, we get

$$M(H;\underline{w}) = M(H';\underline{w}) + w_2M(H^{*};\underline{w}).$$
$$= M(H';\underline{w}) + w_2M(R_1\text{-}\{x_1\}M(R_n\text{-}\{x_n\})M(P_{n-2}(2,3,\ldots,n-1)).$$
$$= M(G';\underline{w}) + w_2M(G^{*};\underline{w}).\ \text{(Using Equation 3)}$$
$$= M(G;\underline{w}).$$

Since H has no proper cycles, it is a D-graph. Hence the result follows. □

An Illustration

Let G be the following graph obtained by attaching trees to the cycle C_6.

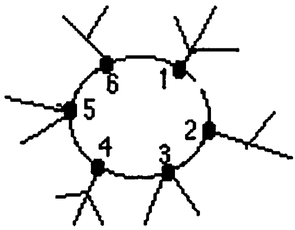

Figure 3

Then the following graph H_1 is the comatching D-graph for G, obtained by using Theorem 4, with node 1 as the symmetric node.

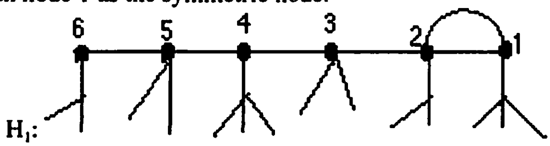

Figure 4

The following graph H_2 is the comatching D-graph for G, obtained by using Theorem 4, with node 4 as the symmetric node.

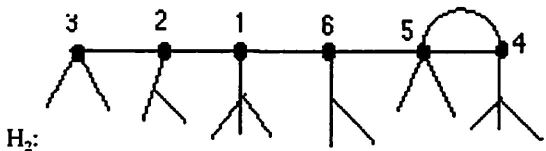

Figure 5

6. Discussion

Given a graph G, it is an interesting problem, to find a comatching D-graph D(G). It is also a difficult problem. In this paper, we have identified graphs D(G), for graphs G, with cyclomatic number 1. However, there are restrictions on the trees which are attached to the basic cycle. One restriction is that two of the attached trees are isomorphic and are separated by a path of length 2. This became necessary, in order to define a symmetric node. Such a node is vital to the construction of the comatching D-graph, since the trees must be attached in the particular order, defined by the symmetric node. Hence the problem has not been entirely solved for all graphs with cyclomatic number 1.

Graphs with cyclomatic number 2, have been investigated. Results have been obtained for the basic graphs; but only for certain kinds of theta graphs. These results have also been extended to general graphs formed by attaching trees to the basic graphs. Again there have been restrictions.

The technique which has been used to construct comatching D-graphs has the advantage of providing more than one such D-graph, for a given graph. The implications of this, on theory of determinants is no doubt, very interesting.

Given an arbitrary graph G, how does one, in general, find a D(G)? Our approach has been via its cyclomatic number. This approach has had limited success so far.

References

[1] E. J. Farrell, An Introduction to Matching Polynomials, J.Comb. Theory B Vol 27, (1979) 75-86.

[2] E. J. Farrell and S. A. Wahid, Some General Classes of Comatching Graphs. Internat.J.Math & Math. Sci.Vol.10. No.3(1987) 519-524.

[3] E. J. Farrell and S. A. Wahid, D-Graphs 1, An Introduction to Graphs Whose Matching Polynomials are Determinants of Matrices, Bulletin of the ICA, Vol.15(1995) 81-86.

[4] F. Harary, Graph Theory, Addison-Wesley Pub. Co. Reading, Mass.(1969).

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