Duality in $lp\gamma$-spaces

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Abstract. Codes in $lp\gamma$-spaces introduced by the author in [3] are a natural generalization of one-dimensional codes in RT-spaces [6] to block coding and has applications in different area of combinatorial/discrete mathematics, e.g. in the theory of uniform distribution, experimental designs, cryptography etc. In this paper, we introduce various types of weight enumerators in $lp\gamma$-codes viz. exact weight enumerator, complete weight enumerator, block weight enumerator and γ-weight enumerator. We obtain the MacWilliams duality relation for the exact and complete weight enumerators of an $lp\gamma$-code.

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1. Introduction

K. Feng and L.Xu and F.J.Hickernesll [2] initiated the concept of linear partition block code which is a natural generalization of the Hamming-metric codes. Also, we know that the Rosenbloom-Tsfasman metric (or RT-metric or ρ-metric) is stronger than the Hamming metric [1, 6, 7]. Motivated by the idea to have linear partition block code endowed with a metric generalizing the RT-metric, the author formulated the concept of linear partition γ-codes (or $lp\gamma$-codes) in [3] and obtained basic results for these codes including various upper and lower bounds on their parameters for the detection and correction of random block errors. In this paper, we introduce various types of weight enumerators in $lp\gamma$-codes viz. exact weight enumerator, complete weight enumerator, block weight enumerator and γ-weight enumerator. We obtain the MacWilliams duality relation [4, 5] for the exact and complete weight enumerators of an $lp\gamma$-code.

2. Definitions and notations

Let q, n be positive integers with $q = p^m$, a power of a prime number p. Let $\mathbf{F}_q$ be the finite field having q elements. A partition P of the positive integer n is defined as:

$$P : n = n_1 + n_2 + \cdots + n_s, \quad 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s, s \geq 1.$$

The partition P is denoted s

$$P : n = [n_1][n_2]\cdots[n_s].$$

In the case, when

$$P : n = \underbrace{[n_1]\cdots[n_1]}_{r_1\text{- copies}}\underbrace{[n_2]\cdots[n_2]}_{r_2\text{- copies}}\cdots\underbrace{[n_t]\cdots[n_t]}_{r_t\text{- copies}},$$

we write

$$P : n = [n_1]^{r_1}[n_2]^{r_2}\cdots[n_t]^{r_t} \qquad \text{where } n_1 < n_2 < \cdots < n_t.$$

Further, given a partition $P : n = [n_1][n_2]\cdots[n_s]$ of a positive integer n, the linear space $\mathbf{F}_q^n$ over $\mathbf{F}_q$ can be viewed as the direct sum

$$\mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s},$$

or equivalently

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s,$$

where $V = \mathbf{F}_q^n$ and $V_i = \mathbf{F}_q^{n_i}$ for all $i \leq i \leq s$.

Consequently, each vector $v \in \mathbf{F}_q^n$ can be uniquely written as a $v = (v_1, v_2, \cdots, v_s)$ where $v_i \in V_i = \mathbf{F}_q{}^{n_i}$ for all $1 \leq i \leq s$. Here v_i is called the i^{th} *block of block size* n_i of the vector v.

Definition 2.1. Let $v = (v_1, v_2, \cdots, v_s) \in \mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ be an s-block vector of length n over $\mathbf{F}_q$ corresponding to the partition $P : n = [n_1][n_2]\cdots[n_s]$ of n. We define the γ-weight of the block vector v as

$$w_\gamma^{(P)}(v) = \max_{i=1}^{s}\{i | v_i \neq 0\}.$$

The γ-distance $d_\gamma^{(P)}(u, v)$ between two s-block vectors of length n viz. $u = (u_1, u_2, \cdots, u_s)$ and $v = (v_1, v_2, \cdots, v_s), u_i, v_i \in \mathbf{F}_q^{n_i} (1 \leq i \leq s)$ corresponding to the partition P is defined as

$$d_\gamma^{(P)}(u, v) = w_\gamma^{(P)}(u - v) = \max_{i=1}^{s}\{i | u_i \neq v_i\}$$

Then $d_\gamma^{(P)}(u, v)$ is a metric on $\mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$.

Note. Once the partition P is specified, we will denote the γ-weight $w_\gamma^{(P)}$ by $w_\gamma(v)$ and γ-distance $d_\gamma^{(P)}$ by d_γ respectively.

Definition 2.2. A linear partition γ-code (or $lp\gamma$-code) V of length n corresponding to the partition $P : n = [n_1][n_2]\cdots[n_s], n_1 \leq n_2 \leq \cdots \leq n_s$ is a $\mathbf{F}_q$-linear subspace of $\mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ equipped with the γ-metric and is denoted as $[n, k, d_\gamma; P]$ code where $k = \dim_{\mathbf{F}_q}(V)$ and $d_\gamma = d\gamma(V) =$ minimum γ-distance of the code V.

Remark 2.1.

1. For $P : n = [1]^n$, the γ-metric (or γ-weight) reduces to the ρ-metric (or ρ-weight) respectively [6].
2. For a partition $P : n = [n_1][n_2]\cdots[n_s]$ of the positive integer n, the γ-distance (or γ-weight) is always greater than or equal to the π-distance (or π-weight) respectively, i.e.

$$\gamma\text{-distance}(\gamma\text{-weight}) \geq \pi\text{-distance}(\pi\text{-weight}).$$

3. Weight enumerators and MacWilliams duality in $lp\gamma$-codes

In this section, we define various types of weight enumerators in $lp\gamma$-codes viz. exact weight enumerator, complete weight enumerator, block weight enumerator and γ-weight enumerator and obtained the exact and complete weight distribution of the dual code of an $lp\gamma$-code V by way of obtaining the MacWiiliams type identity. We begin with few elementary definitions.

Definition 3.1 [5]. Let $a \in \mathbf{F}_q$. The Hamming weight of a is defined as

$$H(a) = \begin{cases} 1 & \text{if} \quad a \neq 0 \\ 0 & \text{if} \quad a = 0. \end{cases}$$

Definition 3.2. Let $P : n = [n_1][n_2]\cdots[n_s]$ be a partition of a positive integer n. Let $v = (v_1, v_2, \cdots, v_s) \in \mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ where $v_i \in \mathbf{F}_q^{n_i}$ for all $1 \leq i \leq s$. The *Hamming block weight* of the block $v_i (1 \leq i \leq s)$ is defined as

$$H_b(v_i) = \begin{cases} 1 & \text{if} \quad v_i \neq 0 \\ 0 & \text{if} \quad v_i = 0. \end{cases}$$

Definition 3.3. Let V be an $[n, k, d; P]$ $lp\gamma$-code over $\mathbf{F}_q$ corresponding to the partition $P : n = [n_1][n_2]\cdots[n_s]$ of n. The *γ-weight spectrum* of the code V is the set $\{A_0, A_1, \cdots, A_s\}$ where for all $0 \leq r \leq s$, A_r is given by

$$A_r = |\{u \in V \mid w_\gamma(u) = r\}|.$$

The γ-*weight enumerator* of the code V is defined as

$$W_V^\gamma(t) = \sum_{r=0}^{s} A_r t^r = \sum_{u \in V} t^{w_\gamma(u)}.$$

Definition 3.4. Let $u = (u_1, u_2, \cdots, u_s)$ and $v = (v_1, v_2, \cdots, v_s)$ be two elements of $\mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ where for all $1 \leq i \leq s, u_i = (u_1^{(i)}, u_2^{(i)}, \cdots, u_{n_i}^{(i)})$ and $v_i = (v_1^{(i)}, v_2^{(i)}, \cdots, v_{n_i}^{(i)})$.
The inner product of u and v is defined as

$$\begin{aligned} < u, v > &= u_1.v_1 + u_2.v_2 + \cdots u_s.v_s \\ &= \sum_{i=1}^{s} (u_1^{(i)} v_1^{(i)} + \cdots + u_{n_i}^{(i)} v_{n_i}^{(i)}). \end{aligned}$$

The dual code of an $lp\gamma$-code $V \subseteq \mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ is defined as

$$V^\perp = \{v \in \mathbf{F}_q^n | < u, v >= 0 \ \forall \ u \in \mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}\}.$$

Then $V^\perp \subseteq \mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ is also an $lp\gamma$-code.

Now we define the character of a finite Abelien group and canonical additive character of a finite field.

Definition 3.5 [5]. Let G be a finite Abelian group with respect to addition. Let U be the multiplicative group of complex numbers having absolute value 1 i.e.

$$U = \{z \in \mathbf{C} : \ |z| = 1\}.$$

A *character* χ *of* G is a group homomorphism from G into U i.e. $\chi : G \to U$ is a map satisfying $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$.

Definition 3.6 [5]. Let $\mathbf{F}_q$ be a finite field having $q = p^m$ elements and U be the group of complex numbers as in Definition 3.5. Consider the additive group $\mathbf{F}_q^+$ of the finite field $\mathbf{F}_q$.

(i) The character $\chi : \mathbf{F}_q^+ \to U$ given by

$$\chi(a) = 1 \text{ for all } a \in \mathbf{F}_q^+$$

is called the trivial additive character of $\mathbf{F}_q^+$.

(ii) The nontrivial canonical additive character χ of the finite field $\mathbf{F}_q$ is a group homomorphism

$$\chi : \mathbf{F}_q^+ = F_{p^m}^+ \longrightarrow U \text{ given by}$$

$$\chi(a) = cos2\pi\left(\frac{a + a^p + \cdots a^{p^{m-1}}}{p}\right) + isin2\pi\left(\frac{a + a^p + \cdots a^{p^{m-1}}}{p}\right)$$
for all $a \in \mathbf{F}_q^+$.

Observations

1. For q prime, the definition of nontrivial canonical additive character χ given in Definition 3.6 (ii) reduces to
$$\begin{aligned} \chi \quad &: \quad \mathbf{F}_q^+ \longrightarrow U \text{ is given by} \\ \chi(a) \quad &= \quad cos\frac{2\pi a}{q} + i\ sin\frac{2\pi a}{q} \text{ for all } a \in \mathbf{F}_q^+. \end{aligned}$$

2. Over F_2, we have
$$\chi(0) = 1, \quad \chi(1) = -1.$$

3. Over $F_3, \chi(0) = 1, \quad \chi(1) = \frac{-1}{2} + \frac{\sqrt{3}}{2}i, \chi(2) = \frac{-1}{2} - \frac{\sqrt{3}}{2}i.$

Now, we define different types of weight enumerators for an $lp\gamma$-code V.

Definition 3.7. Let V be an $[n, k, d; P]$ $lp\gamma$-code corresponding to the partition $P : n = [n_1][n_2]\cdots[n_s]$ of n. Let $u = (u_1, u_2, \cdots, u_s) \in V$ where $u_i = (u_1^{(i)}, u_2^{(i)}, \cdots, u_{n_i}^{(i)}) \in \mathbf{F}_q^{n_i}$ for all $1 \leq i \leq s$. The *exact weight enumerator* of the code V is a polynomial in $n_1 + n_2 + \cdots + n_s = n$ variables given by

$$\begin{aligned} & EW_V(t_{11}, \cdots, t_{1,n_1}, t_{21}, \cdots t_{2,n_2}, \cdots, t_{s1}, \cdots t_{s,n_s}) \\ = & \sum_{u \in v} (t_{11})^{u_{11}} \cdots (t_{1,n_1})^{u_{1,n_1}} (t_{21})^{u_{21}} \cdots (t_{2,n_2})^{u_{2,n_2}} \cdots (t_{s1})^{u_{s1}} \cdots (t_{s,n_s})^{u_{s,n_s}}. \end{aligned}$$

Definition 3.8. The *complete weight enumerator* of the $lp\gamma$-code V is given by

$$\begin{aligned} & CW_V(t_{11}, \cdots, t_{1,n_1}, t_{21}, \cdots, t_{2,n_2}, \cdots, \cdots, t_{s1}, \cdots t_{s,n_s}) \\ = & \sum_{u \in v} (t_{11})^{H(u_{11})} \cdots (t_{1,n_1})^{H(u_{1,n_1})} (t_{21})^{H(u_{21})} \cdots \\ & \cdots (t_{2,n_2})^{H(u_{2,n_2})} \cdots (t_{s1})^{H(u_{s1})} \cdots (t_{s,n_s})^{H(u_{s,n_s})}. \end{aligned}$$

Definition 3.9. The *block weight enumerator* of the $lp\gamma$-code V is a polynomial in s variables viz. $t_1, t_2, \cdots, t_s$ and is obtained from the complete weight enumerator of V on replacing

$$(t_{i,1})^{H(u_{i,1})} (t_{i,2})^{H(u_{i,2})} \cdots (t_{i,n_i})^{H(u_{i,n_i})}$$

by

$$(t_i)^{\max_{j=1}^{n_i}\{H(u_{i,j})\}} \quad \text{for all} 1 \leq i \leq s.$$

Thus

$$BW_V(t_1, t_2, \cdots, t_s) = \sum_{u \in V} (t_1)^{\max_{j=1}^{n_1}\{H(u_{1,j})\}} (t_2)^{\max_{j=1}^{n_2}\{H(u_{2,j})\}} \cdots (t_s)^{\max_{j=1}^{n_s} H(u_{s,j})\}}.$$

Definition 3.10. The *γ-weight enumerator* of the $lp\gamma$-code V is obtained from the block weight enumerator of V on replacing each monomial of the form $t_1^{a_1} t_2^{a_2} \cdots t_s^{a_s}$ in $BW_V(t_1, t_2, \cdots, t_s)$ by $t^{\max_{i=1}^{s}\{i | a_i \neq 0\}}$.

We now illustrate different weight enumerators with the help of following example.

Example 3.1. Let $n = 4, q = 2$. Let $P : 4 = [2][2]$ be a partition of $n = 4$. Let V_1 and V_2 be two $lp\gamma$ codes over $F_2^4 = F_2^2 \oplus F_2^2$ given by

$$V_1\{(00\vdots 00), (10\vdots 10)\}$$

and

$$V_2 = \{(00\vdots 00), (00\vdots 01)\}.$$

Since the codes V_1 and V_2 are over F_2, therefore their exact and complete weight enumerators are same and are given by

$$\begin{aligned} CW_{V_1}(t_{11}, t_{12}, t_{21}, t_{22}) &= EW_{V_1}(t_{11}, t_{12}, t_{21}, t_{22}) \\ &= t_{11}^0 t_{12}^0 t_{21}^0 t_{22}^0 + t_{11}^1 t_{12}^0 t_{21}^1 t_{22}^0 \\ &= 1 + t_{11} t_{21}, \\ CW_{V_2}(t_{11}, t_{12}, t_{21}, t_{22}) &= EW_{V_2}(t_{11}, t_{12}, t_{21}, t_{22}) \\ &= 1 + t_{22}. \end{aligned}$$

The block weight enumerator of V_1 and V_2 are given by

$$\begin{aligned} BW_{V_1}(t_1, t_2) &= t_1^0 t_2^0 + t_1^1 t_2^1 = 1 + t_1 t_2, \\ BW_{V_2}(t_1, t_2) &= t_1^0 t_2^0 + t_1^0 t_2^1 = 1 + t_2. \end{aligned}$$

The γ-weight enumerator of V_1 and V_2 are given by

$$\begin{aligned} W_{V_1}(t) &= 1 + t^2, \\ W_{V_2}(t) &= 1 + t^2. \end{aligned}$$

The dual codes of V_1 and V_2 are given by

$$\begin{aligned} V_1^{\perp} &= \{(pq\dot{:}rz) \in F_2^4 = F_2^2 \oplus F_2^2 \mid p + r = 0\} \\ &= \{(pq\dot{:}rz) \in F_2^4 = F_2^2 \oplus F_2^2 \mid \text{either } p = r = 1 \text{ or } p = r = 0\}, \\ &= \{(00\dot{:}00), (01\dot{:}00), (01\dot{:}01), (00\dot{:}01), (10\dot{:}10), \\ &\quad (11\dot{:}10), (11\dot{:}11), (10\dot{:}11)\}, \end{aligned}$$

and

$$\begin{aligned} V_2^{\perp} &= \{(pq\dot{:}rz) \in F_2^4 = F_2^2 \oplus F_2^2 \mid z = 0\} \\ &= \{(00\dot{:}00), (00\dot{:}10), (01\dot{:}00), (01\dot{:}10), (10\dot{:}00), \\ &\quad (10\dot{:}10), (11\dot{:}00), (11\dot{:}10)\}. \end{aligned}$$

The exact and complete weight enumerators of $V_1^{\perp}$ and $V_2^{\perp}$ are given by

$$\begin{aligned} EW_{V_1^{\perp}}(t_{11}, t_{12}, t_{21}, t_{22}) &= CW_{V_1^{\perp}}(t_{11}, t_{12}, t_{21}, t_{22}) \\ &= 1 + t_{12} + t_{12}t_{22} + t_{22} + t_{11}t_{21} + t_{11}t_{12}t_{21} + \\ &\quad + t_{11}t_{12}t_{21}t_{22} + t_{11}t_{21}t_{22}, \end{aligned}$$

$$\begin{aligned} EW_{V_2^{\perp}}(t_{11}, t_{12}, t_{21}, t_{22}) &= CW_{V_2^{\perp}}(t_{11}, t_{12}, t_{21}, t_{22}) \\ &= 1 + t_{21} + t_{12} + t_{12}t_{21} + t_{11} + t_{11}t_{21} + t_{11}t_{12} + \\ &\quad + t_{11}t_{12}t_{21}. \end{aligned}$$

The block weight enumerators of $V_1^{\perp}$ and $V_2^{\perp}$ are given by

$$\begin{aligned} BW_{V_1^{\perp}}(t_1, t_2) &= 1 + t_1 + t_1t_2 + t_2 + t_1t_2 + t_1t_2 + t_1t_2 + t_1t_2 \\ &= 1 + t_1 + t_2 + 5t_1t_2, \end{aligned}$$

$$\begin{aligned} BW_{V_2^{\perp}}(t_1, t_2) &= 1 + t_2 + t_1 + t_1t_2 + t_1 + t_1t_2 + t_1 + t_1t_2 \\ &= 1 + 3t_1 + t_2 + 3t_1t_2. \end{aligned}$$

The γ-weight enumerators of $V_1^{\perp}$ and $V_2^{\perp}$ are given by

$$\begin{aligned} W^{\gamma}_{V_1^{\perp}}(t) &= t^0 + t^1 + t^2 + 5t^2 = 1 + t + 6t^2, \\ W^{\gamma}_{V_2^{\perp}}(t) &= t^0 + 3t^1 + t^2 + 3t^2 = 1 + 3t + 4t^2. \end{aligned}$$

Remark 3.1. We observe from Example 3.1 that although the γ-weight enumerators of codes V_1 and V_2 are same but γ-weight enumerators of their duals $V_1^{\perp}$ and $V_2^{\perp}$ are different. Therefore, it is difficult to obtain the γ-weight enumerator of the dual of an $lp\gamma$ code V from the γ-weight enumerator of V. However, the same can be achieved for the exact and complete weight enumerators. Keeping this in view, we obtain the exact and complete weight enumerator of the dual code of an $lp\gamma$-code V from the exact and complete weight enumerator respectively of V in the form of MacWilliams duality relation. To prove the MacWilliams duality relation for the exact and weight enumerators in $lp\gamma$ codes, we require the following lemmas:

Lemma 3.1 [5]. *Let χ be the nontrivial canonical additive character of $\mathbf{F}_q$, then*

$$\sum_{\alpha \in \mathbf{F}_q} \chi(\alpha) = 0.$$

Lemma 3.2. *Let χ be the nontrivial cannonical additive character of $\mathbf{F}_q$. Let $V \subseteq \mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ be an $[n,k;P]$ $lp\gamma$ code corresponding to the partition $P: n = [n_1][n_2]\cdots[n_s]$ of n. Then*

$$\sum_{u \in v} \chi(< u, v >) = \begin{cases} 0 & \text{if} \quad v \notin V^{\perp} \\ |V| & \text{if} \quad v \in V^{\perp}. \end{cases}$$

Proof. If $v \in V^{\perp}$, then clearly $< u, v >=0$. This implies that

$$\sum_{u \in v} \chi(< u, v >) = \sum_{u \in v} \chi(0) = |V|.$$

If $v \notin V^{\perp}$, then we claim that in the summation $\sum_{u \in v} \chi(< u, v >)$, the inner product $< u, v >$ assumes every value of $\mathbf{F}_q$, the same number of times. For this, let $r_j \geq 0$ be the number of elements of V whose inner product with v is equal to j for all $0 \leq j \leq q-1$. To be more precise, let $u_1^j, u_2^j, \cdots, u_{r_j}^j$ be all the elements of V such that $< u_i^j, v >= j$ for all $i = 1$ to r_j and for all $j = 0$ to $q = 1$.

Choose j such that $1 \leq j \leq q-1$ and fix it. Then $u_1^j + u_1^0, u_1^j + u_2^0, \cdots, u_1^j + u_{r_0}^0$ are r_0 distinct of V such that

$$\begin{aligned} < u_1^j + u_k^0, v > &= < u_1^j, v > + < u_k^0, v > \\ &= J + 0 = j \text{ for all } k = 1 \text{ to } r_0. \end{aligned}$$

This implies that

$$r_0 \leq r_j. \tag{1}$$

Again, let

$$x = \sum_{\substack{l=1 \\ l \neq j}}^{q-1} u_1^l \in V.$$

Then $u_1^j + x, u_2^j + x, \cdots, u_{r_j}^j + x$ are distinct elements of V such that for every $1 \leq i \leq r_j$, we have

$$\begin{aligned} < u_i^j + x, v > &= < u_i^j, v > + < x, v > \\ &= j + < \sum_{\substack{l=1 \\ l \neq j}}^{q-1} u_1^l, v > \\ &= j + 1 + 2 + \cdots + (j-1) + (j+1) + \cdots + (q-1) \\ &= \text{sum of all the elements of the field } \mathbf{F}_q \\ &= 0. \end{aligned}$$

This implies that

$$r_j \leq r_0. \qquad (2)$$

(1) and (2) give $r_j = r_0$. But j $(1 \leq j \leq q-1)$ is arbitrary and hence the claim. Thus

$$\sum_{u \in V} \chi(u, v) = \text{multiple of } \sum_{\alpha \in \mathbf{F}_q} \chi(\alpha) = 0 \text{ (using Lemma 3.1)}$$

□

Lemma 3.3. *Let V be an $[n, k; P]$ lpγ code in over $\mathbf{F}_q$ corresponding to the partition $P : n = [n_1][n_2] \cdots [n_s]$. Let $f : \mathbf{F}_q^n = \bigoplus_{i=1}^{s} \mathbf{F}_q^{n_i} \to \mathbf{C}[t_{11}, \cdots, t_{1,n_1}, t_{21}, \cdots, t_{2,n_1} \cdots t_{s1}, \cdots, t_{s,n_s}]$ be a map where $\mathbf{C}[t_{11}, \cdots, t_{1,n_1}, t_{21}, \cdots, t_{2,n_1} \cdots t_{s1}, \cdots, t_{s,n_s}]$ is the polynomial ring in $n_1 + n_2 + \cdots + n_s = n$ commutating variables with coefficients from the complex field $\mathbf{C}$. Let χ be the nontrivial canonical additive character $\mathbf{F}_q$. Then*

$$\sum_{v \in V^\perp} f(v) = \frac{1}{|V|} \sum_{u \in V} \hat{f}(u),$$

where $\hat{f}$ is the Hadamard transform of f given by

$$\hat{f}(u) = \sum_{v \in \oplus_{i=1}^{s} \mathbf{F}_q^{n_i}} \chi(< u, v >) f(v).$$

Proof. Consider

$$\begin{aligned}\sum_{u\in V}\hat{f}(u) &= \sum_{u\in V}\sum_{v\in\bigoplus_{i=1}^{s}\mathbf{F}_q^{n_i}} \chi(<u,v)>)f(v)\\ &= \sum_{u\in V}\sum_{v\in V^\perp}\chi(<u,v>)f(v)+\sum_{u\in V}\sum_{v\notin V^\perp}\chi(<u,v>)f(v)\\ &= |V|\sum_{v\in V^\perp} f(v) \quad \text{(using Lemma 3.2).}\end{aligned}$$

This implies

$$\sum_{v\in V^\perp} f(v) = \frac{1}{|V|}\sum_{u\in V}\hat{f}(u).$$

□

The following theorem gives the duality relation for the exact weight enumerator in $lp\gamma$-spaces.

Theorem 3.1. *Let V be an $[n,k,;P]$ $lp\gamma$ code over $\mathbf{F}_q$ corresponding to the partition $P: n=[n_1][n_2]\cdots[n_s], n_1\leq n_2\leq\cdots\leq n_2$. Let $EW[t_{11},\cdots,t_{1,n_1}, t_{21},\cdots,t_{2,n_1}\cdots t_{s1},\cdots,t_{s,n_s}]$ be the exact weight enumerator of the code V. Then the exact weight enumerator of the dual code $V^\perp$ is obtained from the exact weight enumerator of V on replacing $(t_{i_j})^{u_{i_j}}$ by $\sum_{w=0}^{q-1}\chi(u_{i_j}w)(t_{i_j})^w$ and then diving the result by $|V|$.*

Proof. Take $f:\mathbf{F}_q^n=\bigoplus_{i=1}^{s}\mathbf{F}_q^{n_i}\to\mathbf{C}[t_{11},\cdots,t_{1,n_1},t_{21},\cdots,t_{2,n_1}\cdots t_{s1},\cdots,t_{s,n_s}]$ in Lemma 3.3 as

$$\begin{aligned}f(u) &= f(u_1,u_2,\cdots u_3)=(t_{11}^{u_{11}}\cdots(t_{1,n_1})^{u_{1,n_1}}\cdots(t_{21})^{u_{21}}\cdots(t_{2,n_2})^{u_{2,n_2}}\\ &\quad\cdots(t_{s1})^{u_{s1}}\cdots(t_{s,n_s})^{u_{s,n_s}},\end{aligned}$$

where $u_i=(u_{i,1},u_{i,2},\cdots,u_{i,n_2})\in\mathbf{F}_q^{n_i}$ for all $1\leq s$.

Then $\hat{f}(u)$ is given by

$$\hat{f}(u) = \sum_{v\in\bigoplus_{i=1}^{s}\mathbf{F}_q^{n_i}}\chi(<u,v>)f(v)$$

where $v=(v_1,v_2,\cdots,v_s)$ and $v_i=(v_{i,1},v_{i,2},\cdots,v_{i,n_2})\in\mathbf{F}_q^{n_i}$ for all $1\leq i\leq s$.

$$= \sum_{v_1\in\mathbf{F}_q^{n_1}}\sum_{v_2\in\mathbf{F}_q^{n_2}}\cdots\sum_{v_s\in\mathbf{F}_q^{n_s}}\chi(u_1.v_1+u_2.v_2+\cdots+u_s.v_s)f(v_1,\cdots,v_s)$$

$$
\begin{aligned}
&= \sum_{v_{11}\in \mathbf{F}_q} \cdots \sum_{v_{1,n_1}\in \mathbf{F}_q} \sum_{v_{21}\in \mathbf{F}_q} \cdots \sum_{v_{2,n_2}\in \mathbf{F}_q} \cdots \sum_{v_{s1}\in \mathbf{F}_q} \cdots \sum_{v_{s,n_s}\in \mathbf{F}_q} \\
&\quad \left(\chi\left(\sum_{i=1}^{s}\sum_{j=1}^{n_i} u_{ij}v_{ij}\right)\right)\prod_{i=1}^{s}\prod_{j=1}^{n_i}(t_{ij})^{u_{ij}} \\
&= \sum_{i=1}^{s}\sum_{j=1}^{n_i}\sum_{v_{ij}\in \mathbf{F}_q}\left(\chi\left(\sum_{i=1}^{s}\sum_{j=1}^{n_i} u_{ij}v_{ij}\right)\right)\prod_{i=1}^{s}\prod_{j=1}^{n_i}(t_{ij})^{u_{ij}} \\
&= \sum_{i=1}^{s}\sum_{j=1}^{n_i}\sum_{v_{ij}\in \mathbf{F}_q}\left(\prod_{i=1}^{s}\prod_{j=1}^{n_i}\chi\left(u_{ij}v_{ij}\right)(t_{ij})^{u_{ij}}\right) \\
&= \prod_{i=1}^{s}\prod_{j=1}^{n_i}\sum_{w\in \mathbf{F}_q}\chi(u_{ij}w)(t_{ij})^{w} \\
&= \prod_{i=1}^{s}\prod_{j=1}^{n_i}\sum_{w=0}^{q-1}\chi(u_{ij}w)(t_{ij})^{w}. \qquad (3)
\end{aligned}
$$

From Lemma 3.3 and (3), we have

$$
\begin{aligned}
\sum_{v\in V^{\perp}} f(v) &= \frac{1}{|V|}\sum_{u\in V}\hat{f}(u) \\
&= \frac{1}{|V|}\sum_{u\in V}\left(\prod_{i=1}^{s}\prod_{j=1}^{n_i}\left(\sum_{w=0}^{q-1}\chi(u_{ij}w)(t_{ij})^{w}\right)\right).
\end{aligned}
$$

□

The duality relation for the complete weight enumerator in $lp\gamma$-spaces is given by the following theorem:

Theorem 3.2. *Let V be an $[n,k;P]$ $lp\gamma$ code over $\mathbf{F}_q$ corresponding to the partition $P: n=[n_1][n_2]\cdots[n_s], n_1 \le n_2 \le \cdots \le n_2$. Let $CW_V(t_{11},\cdots,t_{1,n_1}, t_{21},\cdots,t_{2,n_2}\cdots t_{s1},\cdots,t_{s,n_s})$ be the complete weight enumerator of the code V given by*

$$
\begin{aligned}
&CW_V(t_{11},\cdots,t_{1,n_1},t_{21},\cdots,t_{2,n_2}\cdots t_{s1},\cdots,t_{s,n_s}) \\
=\; &\sum_{u\in v}(t_11)^{H(u_{11})}\ldots(t_1,n_1)^{H(u_1,n_1)}\ldots(t_21)^{H(u_{21})}\ldots(t_2,n_2)^{H(u_2,n_2)}\ldots \\
&\cdots(t_{s1})^{H(u_{s1})}\ldots(t_s,n_s)^{H(u_s,n_s)}.
\end{aligned}
$$

Then the complete weight enumerator of the dual code $V^{\perp}$ is obtained from the complete weight enumerator of V on replacing $(t_{ij})^{H(u_{ij})}$ by $P(u_{ij},t_{ij})$, $(1\le i\le s, 1\le j\le n_i)$ and then dividing the result by $|V|$ where $P(u_{ij},t_{ij})$

is given by

$$P(u_{ij}, t_{ij}) \quad = \quad \begin{cases} 1 + (q-1)t_{ij} & \text{if} \quad u_{ij} = 0 \\ 1 - t_{ij} & \text{if} \quad u_{ij} \neq 0. \end{cases} \tag{4}$$

Proof. Take $f : \mathbf{F}_q^n = \oplus_{i=1}^s \mathbf{F}_q^{n_i} \to \mathbf{C}[t_{11}, \cdots, t_{1,n_1}, t_{21}, \cdots, t_{2,n_1} \cdots t_{s1}, \cdots, t_{s,n_s}]$ in Lemma 3.3 as

$$\begin{aligned} f(u) &= f(u_1, u_2, \cdots u_s) \\ &= (t_{11})^{H(u_{11})} \cdots (t_{1,n_1})^{H(u_{1,n_1})} \cdots (t_{21})^{H(u_{21})} \cdots \\ &\quad \cdots (t_{2,n_2})^{H(u_{2,n_2})} \cdots (t_{s1})^{H(u_{s1})} \cdots (t_{s,n_s})^{H(u_{s,n_s})}, \end{aligned}$$

where

$$u_i = (u_{i,1}, u_{i,2}, \cdots, u_{i,n_i}) \in \mathbf{F}_q^{n_i} \text{ for all } 1 \leq s.$$

Then $\hat{f}(u)$ is given by

$$\begin{aligned} \hat{f}(u) &= \sum_{v \in \oplus_{i=1}^s \mathbf{F}_q^{n_i}} \chi(< u, v >) f(v) \\ &\quad \text{where } v = (v_1, v_2, \cdots, v_s) \text{ and } v_i = (v_{i,1}, v_{i,2}, \cdots, v_{i,n_i}) \in \mathbf{F}_q^{n_i} \\ &\quad \text{for all } \ 1 \leq i \leq s. \\ &= \sum_{v_1 \in \mathbf{F}_q^{n_1}} \sum_{v_2 \in \mathbf{F}_q^{n_2}} \cdots \sum_{v_s \in \mathbf{F}_q^{n_s}} \chi(u_1.v_1 + u_2.v_2 + \cdots + u_s.v_s) f(v_1, \cdots, v_s) \\ &= \sum_{v_{11} \in \mathbf{F}_q} \cdots \sum_{v_{1,n_1} \in \mathbf{F}_q} \sum_{v_{21} \in \mathbf{F}_q} \cdots \sum_{v_{2,n_2} \in \mathbf{F}_q} \cdots \sum_{v_{s1} \in \mathbf{F}_q} \cdots \sum_{v_{s,n_s} \in \mathbf{F}_q} \\ &\quad \left(\chi\left(\sum_{i=1}^s \sum_{j=1}^{n_i} u_{ij} v_{ij} \right) \right) \prod_{i=1}^s \prod_{j=1}^{n_i} (t_{ij})^{H(v_{ij})} \\ &= \sum_{i=1}^s \sum_{j=1}^{n_i} \sum_{v_{ij} \in \mathbf{F}_q} \left(\chi\left(\sum_{i=1}^s \sum_{j=1}^{n_i} u_{ij} v_{ij} \right) \right) \prod_{i=1}^s \prod_{j=1}^{n_i} (t_{ij})^{H(v_{ij})} \\ &= \sum_{i=1}^s \sum_{j=1}^{n_i} \sum_{v_{ij} \in \mathbf{F}_q} \left(\prod_{i=1}^s \prod_{j=1}^{n_i} \chi(u_{ij} v_{ij}) (t_{ij})^{H(v_{ij})} \right) \\ &= \prod_{i=1}^s \prod_{j=1}^{n_i} \sum_{w \in \mathbf{F}_q} \chi(u_{ij} w) (t_{ij})^{H(w)} \\ &= \prod_{i=1}^s \prod_{j=1}^{n_i} P(u_{ij}, t_{ij}), \end{aligned}$$

where

$$P(u_{ij},t_{ij}) = \sum_{w\in \mathbf{F}_q} \chi(u_{ij}w)(t_{ij})^{H(w)}.$$

In view of Lemma 3.3, it now suffices to show that $P(u_{ij},t_{ij})$ is given by expression (4). There are two cases to consider:

Case 1. When $u_{ij}=0$. In this case

$$\begin{aligned} P(u_{ij},t_{ij}) &= \sum_{w\in \mathbf{F}_q} \chi(u_{ij}w)(t_{ij})^{H(w)} \\ &= \sum_{w\in \mathbf{F}_q} \chi(0)(t_{ij})^{H(w)} \\ &= \sum_{\substack{w\in \mathbf{F}_q \\ w\neq 0}} \chi(0)(t_{ij})^{H(w)} + 1 \\ &= 1+(q-1)t_{ij}. \end{aligned}$$

Case 2. When $u_{ij}\neq 0$. In this case

$$\begin{aligned} P(u_{ij},t_{ij}) &= \sum_{w\in \mathbf{F}_q} \chi(u_{ij}w))(t_{ij})^{(H(w)} \\ &= 1+\sum_{\substack{w\in \mathbf{F}_q \\ w\neq 0}} \chi(u_{ij}w))(t_{ij})^{H(w)} \\ &= 1+t_{ij}\sum_{\substack{w\in \mathbf{F}_q \\ w\neq 0}} \chi(u_{ij}w) \\ &= 1+t_{ij}\sum_{\substack{\alpha\in \mathbf{F}_q \\ \alpha\neq 0}} \chi(\alpha) \\ &= 1-t_{ij} \text{ (using Lemma 3.1)} \end{aligned}$$

□

Example 3.2. Consider the $lp\gamma$ codes V_1 and V_2 of Example 3.1. over $\mathbf{F}_2$ given by

$$V_1 = \{(00\vdots 00),(10\vdots 10)\}, \qquad V_2 = \{(00\vdots 00),(00\vdots 01)\}.$$

The complete weight enumerator of V_1 and V_2 are given by

$$\begin{aligned} CW_{V_1}(t_{11},t_{12},t_{21},t_{22}) &= 1+t_{11}t_{21}, \\ CW_{V_2}(t_{11},t_{12},t_{21},t_{22}) &= 1+t_{22}. \end{aligned}$$

We obtain the complete weight enumerator of the dual codes $V_l^{\perp} (l = 1, 2)$ using Theorem 3.2 on replacing $(t_{ij})^{H(u_{ij})}$ by $P(u_{ij}, t_{ij})$ in the complete weight enumerator of V_l and then dividing the result by $|V_l|$ $(l = 1, 2)$ where $P(u_{ij}, t_{ij})$ is given by (4).

$$\begin{aligned} CW_{V_1^{\perp}}(t_{11}, t_{12}, t_{21}, t_{22}) &= \frac{1}{2}\Big[(1+t_{11})(1+t_{12})(1+t_{21})(1+t_{22}) + \\ &\quad +(1-t_{11})(1+t_{12})(1-t_{21})(1+t_{22})\Big] \\ &= 1 + t_{11}t_{21} + t_{12} + t_{11}t_{12}t_{21} + t_{22} + t_{11}t_{21}t_{22} + \\ &\quad +t_{12}t_{22} + t_{11}t_{12}t_{21}t_{22}. \end{aligned}$$

Similarly,

$$\begin{aligned} CW_{V_2^{\perp}}(t_{11}, t_{12}, t_{21}, t_{22}) &= \frac{1}{2}\Big[(1+t_{11})(1+t_{12})(1+t_{21})(1+t_{22}) + \\ &\quad +(1+t_{11})(1+t_{12})(1+t_{21})(1-t_{22})\Big] \\ &= 1 + t_{11} + t_{12} + t_{11}t_{12} + t_{21} + t_{11}t_{21} + t_{12}t_{21} + \\ &\quad +t_{11}t_{12}t_{21}. \end{aligned}$$

Example 3.3. Let $n = q = 3$. Let $P : n = 3 = [1][2]$ be a partition of $n = 3$. Let $V =< (0\vdots 12) >$ be a $[3, 1; P]$ $lp\gamma$- code over $\mathbf{F}_3$. Clearly, $V = \{(0\vdots 00), (0\vdots 12), (0\vdots 21)\}$. The dual code $V^{\perp}$ of V is given by

$$\begin{aligned} V^{\perp} &= \{(\alpha\vdots\beta\delta) \in \mathbf{F}_3^1 \oplus \mathbf{F}_3^2 | \beta + 2\delta = 0\} \\ &= \{(\alpha\vdots\beta\delta) \in \mathbf{F}_3^1 \oplus \mathbf{F}_3^2 | \beta = \delta\} \\ &= \{(0\vdots 00), (0\vdots 11), (0\vdots 22), (1\vdots 00), (1\vdots 11), (1\vdots 22)\}, \\ &\quad (2\vdots 00), (2\vdots 11), (2\vdots 22)\}. \end{aligned}$$

The exact and complete weight enumerators of V and $V^{\perp}$ are given below:

$$\begin{aligned} EW_V(t_{11}, t_{21}, t_{22}) &= 1 + t_{21}t_{22}^2 + t_{21}^2t_{22}. \\ CW_V(t_{11}, t_{21}, t_{22}) &= 1 + t_{21}t_{22} + t_{21}t_{22} = 1 + 2t_{21}t_{22}. \\ EW_{V^{\perp}}(t_{11}, t_{21}, t_{22}) &= 1 + t_{21}t_{22} + t_{21}^2t_{22}^2 + t_{11} + t_{11}t_{21}t_{22} + t_{11}t_{21}^2t_{22}^2 \\ &\quad +t_{11}^2 + t_{11}^2t_{21}t_{22} + t_{11}^2t_{21}^2t_{22}^2. \\ CW_{V^{\perp}}(t_{11}, t_{21}, t_{22}) &= 1 + t_{21}t_{22} + t_{21}t_{22} + t_{11} + t_{11}t_{21}t_{22} + t_{11}t_{21}t_{22} + t_{11} + \\ &\quad +t_{11}t_{21}t_{22} + t_{11}t_{21}t_{22} \\ &= 1 + 2t_{21}t_{22} + 2t_{11} + 4t_{11}t_{21}t_{22}. \end{aligned}$$

We now compute the exact and complete weight enumerators of $V^{\perp}$ from those of V as shown below:

$$\begin{aligned} EW_{V^{\perp}}(t_{11},t_{21},t_{22}) &= \frac{1}{3}\Bigg[(1+t_{11}+t_{11}^2)(1+t_{21}+t_{21}^2)(1+t_{22}+t_{22}^2)+ \\ &\quad +(1+t_{11}+t_{11}^2)\Bigg(1+\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)t_{21}+ \\ &\quad +\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)t_{21}^2\Bigg)\times\Bigg(1+\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)t_{22}+ \\ &\quad +\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)t_{22}^2\Bigg)+(1+t_{11}+t_{11}^2)\times \\ &\quad \times\left(1+\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)t_{21}+\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)t_{21}^2\right)\times \\ &\quad \times\left(1+\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)t_{22}+(\frac{-1}{2}-\frac{\sqrt{3}}{2}i)t_{22}^2\right)\Bigg] \\ &= 1+t_{21}t_{22}+t_{21}^2t_{22}^2+t_{11}+t_{11}t_{21}t_{22}+t_{11}t_{21}^2t_{22}^2+ \\ &\quad +t_{11}^2+t_{11}^2t_{21}t_{22}+t_{11}^2t_{21}^2t_{22}^2. \\ CW_{V^{\perp}}(t_{11},t_{21},t_{22}) &= \frac{1}{3}\Bigg[(1+2t_{11})(1+2t_{21})(1+2t_{22})+2(1+2t_{11}) \\ &\quad (1-t_{21})(1-t_{22})\Bigg] \\ &= \frac{1}{3}(1+2t_{11})(3+6t_{21}t_{22}) \\ &= 1+2t_{21}t_{22}+2t_{11}+4t_{11}t_{21}t_{22}. \end{aligned}$$

4. Conclusion

In this paper, we have introduced various types of weight enumerators for $lp\gamma$-codes viz. exact weight enumerator, complete weight enumerator, block weight enumerator and γ-weight enumerator. It was observed that in spite of the fact that two $lp\gamma$-codes V_1 and V_2 are having the same γ-weight enumerator but the γ-weight enumerators of their duals are different. However, the same is not true for the exact and complete weight enumerators. In other words, if two $lp\gamma$-codes have the same exact (or complete) weight enumerator, then their duals also have the same exact (or complete) weight enumerator. Keeping in this view, in this paper, we have obtained the MacWilliams duality relation for the exact and complete weight enumerators of $lp\gamma$-codes.

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