Total domination in generalized θ graphs and ladder graphs *

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Abstract

A set of vertices in a graph G without isolated vertices is a total dominating set (TDS) of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a TDS of G is the total domination number $\gamma_t(G)$ of G. In this paper, the total domination number of generalized θ graphs and $m \times n$ ladder graphs is determined.

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1 Introduction

Throughout this paper, we only consider finite and simple undirected graphs without isolated vertices. For $m \in N$, set $I_m = \{n \in N : 1 \leq n \leq m\}$. Set $I_0 = \emptyset$. For a graph G, $V = V(G)$ and $E = E(G)$ will denote its sets of vertices and edges. For each vertex $v \in V$, let $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from the context. For $v \in V$ and $S \subseteq V$, let $d_S(v) = N(v) \cap S$.

For $S \subseteq V$, let $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. For $S \subseteq V$, an induced subgraph of G, denoted by $G - S$, is a graph obtained from G by deleting all vertices in S and all edges with at least one end vertex in S. For a graph $G = (V, E)$ and an edge set E_1, we define $G + E_1 = (V, E \cup E_1)$ and $G - E_1 = (V, E \backslash E_1)$. For each vertex $u \in V$, let $G - u = G - \{u\}$. For $S_1, S_2 \subseteq V$, we set $S_1 - S_2 = S_1 \setminus S_2$ and $S_1 + S_2 = S_1 \cup S_2$. A path(cycle) on n vertices is denoted by $P_n(C_n)$.

A total dominating set, abbreviated TDS, of a graph G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)-$set. In 1980, E. J. Cockayne[3] introduced the subject of total dominating set (TDS) in graphs. In[1,2,7], the authors proved that $\gamma_t(G) \leq n/2$ if G is a graph of order n with minimum degree $\delta \geq 3$. In [5], M. A. Henning proved that $\gamma_t(G) \leq (1/2+1/g)n$ if G is a graph of order n with minimum degree $\delta \geq 2$ and girth $g \geq 3$. We refer to [1-7] for more background on the historical importance of this problem and other results not mentioned here.

In [4], the authors defined a function rd counting the times v is re-dominated as $rd(v) = |N[v] \cap S| - 1$. In this paper, we defined a function rtd counting the times v is re-total-dominated as $rtd(v) = d_S(v) - 1$.

$G = (V,\ E)$ is called a generalized θ graph[6] if G is a simple connected graph obtained from two vertices x and y by adding at least two paths joining x and y, such that $d(v) = 2$ for each $v \in V \setminus \{x,\ y\}$.

For each $m,\ n \geq 3$, $G = L_{mn} = (V,\ E)$ is called a $m \times n$ ladder graph if G is a simple connected graph obtained from two paths $u_1u_2\cdots u_m$ and $v_1v_2\cdots v_m$ by adding a path of $n-1$ edges joining u_i and v_i for each $i \in I_m$, such that $d(v) = 2$ for each $v \in V \setminus \{u_i,\ v_i :\ i \in I_m\}$.

2 Main Result

In this paper, we have the following results:

Theorem 2.1 Let $G = (V,\ E)$ be a generalized θ graph, where $V = \{x_{ij} :\ i \in I_m,\ j \in I_{k_i}\} \cup \{x,\ y\}$ and $E \setminus \{xy\} = \{xx_{i1},\ yx_{ik_i} :\ i \in I_m\} \cup \{x_{ij}x_{i,j+1} :\ i \in I_m,\ j \in I_{k_i-1}\}$ (Note that $I_0 = \emptyset$.) such that $k_i \in N$ for each $i \in I_m$. Let $\varphi(k) = \gamma_t(P_{k+2}) - 2$ for each $k \in N$. For $l = 0,\ 1,\ 2,\ 3$, let $J_l = \{i \in I_m :\ k_i \equiv l(mod 4)\}$. Then we have the following results.

(A) $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 2$ if $xy \in E$.

(B) $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 2 = (|V| - m + 2)/2$ if $J_1 = I_m$.

(C) $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 2$ if $J_0 \neq \emptyset$.

(D) $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 2$ if $|J_3| \geq 2$.

(E) $\gamma_t(G) \geq \sum_{i=1}^{m} \varphi(k_i) + 3$ if and only if $xy \notin E,\ J_0 = \emptyset,\ |J_3| \leq 1$ and $J_1 \neq I_m$.

(F) $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 4 = |V|/2 - m + 3$ if $xy \notin E$ and $J_2 = I_m$.

(G) $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 3$ if $xy \notin E,\ J_0 = \emptyset$ and $|J_3| = 1$.

(H) $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 3$ if $xy \notin E,\ J_0 = J_3 = \emptyset,\ J_1 \neq \emptyset,\ J_2 \neq \emptyset$.

Theorem 2.2 Let $G = L_{mn} = (V,\ E)$ be a $m \times n$ ladder graph. Then we have the following results.

(A) $\gamma_t(G) = mn/2$ if $n \equiv 0(mod4)$.

(B) $\gamma_t(G) = mn/2 - m/2 + 2$ if $n \equiv 3(mod4)$ and $m \geq 4$. Moreover, $\gamma_t(L_{3n}) = (3n-1)/2$ if $n \equiv 3(mod4)$.

(C) $\gamma_t(G) = mn/2$ if $n \equiv 1(mod4)$ and $m \equiv 0(mod4)$.

(D) $\gamma_t(G) = mn/2{+}1/2$ if $n \equiv 1(mod4)$ and $m \equiv 1,\ 3(mod4)$.

(E) $\gamma_t(G) = mn/2 + 1$ if $n \equiv 1(mod4)$ and $m \equiv 2(mod4)$.

(F) $\gamma_t(G) = mn/2 - m/3$ if $n \equiv 2(mod4)$ and $m \equiv 0(mod3)$.

(G) $\gamma_t(G) = mn/2 - m/3 + 4/3$ if $n \equiv 2(mod4)$ and $m \equiv 1(mod3)$.

(H) $\gamma_t(G) = mn/2 - m/3 + 2/3$ if $n \equiv 2(mod4)$ and $m \equiv 2(mod3)$.

3 Preliminaries

In [6], the following results is proposed.

Lemma 3.1[6] For each $n \geq 3$,

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} n/2, & n \equiv 0(mod4); \\ [n/2]{+}1, & \text{otherwise.} \end{cases}$$

Lemma 3.2[6] Let $G = (V,\ E)$ be a generalized θ graph as in Theorem 2.1, there must be some $\gamma_t(G)-$set containing x or y.

Lemma 3.3[6] $\gamma_t(G) \leq \alpha(G) + 1$ for each generalized θ graph G, where $\alpha(G)$ denoted the matching number of G.

By Lemma 3.1, we have the following result.

Lemma 3.4 For each $k \in N \cup \{0\}$, let $\varphi(k) = \gamma_t(P_{k+2}) - 2$, then

$$\varphi(k) = \begin{cases} k/2-1, & k \equiv 2(mod4); \\ (k-1)/2, & k \equiv 1,\ 3(mod4); \\ k/2, & k \equiv 0(mod4). \end{cases}$$

Moreover, $0 \leq \varphi(k+1)-\varphi(k) \leq 1$, $\varphi(k+4)-\varphi(k) = 2$ and $\varphi(k+3)+\varphi(k+1) = \varphi(k+2)+\varphi(k)+1$ for each $k \in N \cup \{0\}$.

Lemma 3.5 Let $G = (V,\ E)$ be a simple connected graph. Let $P = xu_1u_2 \cdots u_ky$ be a path of G such that $d(u_i) = 2$ for each $i \in I_k (k \geq 0)$. Let $U = \{u_i :\ i \in I_k\}$, $U_0 = \{x,\ y\}$, $U_1 = \{u_1,\ u_k\}$. Let S be a $\gamma_t(G)$−set. Then we have the following results.

(A) $\sum_{u \in U} rtd(u,\ S) = 2|U \cap S| - |U_1 \cap S| + |U_0 \cap S| - k$ if $k \geq 1$.

(B) $|U \cap S| \geq \lceil k/2 \rceil - 1$.

(C) $|U \cap S| \geq \varphi(k) = \gamma_t(P_{k+2}) - 2$ if $U_0 \subseteq S$.

(D) $|U \cap S| \geq \gamma_t(P_{k+1}) - 1$ if $k \geq 1$ and $x,\ u_1 \in S$.

(E) $|U \cap S| \geq \gamma_t(P_{k-1}) + 1$ if $k \geq 3$ and $u_j \in S$ for $j = 1,\ k-1,\ k$.

(F) $|U \cap S| \geq \gamma_t(P_{k-1})$ if $k \geq 3$, $x \in S$ and $y \notin S$.

(G) $|U \cap S| \geq \gamma_t(P_k)$ if $k \geq 2$ and $x,\ y \notin S$.

Proof (A) Note that $\sum_{u \in U} rtd(u,\ S) = \sum_{u \in U} d_S(u) - k = \sum_{s \in S} d_U(s) - k = 2|U \cap S| - |U_1 \cap S| + |U_0 \cap S| - k$. (A) is proved.

(B) We may suppose that $k \geq 1$. By (A), $2|U \cap S| \geq k - |U_0 \cap S| + |U_1 \cap S| \geq k-2$, (B) is proved.

(C) We may suppose that $k \geq 1$. If $U_0 \subseteq S$, then $S \cap V(P)$ is a TDS of $P + \{xy\}$, (C) is proved.

(D) We may suppose that $k \geq 2$. If $x,\ u_1 \in S$, then $U \cap S + \{x\}$ is a TDS of $G[U \cup \{x\}] + \{xu_k\}$, (D) is proved.

(E) We may suppose that $k \geq 4$. If $u_j \in S$ for $j = 1,\ k-1,\ k$, then $U \cap S - \{u_k\}$ is a TDS of $G[U] - u_k + \{u_1u_{k-1}\}$, (E) is proved.

(F) Let $x \in S$ and $y \notin S$. If $k = 3$, note that $N[u_3] \cap S \neq \emptyset$, then $|U \cap S| \geq 2$, the result follows. We may suppose that $k \geq 4$. If $u_1 \notin S$, then $U \cap S$ is a TDS of $G[U] - u_1$, the result follows. If $u_1, u_k \in S$, then by (C), we have $|U \cap S| \geq \gamma_t(P_k) \geq \gamma_t(P_{k-1})$. If $u_1 \in S$ and $u_k \notin S$, then $u_{k-1} \in S$, and then by (C), we have $|U \cap S| \geq \gamma_t(P_{k-1})$. (F) is proved.

(G) If $k \geq 2$ and $x, y \notin S$, then $U \cap S$ is a TDS of $G[U]$, the result follows. Lemma 3.5 is proved.

4 Generalized θ graphs

Let $G = (V, E)$, φ, $J_l (l = 0, 1, 2, 3)$ be as in Theorem 2.1. Let $A_i = \{x_{ij} : j \in I_{k_i}\}$ for $i \in I_m$. Let $U_0 = \{x, y\}$, $U_1 = N(U_0) - U_0$ and $U_2 = N(U_1) - U_0 - U_1$. Let $X = \{S \subseteq V : S$ is a $\gamma_t(G)$-set$\}$, $X_0 = \{S_0 \in X : |S_0 \cap U_0| = \max\{|S \cap U_0| : S \in X\}\}$, $X_1 = \{S_1 \in X_0 : |S_1 \cap U_1| = \max\{|S_0 \cap U_1| : S_0 \in X_0\}\}$, $X_2 = \{S_2 \in X_1 : |S_2 \cap U_2| = \max\{|S_1 \cap U_2| : S_1 \in X_1\}\}$. We may suppose that $S \in X_2$.

Lemma 4.1 If $|U_0 \cap S| = 1$, then $xy \notin E$ and $U_1 \cap S = U_1 \cap A_{i_0}$ for some $i_0 \in I_m$.

Proof Note that S contains no isolated vertices, we may suppose that $x, x_{i_0 1} \in S$ for some $i_0 \in I_m$.

Claim 1 $xy \notin E$.

Otherwise, let $T = S - A_{i_0} + \{x_{i_0,j} \in A_{i_0} : x_{i_0,j+1} \in S\} + \{y\}$, then $T \in X$ and $|T \cap U_0| > |S \cap U_0|$, and then $S \notin X_0$, a contradiction. Claim 1 is proved.

Claim 2 $N(y) \cap S = \{x_{i_0 k_{i_0}}\}$.

Suppose that $N(y) \cap S \neq \{x_{i_0 k_{i_0}}\}$, note that $N(y) \cap S \neq \emptyset$, then $x_{i_1 k_{i_1}} \in N(y) \cap S$ for some $i_1 \neq i_0$. Let $T = S - A_{i_0} - A_{i_1} + \{x_{i_0,j-1} \in A_{i_0} : x_{i_0 j} \in S\} + \{x_{i_1,j+1} \in A_{i_1} : x_{i_1 j} \in S\} + \{y, x_{i_1 1}\}$, then $T \in X$ and $|T \cap U_0| > |S \cap U_0|$, then $S \notin X_0$, a contradiction.

Claim 2 is proved.

Claim 3 $N(x) \cap S = \{x_{i_0 1}\}$.

Otherwise, let $x_{i_2 1} \in N(x) \cap S$ for some $i_2 \neq i_0$. Let $T = S - A_{i_0} - A_{i_2} + \{x_{i_0, j+1} \in A_{i_0} : x_{i_0 j} \in S\} + \{x_{i_2, j-1} \in A_{i_2} : x_{i_2 j} \in S\} + \{y, x_{i_0 1}\}$, then $T \in X$ and $|T \cap U_0| > |S \cap U_0|$, a contradiction. Claim 3 is proved. Lemma 4.1 is proved.

Lemma 4.2 Let $U_0 \cap S = \{x\}$ and $U_1 \cap S = U_1 \cap A_1$, then $1 \in J_1 \cup J_2$. Moreover, if $1 \in J_1$, then $|A_1 \cap S| = (k_1+1)/2$ and $A_1 \cap S = \{x_{1j} : j \equiv 0, 1(mod 4)\}$. If $1 \in J_2$, then $|A_1 \cap S| = k_1/2 + 1$ and $A_1 \cap S = \{x_{1j} : j \equiv 1, 2(mod 4)\}$.

Proof **Claim 1** $|A_1 \cap S| \geq \gamma_t(P_{k_1-1}) + 1$ if $k_1 \geq 3$.

Note that $x_{1j} \in S$ for $j = 1, k_1 - 1, k_1$, by Lemma 3.5 (E), Claim 1 is proved.

Claim 2 $1 \notin J_3$.

Otherwise, by Lemma 3.1, $|A_1 \cap S| \geq \gamma_t(P_{k_1-1}) + 1 = [(k_1 - 1)/2] + 2 = (k_1+3)/2$. Let $T = S - A_1 + \{x_{1j} : j \equiv 0, 1(mod 4)\} + \{y, x_{1k_1}\}$. Then $T \in X$ and $|T \cap U_0| > |S \cap U_0|$, then $S \notin X_0$, a contradiction. Claim 2 is proved.

Claim 3 $1 \notin J_0$.

Otherwise, note that $k_1 \geq 4$. By Lemma 3.1, $|A_1 \cap S| \geq \gamma_t(P_{k_1-1}) + 1 = k_1/2 + 1$. Let $T = S - A_1 + \{x_{1j} : j \equiv 0, 1(mod 4)\} + \{y\}$. Then $T \in X$ and $|T \cap U_0| > |S \cap U_0|$, a contradiction. Claim 3 is proved.

Claim 4 If $1 \in J_1$, then $|A_1 \cap S| = (k_1+1)/2$ and $A_1 \cap S = \{x_{1j} : j \equiv 0, 1(mod 4)\}$.

We may suppose that $k_1 \geq 5$. By Lemma 3.1, $|A_1 \cap S| \geq \gamma_t(P_{k_1-1}) + 1 = (k_1 - 1)/2 + 1 = (k_1 + 1)/2$. Let $T = S - A_1 + \{x_{1j} : j \equiv 0, 1(mod 4)\}$, then T is a TDS of G. Note that $|S| \leq |T|$ and $|A_1 \cap T| = (k_1+1)/2$, then $|A_1 \cap S| = (k_1+1)/2$. Since $U_1 \cap S = U_1 \cap A_1$, by Lemma 3.5 (A), $\sum_{u \in A_1} rtd(u, S) = 2|A_1 \cap S| - |U_1 \cap S| + |U_0 \cap S| - k_1 = (k_1+1) - 2 + 1 - k_1 = 0$, then each connected component of $G[A_1 \cap S] - x_{11}(G[A_1] - S)$ is K_2. Claim 4 is proved.

Claim 5 If $1 \in J_2$, then $|A_1 \cap S| = k_1/2 + 1$ and $A_1 \cap S = \{x_{1j} : j \equiv 1, 2(mod4)\}$.

We may suppose that $k_1 \geq 6$. By Lemma 3.1, $|A_1 \cap S| \geq \gamma_t(P_{k_1-1}) + 1 = [(k_1-1)/2] + 2 = k_1/2+1$. Let $T = S - A_1 + \{x_{1j} : j \equiv 1, 2(mod4)\}$, then T is a TDS of G. Note that $|S| \leq |T|$ and $|A_1 \cap T| = k_1/2 + 1$, then $|A_1 \cap S| = k_1/2+1$. Since $S, T \in X_2$, then $x_{1j} \in S$ for $j = 1, 2, k_1 - 1, k_1$. By Lemma 3.5 (A), $\sum_{u \in A_1} rtd(u, S) = 2|A_1 \cap S| - |U_1 \cap A_1 \cap S| + |U_0 \cap S| - k_1 = (k_1 + 2) - 2 + 1 - k_1 = 1$. Note that $rtd(x_{11}, S) = 1$, then each connected component of $G[A_1 \cap S](G[A_1] - S)$ is K_2, Claim 5 is proved. Lemma 4.2 is proved.

Lemma 4.3 Let $U_0 \cap S = \{x\}$ and $U_1 \cap S = U_1 \cap A_1$, then $I_m - \{1\} \subseteq J_1$ and $A_i \cap S = \{x_{ij} \in A_i : j \equiv 0, 3(mod4)\}$ for each $i \in I_m - \{1\}$.

Proof **Claim 1** $|A_i \cap S| \geq \gamma_t(P_{k_i-1})$ if $i \in I_m - \{1\}$ and $k_i \geq 3$.

By Lemma 3.5 (F), Claim 1 is proved.

Claim 2 $2 \notin J_0$.

Otherwise, since $k_2 \geq 4$, then $|A_2 \cap S| \geq \gamma_t(P_{k_2-1}) = [(k_2 - 1)/2]+1 = k_2/2$. Let $T = S - A_2 + \{x_{2j} \in A_2 : j \equiv 0, 3(mod4)\}$, then $T \in X_0$ and $|T \cap U_1| > |S \cap U_1|$, a contradiction. Claim 2 is proved.

Claim 3 $2 \notin J_2$.

Otherwise, since $U_1 \cap S = U_1 \cap A_1$ and $N[x_{2k_2}] \cap S \neq \emptyset$, then $k_2 \geq 6$, and then $|A_2 \cap S| \geq \gamma_t(P_{k_2-1}) = k_2/2$. Let $T = S - A_2 + \{x_{2j} \in A_2 : j \equiv 0, 1(mod4)\}$, then $T \in X_0$ and $|T \cap U_1| > |S \cap U_1|$, a contradiction. Claim 3 is proved.

Claim 4 $2 \notin J_3$.

Otherwise, $|A_2 \cap S| \geq \gamma_t(P_{k_2-1}) = (k_2 + 1)/2$. Let $T =$

$S - A_2 + \{x_{2j} \in A_2 : \ j \equiv 0, \ 3(mod4)\} + \{y\}$, then $T \in X$ and $|T \cap U_0| > |S \cap U_0|$, a contradiction. Claim 4 is proved.

Claim 5 $2 \in J_1$, $A_2 \cap S = \{x_{2j} \in A_2 : \ j \equiv 0, \ 3(mod4)\}$.

By the above claims, we have $2 \in J_1$ and $|A_2 \cap S| \geq (k_2 - 1)/2$. Let $T = S - A_2 + \{x_{2j} \in A_2 : \ j \equiv 0, \ 3(mod4)\}$, then T is a TDS of G. Note that $|S| \leq |T|$ and $|A_2 \cap T| = (k_2 - 1)/2$, then $|A_2 \cap S| = (k_2 - 1)/2$. We may suppose that $k_2 \neq 1$. Since $U_1 \cap S = U_1 \cap A_1$ and $N[x_{2k_2}] \cap S \neq \emptyset$, then $x_{2,k_2-1} \in S$. By Lemma 3.5 (A), $\sum_{u \in A_2} rtd(u, \ S) = 2|A_2 \cap S| - |U_1 \cap A_2 \cap S| + |U_0 \cap S| - k_2 = 0$. Claim 5 is proved.

By symmetry, Lemma 4.3 is proved.

Lemma 4.4 $|U_0 \cap S| = 1$ if and only if $xy \notin E$ and $I_m = J_1$.

Proof $\Rightarrow$ Otherwise, suppose that $U_0 \cap S = \{x\}$ and $1 \notin J_1$. By Lemma 4.1, $xy \notin E$. Moreover, $U_1 \cap S = U_1 \cap A_{i_0}$ for some $i_0 \in I_m$. By Lemma 4.3, $i_0 = 1$ and $A_i \cap S = \{x_{ij} \in A_i : \ j \equiv 0, \ 3(mod4)\}$ for each $i \in I_m - \{1\}$. By Lemma 4.2, $1 \in J_2$ and $A_1 \cap S = \{x_{1j} \in A_1 : \ j \equiv 1, \ 2(mod4)\}$. Now, $\gamma_t(G) = |S| = \sum_{i=2}^{m}(k_i - 1)/2 + (k_1/2 + 1) + 1 = (|V| - m + 3)/2$. Let $T = U_0 + \{x_{1j} : \ j \equiv 0, \ 3(mod4)\} + \{x_{2j} : \ j \equiv 1, \ 2(mod4)\} + \{x_{ij} : i \geq 3, \ j \equiv 2, \ 3(mod4)\}$, then $T \in X$ and $|T \cap U_0| > |S \cap U_0|$, a contradiction. the result follows.

$\Leftarrow$ Otherwise, we have $U_0 \subseteq S$ by Lemma 3.2. By Lemma 3.5 (B), $|A_i \cap S| \geq \lceil k_i/2 \rceil - 1 = (k_i - 1)/2$ for each $i \in I_m$. Note that $U_1 \cap S \neq \emptyset$, we may suppose that $x_{11} \in S$, then by Lemma 3.5 (D), $|A_1 \cap S| \geq \gamma_t(P_{k_1+1}) - 1 = (k_1 + 1)/2$. Now, $\gamma_t(G) = |S| \geq \sum_{i=1}^{m}(k_i - 1)/2 + 3 = (|V| - m + 4)/2$. Let $T = \{x_{1j} : \ j \equiv 0, \ 1(mod4)\} + \{x_{ij} : \ i \geq 2, \ j \equiv 0, \ 3(mod4)\} + \{x\}$, then T is a TDS of G with $|T| < |S|$, a contradiction. Lemma 4.4 is proved.

Lemma 4.5 $\gamma_t(G) = (|V| - m + 2)/2$ if $xy \notin E$ and $I_m = J_1$.

Proof By Lemma 4.1 and Lemma 4.4, we may suppose that $U_0 \cap S = \{x\}$ and $U_1 \cap S = U_1 \cap A_1$. By Lemma 3.5 (E) and (F), $\gamma_t(G) = |S| \geq \sum_{i=1}^{m} \gamma_t(P_{k_i-1}) + 2 = \sum_{i=1}^{m}(k_i - 1)/2 + 2 = (|V| - m + 2)/2$. Let $T = \{x\} + \{x_{1j} : j \equiv 0, 1(mod4)\} + \{x_{ij} : i \geq 2, j \equiv 0, 3(mod4)\}$, then T is a TDS of G with $|T| = (|V| - m + 2)/2$. Lemma 4.5 is proved.

By Lemma 4.4 and Lemma 4.5, the case for $|U_0 \cap S| = 1$ is solved completely.

Lemma 4.6 Let $U_0 \subseteq S$ and $1 \in J_1$, then we have the following results.

(A) $|A_1 \cap S| \geq \varphi(k_1) = (k_1 - 1)/2$.

(B) $|A_1 \cap S| = \varphi(k_1) = (k_1 - 1)/2$ if $A_1 \cap U_1 \cap S = \emptyset$.

(C) $|A_1 \cap S| = \varphi(k_1) + 1 = (k_1 + 1)/2$ if $A_1 \cap U_1 \cap S \neq \emptyset$.

Proof (A) The result follows by Lemma 3.4 and Lemma 3.5 (C).

(B) Let $T = S - A_1 + \{x_{1j} \in A_1 : j \equiv 0, 3(mod4)\}$, then T is a TDS of G with $|A_1 \cap T| = \varphi(k_1)$. (B) is proved.

(C) We may suppose that $x_{11} \in S$, then by Lemma 3.5 (D), $|A_1 \cap S| \geq \gamma_t(P_{k_1+1}) - 1 = (k_1+1)/2$. Let $T = S - A_1 + \{x_{1j} \in A_1 : j \equiv 1, 2(mod4)\}$, then T is a TDS of G with $|A_1 \cap T| = (k_1+1)/2$. (C) is proved. Lemma 4.6 is proved.

Lemma 4.7 Let $U_0 \subseteq S$ and $1 \in J_2$, then we have the following results.

(A) $|A_1 \cap S| \geq \varphi(k_1) = k_1/2 - 1$.

(B) $|A_1 \cap S| = \varphi(k_1) = k_1/2 - 1$ if $A_1 \cap U_1 \cap S = \emptyset$.

(C) $|A_1 \cap S| = \varphi(k_1) + 1 = k_1/2$ if $|A_1 \cap U_1 \cap S| = 1$.

(D) $|A_1 \cap S| = \varphi(k_1) + 2 = k_1/2 + 1$ if $|A_1 \cap U_1 \cap S| = 2$.

Proof (A) The result follows by Lemma 3.4 and Lemma 3.5 (C).

(B) Let $T = S - A_1 + \{x_{1j} \in A_1 : \ j \equiv 0,\ 3(mod4)\}$, then T is a TDS of G with $|A_1 \cap T| = k_1/2 - 1$. (B) is proved.

(C) We may suppose that $x_{11} \in S$, then by Lemma 3.5 (D), $|A_1 \cap S| \geq \gamma_t(P_{k_1+1}) - 1 = k_1/2$. Let $T = S - A_1 + \{x_{1j} \in A_1 : j \equiv 0,\ 1(mod4)\}$, then T is a TDS of G with $|A_1 \cap T| = k_1/2$. (C) is proved.

(D) By Lemma 3.5 (C), $|A_1 \cap S| \geq \gamma_t(P_{k_1}) = k_1/2 + 1$. Let $T = S - A_1 + \{x_{1j} \in A_1 : \ j \equiv 1,\ 2(mod4)\}$, then T is a TDS of G with $|A_1 \cap T| = k_1/2 + 1$. (D) is proved. Lemma 4.7 is proved.

Lemma 4.8 Let $U_0 \subseteq S$ and $1 \in J_3$, then we have the following results.

(A) $|A_1 \cap S| \geq \varphi(k_1) = (k_1 - 1)/2$ and $A_1 \cap U_1 \cap S \neq \emptyset$.

(B) $|A_1 \cap S| = \varphi(k_1) = (k_1 - 1)/2$ if $|A_1 \cap U_1 \cap S| = 1$.

(C) $|A_1 \cap S| = \varphi(k_1) + 1 = (k_1 + 1)/2$ if $|A_1 \cap U_1 \cap S| = 2$.

Proof (A) By Lemma 3.4 and Lemma 3.5 (C), $|A_1 \cap S| \geq \varphi(k_1) = (k_1 - 1)/2$. Let $T = S - A_1 + \{x_{1j} \in A_1 : \ j \equiv 0,\ 1(mod4)\}$. Suppose that $A_1 \cap U_1 \cap S = \emptyset$, then $T \in X_0$ and $|A_1 \cap U_1 \cap T| = 1$, then $S \notin X_1$, a contradiction. (A) is proved.

(B) We may suppose that $x_{11} \in S$, let T be defined as in (A), then $T \in X_1$, and then $|A_1 \cap S| = |A_1 \cap T| = (k_1 - 1)/2$, the result follows.

(C) By Lemma 3.5 (C), $|A_1 \cap S| \geq \gamma_t(P_{k_1}) = (k_1 + 1)/2$. Let $T = S - A_1 + \{x_{1j} \in A_1 : \ j \equiv 0,\ 1(mod4)\} + \{x_{1k_1}\}$, then T is a TDS of G with $|A_1 \cap T| = (k_1 + 1)/2$. (C) is proved. Lemma 4.8 is proved.

Lemma 4.9 Let $U_0 \subseteq S$ and $1 \in J_0$, then $|A_1 \cap S| = \varphi(k_1) = k_1/2$ and $A_1 \cap S = \{x_{1j} \in A_1 : \ j \equiv 0,\ 1(mod4)\}$.

Proof By Lemma 3.5 (C), $|A_1 \cap S| \geq \varphi(k_1) = k_1/2$. Let $T = S - A_1 + \{x_{1j} \in A_1 : \ j \equiv 0,\ 1(mod4)\}$, then $T \in X_1$, then $|A_1 \cap S| = k_1/2$ and $x_{1j} \in S$ for $j = 1,\ k_1$. By Lemma 3.5 (A), $\sum_{u \in A_1} rtd(u,\ S) = 2|A_1 \cap S| - |A_1 \cap U_1 \cap S| + |U_0 \cap S| - k_1 = 0$, the result follows.

By the above six lemmas, we have the following result.

Lemma 4.10 $\gamma_t(G) \geq \sum_{i=1}^{m} \varphi(k_i) + 2$. For each $i \in I_m$, $|A_i \cap S| \geq \varphi(k_i)$.

By Lemma 3.4, we have the following result.

Lemma 4.11 $\sum_{i=1}^{m} \varphi(k_i) = (|V| - |J_1| - 2|J_2| - |J_3| - 2)/2$.

5 Proof of Theorem 2.1

Let $G = (V, E)$, φ, $J_l(l = 0, 1, 2, 3)$ be as in Theorem 2.1. Let $A_i(i \in I_m)$, $U_l(l = 0, 1, 2)$, X, $X_l(l = 0, 1, 2)$ be as in Section 4. We may suppose that $S \in X_2$.

(A) Let $xy \in E$. Let $T = U_0 + \{x_{ij} \in V : i \in J_0 \cup J_3, j \equiv 0, 1(mod4)\} + \{x_{ij} \in V : i \in J_1 \cup J_2, j \equiv 0, 3(mod4)\}$, then T is a TDS of G, then $\gamma_t(G) \leq |T| = \sum_{i=1}^{m} \varphi(k_i) + 2$. By Lemma 4.10, (A) is proved.

(B) Let $J_1 = I_m$, the result follows by (A) and Lemma 4.5.

(C) Let $J_0 \neq \emptyset$. Similar to the proof of (A), (C) is proved.

(D) Let $|J_3| \geq 2$, we may suppose that $1, 2 \in J_3$. Let T be as in (A) and $T_0 = T - A_2 + \{x_{2j} \in A_2 : j \equiv 0, 3(mod4)\}$, then T_0 is a TDS of G, then $\gamma_t(G) \leq |T_0| = \sum_{i=1}^{m} \varphi(k_i) + 2$. By Lemma 4.10, (D) is proved.

(E) $\Rightarrow$ By (A), (B), (C) and (D), the result follows.

$\Leftarrow$ By Lemma 3.2 and Lemma 4.4, $U_0 \subseteq S$. Suppose that $\gamma_t(G) = \sum_{i=1}^{m} \varphi(k_i) + 2$, then $|A_i \cap S| = \varphi(k_i)$ for each $i \in I_m$ by Lemma 4.10. Since $xy \notin E$, then $N(x) \cap S \neq \emptyset$ and $N(y) \cap S \neq \emptyset$, we may suppose that $x_{11} \in S$. By Lemma 4.6, Lemma 4.7 and Lemma 4.8, $x_{1k_1} \notin S$, we may suppose that $x_{2k_2} \in S$, then $1, 2 \in J_3$, a contradiction. (E) is proved.

(F) By Lemma 3.2 and Lemma 4.4, $U_0 \subseteq S$. By Lemma

4.7, $|A_i \cap S| = \varphi(k_i) + |A_i \cap S \cap U_1|$ for each $i \in I_m$. Note that $N(x) \cap S \neq \emptyset$ and $N(y) \cap S \neq \emptyset$, then $\gamma_t(G) = |S| = \sum_{i=1}^{m} \varphi(k_i) + 2 + |U_1 \cap S| \geq \sum_{i=1}^{m} \varphi(k_i) + 4$. Let $T = \{x_{ij} \in V : i \in I_m,\ j \equiv 0,\ 3(mod4)\} + \{x,\ y,\ x_{11},\ x_{1k_1}\}$, then T is a TDS of G with $|T| = \sum_{i=1}^{m} \varphi(k_i) + 4$, (F) is proved.

(G) By (E), $\gamma_t(G) \geq \sum_{i=1}^{m} \varphi(k_i) + 3$. Let T be defined as in (A), let $T_0 = T + \{x_{1k_1}\}$, then T_0 is a TDS of G with $|T_0| = \sum_{i=1}^{m} \varphi(k_i) + 3$. (G) is proved.

(H) By (E), $\gamma_t(G) \geq \sum_{i=1}^{m} \varphi(k_i) + 3$. We may suppose that $1 \in J_1$. Let T be defined as in (A), let $T_0 = T - A_1 + \{x_{1j} \in A_1 :\ j \equiv 1,\ 2(mod4)\}$, then T_0 is a TDS of G with $|T_0| = \sum_{i=1}^{m} \varphi(k_i) + 3$. (H) is proved. Theorem 2.1 is proved.

6 $m \times n$ ladder graph

Let $G = L_{mn} = (V,\ E)$ be a $m \times n$ ladder graph, where $V = \{x_{ij} :\ i \in I_m,\ j \in I_n\}$ and $E = \{x_{ij}x_{i+1,j} :\ i \in I_{m-1},\ j = 1,\ n\} \cup \{x_{ij}x_{i,j+1} :\ i \in I_m,\ j \in I_{n-1}\}$. Let $A_i = \{x_{ij} \in V :\ j \in I_n\}$ for $i \in I_m$. Let $B_j = \{x_{ij} \in V :\ i \in I_m\}$ for $j \in I_n$. Let $U_1 = B_1 \cup B_n$ and $U_2 = B_2 \cup B_{n-1}$. Let $X = \{S \subseteq V :\ S$ is a $\gamma_t(G)-$set$\}$, $X_1 = \{S_1 \in X :\ |S_1 \cap U_1| = \max\{|S \cap U_1| :\ S \in X\}\}$, $X_2 = \{S_2 \in X_1 :\ |S_2 \cap U_2| = \max\{|S_1 \cap U_2| :\ S_1 \in X_1\}\}$. We may suppose that $S \in X_2$.

Lemma 6.1 Let $n \geq 5$. For each $i \in I_m$, $|S \cap A_i| \geq \gamma_t(P_{n-2})$. Moreover, $|S \cap A_i| \geq \gamma_t(P_n)$ if $x_{i1},\ x_{in} \in S$, $|S \cap A_i| \geq \gamma_t(P_{n-3}) + 1$ if $|A_i \cap U_1 \cap S| = 1$.

Proof If $x_{i1},\ x_{in} \notin S$, since $S \cap A_i$ is a TDS of $G[A_i] - U_1$, the result follows. If $x_{i1},\ x_{in} \in S$, then $|S \cap A_i| \geq \gamma_t(P_n) \geq \gamma_t(P_{n-2})$ by Lemma 3.5 (C). By Lemma 3.4, $0 \leq \varphi(k+1) - \varphi(k) \leq 1$ for each $k \in N$. If $|A_i \cap U_1 \cap S| = 1$, then $|S \cap A_i| \geq \gamma_t(P_{n-3}) + 1 \geq \gamma_t(P_{n-2})$ by Lemma 3.5 (F). Lemma 6.1 is proved.

Lemma 6.2 $\gamma_t(L_{mn}) = mn/2$ if $n \equiv 0(mod4)$.

Proof The case for $n = 4$ is trivial, we may suppose that $n \geq 8$. By Lemma 6.1, $\gamma_t(L_{mn}) \geq mn/2$. Let $T = \{x_{ij} \in V : i \in I_m, j \equiv 0, 1(mod4)\}$, then T is a TDS of G with $|T| = mn/2$. Lemma 6.2 is proved.

Lemma 6.3 $|S \cap A_1| + |S \cap A_2| \geq \gamma_t(P_n) + \gamma_t(P_{n-2})$ if $n \geq 5$.

Proof If $x_{21}, x_{2n} \notin S$, since $S \cap A_1$ is a TDS of $G[A_1]$ and $S \cap A_2$ is a TDS of $G[A_2] - U_1$, the result follows. If $|A_2 \cap U_1 \cap S| = 1$, then $|S \cap A_1| + |S \cap A_2| \geq \gamma_t(P_{n-1}) + \gamma_t(P_{n-3}) + 1 = \gamma_t(P_n) + \gamma_t(P_{n-2})$ by Lemma 3.4 and Lemma 3.5 (F). If $x_{21}, x_{2n} \in S$, then $|S \cap A_1| + |S \cap A_2| \geq \gamma_t(P_{n+2}) + \gamma_t(P_n) - 2 = \gamma_t(P_n) + \gamma_t(P_{n-2})$ by Lemma 3.5 (C). Lemma 6.3 is proved.

Lemma 6.4 $\gamma_t(L_{mn}) = m(n-1)/2 + 2$ if $n \equiv 3(mod4)$ and $m \geq 4$. Moreover, $\gamma_t(L_{3n}) = (3n-1)/2$ if $n \equiv 3(mod4)$.

Proof Since the result follows for the case $m = 3$ by Theorem 2.1 (D), we may suppose that $m \geq 4$.

Claim 1 $\gamma_t(L_{mn}) \geq m(n-1)/2 + 2$ if $n \geq 7$.

By Lemma 6.1, $|S \cap A_i| \geq \gamma_t(P_{n-2}) = (n-1)/2$ for each $i \in I_m$. By Lemma 6.3, $|S \cap A_1| + |S \cap A_2| \geq \gamma_t(P_n) + \gamma_t(P_{n-2}) = n$. By symmetry, $|S \cap A_{m-1}| + |S \cap A_m| \geq n$. Claim 1 is proved.

Claim 2 $\gamma_t(L_{m3}) \geq m + 2$.

For each $i \in I_m$, note that $N[x_{i2}] = A_i$, then $|A_i \cap S| \geq 1$. Suppose that $|S \cap A_1| = |S \cap A_2| = 1$, note that S contains no isolated vertices, we may suppose that $x_{11}, x_{21} \in S$, then $N[x_{13}] \cap S = \emptyset$, a contradiction. Therefore, $|S \cap A_1| + |S \cap A_2| \geq 3$. By symmetry, $|S \cap A_{m-1}| + |S \cap A_m| \geq 3$. Claim 2 is proved.

Now, let $T = \{x_{ij} \in V : i \equiv 1, 2(mod4), j \equiv 0, 1(mod4)\} + \{x_{ij} \in V : i \equiv 0, 3(mod4), j \equiv 0, 3(mod4)\} + \{x_{2n}, x_{m-1,1}, x_{m-1,n}\}$, then T is a TDS of G with $|T| = m(n-1)/2 + 2$. Lemma 6.4 is proved.

The cases for $n \equiv 0, 3(mod4)$ is simple. In order to consider the cases for $n \equiv 1,\ 2(mod4)$, some notations is added. Let $K_l = \{i \in I_m :\ |A_i \cap U_1 \cap S| = l\}$ and $J_l = \{i \in I_m :\ |A_i \cap S| = \gamma_t(P_{n-2}) + l\}$ for $l = 0,\ 1,\ 2$. Let $\phi(T) = \sum_{i=1}^{m} i^2 |A_i \cap T|$ for each $T \in X$, we may suppose that $\phi(S) = \max\{\phi(T) :\ T \in X_2\}$.

Lemma 6.5 Let $n \equiv 1(mod4)$, then we have the following result.

(A) $J_2 = \emptyset$.

(B) $J_0 \subseteq K_0$. Moreover, $|A_i \cap U_2 \cap S| \leq 1$ for each $i \in J_0$.

(C) $2,\ m-1 \notin J_0$.

(D) $\{i,\ i+2\} - J_0 \neq \emptyset$ for each $i \in I_{m-2}$.

Proof (A) Suppose that $i_0 \in J_2$, then $|A_{i_0} \cap S| = (n+3)/2$. We may suppose that $i_0 \neq 1$. Let $T = S - A_{i_0} + \{x_{i_0 j} \in A_{i_0} :\ j \equiv 1,\ 2(mod4)\} + \{x_{i_0-1,n}\}$. then $T \in X$ and $|T \cap U_1| > |S \cap U_1|$, and then $S \notin X_1$, a contradiction. (A) is proved.

(B) By Lemma 6.1, $J_0 \subseteq K_0$. Suppose that $A_i \cap U_2 \subseteq S$, then $A_i \cap S$ is a TDS of $G[A_i]$, then $|A_i \cap S| \geq \gamma_t(P_n) = (n+1)/2$, and then $i \notin J_0$, (B) is proved.

(C) Otherwise, we may suppose that $2 \in J_0$, then $2 \in K_0$, and then $x_{21},\ x_{2n} \notin S$.

Claim 1 $|A_1 \cap S| + |A_2 \cap S| \leq n$.

Otherwise, let $T = S - A_1 - A_2 + \{x_{ij} \in V :\ i = 1,\ 2,\ j \equiv 1,\ 2(mod4)\}$, then T is a TDS of G with $|A_1 \cap T| + |A_2 \cap T| = n+1 \leq |A_1 \cap S| + |A_2 \cap S|$, and then $T \in X$ and $|T \cap U_1| > |S \cap U_1|$, a contradiction. Claim 1 is proved.

Claim 2 $|A_1 \cap S| + |A_2 \cap S| = n$.

Since $|A_1 \cap S| + |A_2 \cap S| \geq \gamma_t(P_n) + \gamma_t(P_{n-2}) = n$ by Lemma 6.3, by Claim 1, Claim 2 is proved.

Claim 3 $(A_1 \cup A_2) \cap S$ is not a TDS of $G[A_1 \cup A_2]$. Moreover, $\{x_{31},\ x_{3n}\} \cap S \neq \emptyset$.

Note that $\gamma_t(P_{2n}) = n + 1$, by Claim 2, Claim 3 is proved.

By Claim 3, we may suppose that $x_{3n} \in S$. Let $T = S - A_1 - A_2 + \{x_{1j} \in A_1 : j \equiv 2, 3(mod4)\} + \{x_{2j} \in A_2 : j \equiv 1, 2(mod4)\}$. By Claim 2, $T \in X_1$ and $|S \cap U_1 \cap A_1| = 2$, and then $x_{1j} \in S$ for $j = 1, 2, n-1, n$. But now, $(A_1 \cup A_2) \cap S$ is a TDS of $G[A_1 \cup A_2]$, a contradiction by Claim 3. (C) is proved.

(D) Otherwise, suppose that $i, i+2 \in J_0$ for some $i \in I_{m-2}$, then $A_{i+1} \cap S$ is a TDS of $G[A_{i+1}]$, then $|A_{i+1} \cap S| \geq \gamma_t(P_n) = (n+1)/2$, then $i+1 \in J_1$. Since $|A_i \cap S| + |A_{i+1} \cap S| = n < \gamma_t(P_{2n})$, then $(A_i \cup A_{i+1}) \cap S$ is not a TDS of $G[A_i \cup A_{i+1}]$. We may suppose that $N[x_{i1}] \cap S = \{x_{i-1,1}\}$. There are two cases.

Case 1 $i+1 \in J_1 - K_0$.

In this case, $x_{i+1,j} \in S$ for $j = 2, 3, n-1, n$. Note that $A_{i+1} \cap S$ is a TDS of $G[A_{i+1} \cup \{x_{in}\}]$, then $|A_{i+1} \cap S| \geq \gamma_t(P_{n+1}) = (n+3)/2$, a contradiction.

Case 2 $i+1 \in J_1 \cap K_0$.

Let $T = S - A_{i+1} + \{x_{i+1,j} \in A_{i+1} : j \equiv 0, 3(mod4)\} + \{x_{i1}\}$, then $T \in X$ and $|T \cap U_1| > |S \cap U_1|$, a contradiction. Lemma 6.5 is proved.

Lemma 6.6 $K_0 = J_0$, $K_2 = J_1$, $K_1 = \emptyset$ if $n \equiv 1(mod4)$.

Proof Claim 1 $i_0 - 1 \in K_0$ if $i_0 \in K_0 - J_0 - \{1\}$.

Otherwise, we may suppose that $x_{i_0-1,1} \in S$. Let $T = S - A_{i_0} + \{x_{i_0j} \in A_{i_0} : j \equiv 0, 1(mod4)\}$. Since $i_0 \notin J_0$, then $|A_{i_0} \cap S| \geq \gamma_t(P_{n-2}) + 1 = (n+1)/2$, then $T \in X$ and $|T \cap U_1| > |S \cap U_1|$, and then $S \notin X_1$, a contradiction. Claim 1 is proved.

By symmetry, we have the following result.

Claim 2 $i_0 + 1 \in K_0$ if $i_0 \in K_0 - J_0 - \{m\}$.

Claim 3 $K_0 = J_0$.

By Lemma 6.5 (B), $J_0 \subseteq K_0$. Suppose that $i_0 \in K_0 - J_0$. Then by Claim 1 and Claim 2, we have $N[x_{i_0,1}] \cap S = \{x_{i_0,2}\}$ and $N[x_{i_0,n}] \cap S = \{x_{i_0,n-1}\}$. We may suppose that $i_0 \neq 1$. Let

$T = S - A_{i_0} + \{x_{i_0 j} \in A_{i_0} : j \equiv 0, 3(mod 4)\} + \{x_{i_0-1,1}\}$. Note that $N[x_{i_0-1,1}] \cap S \neq \emptyset$, then $x_{i_0-1,1}$ is not an isolated vertex in T, and then T is a TDS of G with $|T| \leq |S|$. Now, $T \in X$ and $|T \cap U_1| > |S \cap U_1|$, then $S \notin X_1$, a contradiction. Claim 3 is proved.

Claim 4 $K_2 = J_1$ and $K_1 = \emptyset$.

Since $J_2 = \emptyset$, by Claim 3, $J_1 = K_1 \cup K_2$. Suppose that $K_1 \neq \emptyset$, then $|A_i \cap S| = (n+1)/2$ for each $i \in K_1$. Let $T = S - (\cup\{A_i : i \in K_1\}) + \{x_{ij} : i \in K_1, j \equiv 1, 2(mod 4)\}$, note that $K_0 = J_0$, then T contains no isolated vertices by Lemma 6.5, and then T is a TDS of G. Since $T \in X$ and $|T \cap U_1| > |S \cap U_1|$, then $S \notin X_1$, a contradiction. Therefore, $K_1 = \emptyset$. Lemma 6.6 is proved.

Lemma 6.7 $\gamma_t(L_{mn}) = m(n-1)/2 + \gamma_t(P_m)$ if $n \equiv 1(mod 4)$.

Proof Let $L_l = \{i \in I_m : i \equiv l(mod 2)\}$ for $l = 0, 1$, then $|L_0| = \lfloor m/2 \rfloor$ and $|L_1| = \lceil m/2 \rceil$, and then $\gamma_t(P_m) = \lceil |L_0|/2 \rceil + \lceil |L_1|/2 \rceil$ by Lemma 3.1. By Lemma 6.5 (D), we have the following result.

Claim 1 $|J_0 \cap L_l| \leq \lceil |L_l|/2 \rceil$ for $l = 0, 1$.

Claim 2 $|J_0| \leq m - \gamma_t(P_m)$ if $m \equiv 0(mod 4)$.

$|J_0| = |J_0 \cap L_0| + |J_0 \cap L_1| \leq \lceil |L_0|/2 \rceil + \lceil |L_1|/2 \rceil = \gamma_t(P_m) = m/2 = m - \gamma_t(P_m)$, Claim 2 is proved.

Claim 3 $|J_0| \leq m - \gamma_t(P_m)$ if $m \equiv 1, 3(mod 4)$.

By Lemma 6.5 (C), 2, $m-1 \notin J_0$, then $|J_0 \cap L_0| \leq \lceil (|L_0| - 2)/2 \rceil$. Since $|J_0 \cap L_1| \leq \lceil |L_1|/2 \rceil$, then $|J_0| = |J_0 \cap L_0| + |J_0 \cap L_1| \leq \lceil |L_0|/2 \rceil + \lceil |L_1|/2 \rceil - 1 = \gamma_t(P_m) - 1 = (m-1)/2 = m - \gamma_t(P_m)$. Claim 3 is proved.

Claim 4 $|J_0| \leq m - \gamma_t(P_m)$ if $m \equiv 2(mod 4)$.

By Lemma 6.5 (C), 2, $m-1 \notin J_0$, then $|J_0 \cap L_l| \leq \lceil (|L_l| - 1)/2 \rceil = \lceil (m/2 - 1)/2 \rceil = (m-2)/4$ for $l = 0, 1$, then $|J_0| =$

$|J_0 \cap L_0| + |J_0 \cap L_1| \leq m/2 - 1 = m - \gamma_t(P_m)$. Claim 4 is proved.

Claim 5 $|J_0| \leq m - \gamma_t(P_m)$ and $|J_1| \geq \gamma_t(P_m)$.

Note that $J_2 = \emptyset$ by Lemma 6.5 (A), Claim 5 is proved by the above claims.

Claim 6 $\gamma_t(L_{mn}) \geq m(n-1)/2 + \gamma_t(P_m)$.

Since $J_2 = \emptyset$, then $I_m = J_0 \cup J_1$. By Claim 5, $\gamma_t(L_{mn}) = m(n-1)/2 + |J_1| \geq m(n-1)/2 + \gamma_t(P_m)$. Claim 6 is proved.

Let $T = \{x_{ij} \in V : j \equiv 2, 3(mod4)\} + \{x_{in} : i \equiv 2, 3(mod4)\} + \{x_{m-1,n}\}$, then T is a TDS of G with $|T| = m(n-1)/2 + \gamma_t(P_m)$. Lemma 6.7 is proved.

Lemma 6.8 Let $n \equiv 2(mod4)$, then we have the following result.

(A) $J_0 \subseteq K_0$. Moreover, $A_i \cap S = \{x_{ij} \in A_i : j \equiv 0, 3(mod4)\}$ for each $i \in J_0$.

(B) $J_2 = K_2$. Moreover, $A_i \cap S = \{x_{ij} \in A_i : j \equiv 1, 2(mod4)\}$ for each $i \in J_2$.

(C) $1 \in J_0$ and $2 \in J_2$.

(D) $m \in J_2$ if $m - 1 \in J_0$.

(E) For each $i \in I_{m-2}$, we have $i + 1 \in J_2$ if $i,\ i + 2 \in J_0$.

(F) $|A_i \cap U_2 \cap S| \leq 1$ for each $i \in J_1$.

Proof (A) Let $i \in J_0$, then $|A_i \cap S| = \gamma_t(P_{n-2}) = n/2 - 1$. Note that by Lemma 3.5 (A), $\sum_{u \in A_i - U_1} rtd(u,\ S) = 2|A_i \cap S - U_1| - |A_i \cap U_2 \cap S| + |A_i \cap U_1 \cap S| - (n-2) = 2|A_i \cap S| - |A_i \cap U_2 \cap S| - |A_i \cap U_1 \cap S| - (n-2) \leq 2|A_i \cap S| - (n-2) = 0$, then we have $|A_i \cap U_1 \cap S| = |A_i \cap U_2 \cap S| = 0$ and $rtd(u,\ S) = 0$ for each $u \in A_i - U_1$. Note that each component of $G[A_i] - S(G[A_i \cap S])$ is K_2, (A) is proved.

(B) By Lemma 3.5 (C), $K_2 \subseteq J_2$. Let $i \in J_2$, then $|A_i \cap S| = \gamma_t(P_{n-2}) + 2 = n/2 + 1$. Let $T = S - A_i + \{x_{ij} \in A_i : j \equiv 1,\ 2(mod4)\}$. Note that $S,\ T \in X_2$, then $x_{ij} \in S$ for $j =$

1, 2, $n-1$, n. By Lemma 3.5 (A), $\sum_{u\in A_i-U_1} rtd(u, S) = 2|A_i\cap S|-|A_i\cap U_2\cap S|-|A_i\cap U_1\cap S|-(n-2) = 2|A_i\cap S|-4-(n-2) = 0$, then we have $rtd(u, S) = 0$ for each $u \in A_i - U_1$. Note that each component of $G[A_i] - S(G[A_i \cap S])$ is K_2, (B) is proved.

(C) Let $T = S - A_1 - A_2 + \{x_{1j} : j \equiv 0, 3(mod4)\} + \{x_{2j} : j \equiv 1, 2(mod4)\}$, then we have the following result.

Claim 1 T is a TDS of G. Moreover, $|S| \leq |T|$.

Claim 2 $1 \notin J_2$ and $2 \notin J_0$.

Suppose that $1 \in J_2$, then $2 \in J_0$ and $|T| = |S|$ by Claim 1, then $T \in X_2$ and $\phi(T) > \phi(S)$, a contradiction. Therefore, $1 \notin J_2$. Suppose that $2 \in J_0$, then x_{21}, $x_{2n} \notin S$ by (A), then $|A_1 \cap S| \geq \gamma_t(P_n) = n/2 + 1$ by Lemma 3.5 (G), then $1 \in J_2$, a contradiction. Claim 2 is proved.

Claim 3 $1 \in J_0$.

Suppose that $1 \in J_1$. Note that $2 \notin J_0$ and $|S| \leq |T|$, then $2 \in J_1$, then $T \in X_2$ and $\phi(T) > \phi(S)$, a contradiction. Claim 3 is proved.

By Claim 3 and (A), note that $N[x_{1j}] \cap S = \{x_{2j}\}$ for $j = 1, n$, then $2 \in K_2$. By (B), (C) is proved.

(D) If $m - 1 \in J_0$, then $x_{m-1,1}$, $x_{m-1,n} \notin S$ by (A), then $|A_m \cap S| \geq \gamma_t(P_n) = n/2 + 1$ by Lemma 3.5 (G), then $m \in J_2$. (D) is proved.

(E) Let $i \in I_{m-2}$ and i, $i + 2 \in J_0$, then i, $i + 2 \in K_0$ by (A), then $A_{i+1} \cap S$ is a TDS of $G[A_{i+1}]$, and then $|A_{i+1} \cap S| \geq \gamma_t(P_n) = n/2 + 1$, (E) is proved.

(F) Otherwise, $A_i \cap S$ is a TDS of $G[A_i]$, then $|A_i \cap S| \geq \gamma_t(P_n) = n/2+1$, and then $i \notin J_1$, a contradiction. (F) is proved. Lemma 6.8 is proved.

Lemma 6.9 $i + 1 \in J_0$ if $n \equiv 2(mod4)$ and $i \in J_2 \cap I_{m-1}$.

Proof **Claim 1** $m \in J_0$ if $m - 1 \in J_2$.

Otherwise, let $T_1 = S - A_m + \{x_{mj} \in A_m : j \equiv 0,\ 3(mod4)\}$, then T_1 is a TDS of G with $|T_1| < |S|$, a contradiction. Claim 1 is proved.

Now, Suppose that $i \in J_2$ and $i+1 \notin J_0$. By Claim 1, $i \neq m-1$. Let $T_2 = S - A_{i+1} - A_{i+2} + \{x_{i+1,j} \in A_{i+1} : j \equiv 0,\ 3(mod4)\} + \{x_{i+2,j} \in A_{i+2} : j \equiv 1,\ 2(mod4)\}$.

Claim 2 $|(A_{i+1} \cup A_{i+2}) \cap S| = n-1,\ n$.

Note that T_2 is a TDS of G, then $|S| \leq |T_2|$, then $|(A_{i+1} \cup A_{i+2}) \cap S| \leq n$. Since $i+1 \notin J_0$, then $|A_{i+1} \cap S| \geq n/2$. Claim 2 is proved.

Claim 3 $|(A_{i+1} \cup A_{i+2}) \cap S| = n-1$.

Suppose that $|(A_{i+1} \cup A_{i+2}) \cap S| = n$, then $|S| = |T_2|$, then $T_2 \in X$. Since $i+1 \notin J_0$, we consider two cases.

Case 1 $i+1 \in J_2$ and $i+2 \in J_0$.

In this case, By Lemma 6.8 (A) and (B), $|T_2 \cap U_k| = |S \cap U_k|$ for $k = 1,\ 2$, then $T_2 \in X_2$ and $\phi(T_2) > \phi(S)$, a contradiction.

Case 2 $i+1,\ i+2 \in J_1$.

By Lemma 6.8 (B), $|(A_{i+1} \cup A_{i+2}) \cap U_1 \cap S| \leq 2 = |(A_{i+1} \cup A_{i+2}) \cap U_1 \cap T_2|$, then $T_2 \in X_1$. Since $|A_k \cap U_2 \cap S| \leq 1$ for $k = i+1,\ i+2$ by Lemma 6.8 (F), then $|S \cap U_2| \leq |T_2 \cap U_2|$, then $T_2 \in X_2$ and $\phi(T_2) > \phi(S)$, a contradiction. Claim 3 is proved.

Claim 4 $i+1 \in J_1$ and $i+2 \in J_0$.

Since $i+1 \notin J_0$, $|A_{i+1} \cap S| \geq n/2$. By Claim 3, Claim 4 is proved.

By Lemma 6.8 (B), $i+1 \notin K_2$, we may suppose that $x_{i+1,1} \notin S$. By Lemma 6.8 (A), $x_{i+2,j} \notin S$ for $j = 1,\ 2,\ n-1,\ n$, then $x_{i+3,1} \in S$. Now, let $T_3 = S - A_{i+1} - A_{i+2} + \{x_{i+1,j} \in A_{i+1} : j \equiv 0,\ 3(mod4)\} + \{x_{i+2,j} \in A_{i+2} : j \equiv 0,\ 1(mod4)\}$. Note that $T_3 \in X_2$ and $\phi(T_3) > \phi(S)$, a contradiction. Lemma 6.9 is proved.

Lemma 6.10 $i+1,\ i+2 \in J_0$ and $i+3 \in J_2$ if $n \equiv 2(mod4)$ and $i \in J_2 \cap I_{m-3}$.

Proof Let $i \in J_2 \cap I_{m-3}$. By Lemma 6.9, $i+1 \in J_0$.

Claim 1 $i+2 \in J_0$.

Let $T = S - A_{i+2} - A_{i+3} + \{x_{i+2,j} \in A_{i+2} :\ j \equiv 0,\ 3(mod4)\} + \{x_{i+3,j} \in A_{i+3} :\ j \equiv 1,\ 2(mod4)\}$. Then T is a TDS of G, then $|S| \leq |T|$. Suppose that $i+2 \in J_2$, then $i+1 \in J_0$, $T \in X_2$ and $\phi(T) > \phi(S)$, a contradiction. Suppose that $i+2 \in J_1$, then $i+3 \in J_1$ by Lemma 6.8 (E). Similar to the proof in Case 2 of Lemma 6.9, $T \in X_2$ and $\phi(T) > \phi(S)$, a contradiction. Claim 1 is proved.

Since $i+1,\ i+2 \in J_0$, then $x_{kj} \notin S$ for $k = i+1,\ i+2$ and $j = 1,\ 2,\ n-1,\ n$, then $x_{i+3,j} \in S$ for $j = 1,\ n$. By Lemma 6.8 (B), $i+3 \in K_2 = J_2$. Lemma 6.10 is proved.

Lemma 6.11 Let $n \equiv 2(mod4)$, then we have the following results.

(A) $J_1 = \emptyset$.

(B) $J_0 = \{i \in I_m :\ i \equiv 0,\ 1(mod3)\}$ and $J_2 = \{i \in I_m :\ i \equiv 2(mod3)\}$ if $m \equiv 0,\ 2(mod3)$.

(C) $J_0 = \{i \in I_m :\ i \equiv 0,\ 1(mod3)\} - \{m\}$ and $J_2 = \{i \in I_m :\ i \equiv 2(mod3)\} + \{m\}$ if $m \equiv 1(mod3)$.

Proof By Lemma 6.8 (C), $1 \in J_0$ and $2 \in J_2$. By Lemma 6.8 (D), $m \in J_2$ if $m-1 \in J_0$. The result follows by Lemma 6.9 and Lemma 6.10. Lemma 6.11 is proved.

Lemma 6.12 Let $G = L_{mn} = (V,\ E)$ and $n \equiv 2(mod4)$, then we have the following results.

(A) $\gamma_t(G) = mn/2 - m/3$ if $m \equiv 0(mod3)$.

(B) $\gamma_t(G) = mn/2 - m/3 + 4/3$ if $m \equiv 1(mod3)$.

(C) $\gamma_t(G) = mn/2 - m/3 + 2/3$ if $m \equiv 2(mod3)$.

Proof By Lemma 6.11, $\gamma_t(G) = m(n/2-1) + 2|J_2|$.

(A) Since $|J_2| = m/3$, then $\gamma_t(G) = mn/2 - m/3$.

(B) Since $|J_2| = (m+2)/3$, then $\gamma_t(G) = mn/2 - m/3 + 4/3$.

(C) Since $|J_2| = (m+1)/3$, then $\gamma_t(G) = mn/2 - m/3 + 2/3$.
Lemma 6.12 is proved.

7 Proof of Theorem 2.2

Proof Let $G = L_{mn} = (V,\ E)$ be a $m \times n$ ladder graph. By Lemma 6.2, Lemma 6.4, Lemma 6.7 and Lemma 6.12, Theorem 2.2 is proved.

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