Parallel Packing and Covering of an Isosceles Right Triangle with Sequences of Squares *

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Abstract. Let T be an isosceles right triangle and let $S_1, S_2, S_3, \dots$ be the homothetic copies of a square S. In this paper we consider the parallel covering and packing of T with the sequences $\{S_n\}$ of squares.
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1 Introduction

By a *convex body* we mean a compact convex set with non-empty interior. Let us recall some definitions from [8]. Let $C_1, C_2, C_3, \dots$ be plane convex bodies. We say that the sequence $\{C_n\}$ *permits a translative covering* of a plane convex body C if there exist translations σ_n such that $C \subset \bigcup \sigma_n C_n$. We say that $\{C_n\}$ can be *translatively packed* into C if there exist translations σ_n such that $C \supset \bigcup \sigma_n C_n$ and that they have pairwise disjoint interiors. In particular, a covering or packing of a polygon P with squares $\{S_n\}$ is *parallel* if there is a side of P such that each square $\sigma_n S_n$ has a side parallel to this side of P.

Denote by $A(C)$ the area of a plane convex body C. Let D and K be two plane convex bodies. Denote by $f(D,K)$ the smallest positive number such that any sequence of positive homothetic copies of K whose total area is not less than $f(D,K) \cdot A(D)$ permits a translative covering of D and denote by $p(D,K)$ the greatest number such that any sequence of positive homothetic copies of K whose total area does not exceed $p(D,K) \cdot A(D)$ can be translatively packed into D (see [8]).

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There are many results about covering of some special convex bodies, such as squares and triangles, for these results one can refer to [3], [5], [10], and [12]. For packing of squares and triangles one can see [6], [7], and [10]. In addition, various results concerning packing and covering are discussed in [1], [2], [4], [8], [9], and [11].

Let T be an isosceles right triangle and let S be a square. The aim of this paper is to show that $f(T,S) = 4$ and $p(T,S) = \frac{4}{9}$.

2 Parallel covering of an isosceles right triangle with squares

Moon and Moser gave the following lemma (see [10]).

Lemma 1. *Assume that S is a square and that $\{S_n\}$ is a sequence of squares. If $\sum A(S_n) \geq 3A(S)$, then $\{S_n\}$ permits a parallel covering of S.*

The following lemma is a direct consequence of Lemma 1 (see [8]).

Lemma 2. *Suppose that S is a square and that $\{S_n\}$ is a sequence of squares of side lengths not greater than a. Moreover, let $\sum A(S_n) > 3A(S)$. Then there exists an integer k such that it is possible to parallel cover S with $S_1, S_2, \dots, S_k$ and that $\sum_{n=1}^{k} A(S_n) \leq 3A(S) + a^2$.*

2.1 The square S has a side parallel to the hypotenuse of T

Without loss of generality we may assume that the hypotenuse length of T is 2. Then the area $A(T)$ of T is 1.

Theorem 1. Any (finite or infinite) sequence of squares permits a parallel covering of T provided the total area of the squares is not less than 4.

Proof. Let S be a square with a side parallel to the hypotenuse of T (where the hypotenuse length of T is 2), let $\{S_n\}$ be the homothetic copies of S, and let $\sum A(S_n) \geq 4$. Denote by a_n the side length of S_n, for $n = 1, 2, \dots$. Without loss of generality we may assume that $a_1 \geq a_2 \geq \dots$. We claim that $\{S_n\}$ permits a parallel covering of T.

We may assume that $a_1 < 2$, otherwise T can be covered with S_1, then we need to consider the following two cases.

Case 1: $1 \leq a_1 < 2$.

We place S_1 as in Figure 1. The remaining squares are used for the covering of the square $R_1 \supset T \setminus \sigma_1 S_1$ of side length $2 - a_1$. By Lemma 1

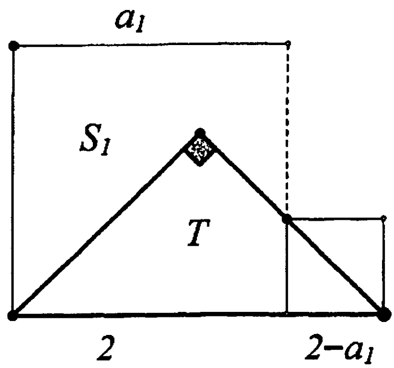

Figure 1: $1 \le a_1 < 2$

we deduce that if T cannot be covered, then

$$\sum A(S_n) < a_1^2 + 3(2-a_1)^2 = 4a_1^2 - 12a_1 + 12.$$

This upper bound does not exceed 4, which is a contradiction.
Case 2: $0 < a_1 < 1$.

A covering method used in this case is based on the method from [8] and [10], which is illustrated in Figure 2. The hypotenuse of T parallel to a side of S is called the *base*. The remaining sides of T are called the *left* and the *right* side, respectively. We place the squares from the sequence in layers, whose bases are parallel to the base of T. The base of the first layer contains the base of T. The first square is placed as far to the right as possible but so that $\sigma_1 S_1$ covers the common point of the left side of T and the base of the first layer. We place the squares, side by side, beginning from the left side of T along the base of the layer. If a square S_n is not the last in the sequence and if $\sigma_n S_n$ covers the common point of the base and the right side of T, then the new layer is created directly above $\sigma_n S_n$. The square S_{n+1} is placed as far to the right as possible but so that $\sigma_{n+1} S_{n+1}$ covers the common point of the left side of T and the base of the layer. We stop the covering process when T is covered.

Contrary to the statement, we assume that it is impossible to cover T with the sequence $\{S_n\}$ by this method. Denote by $\sigma_{n_j} S_{n_j}$ the last square lying in the j-th layer.

We first consider the case that there is a finite number of created layers. A layer is *full* if each point of its base that is contained in T is covered by a placed square. Denote by k the number of full layers.

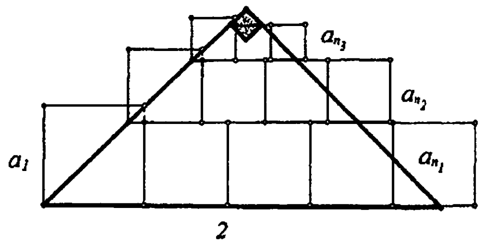

Figure 2: $0 < a_1 < 1$

Clearly,

$$a_{n_1} + a_{n_2} + \cdots + a_{n_k} < 1. \tag{1}$$

The total area of the squares lying in the first layer is less than $2 \cdot a_1 + a_{n_1}^2$. The total area of the squares lying in the j-th layer, for $j \in \{2, 3, \ldots, k\}$, is less than

$$(2 - 2a_{n_1} - 2a_{n_2} - \cdots - 2a_{n_{j-1}})a_{n_{j-1}} + a_{n_j}^2.$$

Moreover, the total area of the squares lying in the $(k+1)$-th layer, if it has been created, is less than

$$(2 - 2a_{n_1} - 2a_{n_2} - \cdots - 2a_{n_k})a_{n_k}.$$

By (1), we know that the sum

$$(2 - 2a_{n_1})a_{n_1} + a_{n_1}^2 + (2 - 2a_{n_1} - 2a_{n_2})a_{n_2} + a_{n_2}^2 + \cdots$$
$$+(2 - 2a_{n_1} - 2a_{n_2} - \cdots - 2a_{n_k})a_{n_k} + a_{n_k}^2$$

does not exceed the area of a right triangle of leg lengths 1 and 2, that is, does not exceed 1.

Hence,

$$\sum A(S_n) < 2a_1 + 1.$$

This upper bound is less than 3 provided $a_1 < 1$, which is a contradiction.

Then we consider the case that there are infinitely many created layers. Also in this case the sum

$$(2 - 2a_{n_1})a_{n_1} + a_{n_1}^2 + (2 - 2a_{n_1} - 2a_{n_2})a_{n_2} + a_{n_2}^2 + \cdots$$

does not exceed the area of a right triangle of leg lengths 1 and 2, that is, does not exceed 1.

Hence,

$$\sum A(S_n) < 2a_1 + 1 < 3,$$

which is a contradiction. The proof is complete. □

2.2 The square S has a side parallel to some leg of T

Without loss of generality we suppose that the leg length of T is 1. Then the area $A(T)$ of T is $\frac{1}{2}$.

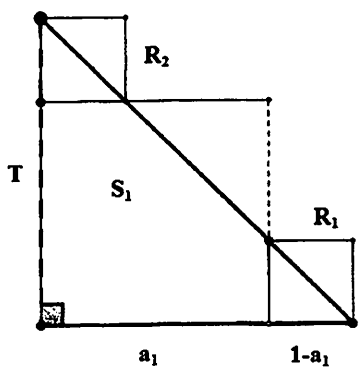

Figure 3: $\frac{1}{2} \leq a_1 < 1$

Theorem 2. Any (finite or infinite) sequence of squares permits a parallel covering of T provided the total area of the squares is not less than 2.

Proof. Let $\{S_n\}$ be a sequence of squares, let $\sum A(S_n) \geq 2$, and let S be a square with a side parallel to some leg of T (where the leg length of T is 1). Denote by a_n the side length of S_n, for $n = 1, 2, \ldots$. Without loss of generality we can assume that $a_1 \geq a_2 \geq \ldots$ and that each S_n is a homothetic copy of S. We show that $\{S_n\}$ permits a parallel covering of T.

We may assume that $a_1 < 1$, otherwise T can be covered by S_1. We need to consider the following two cases.

Case 1: $\frac{1}{2} \leq a_1 < 1$.

We place S_1 as in Figure 3. The remaining squares are used for the coverings of the squares R_1 and R_2 of side length $1 - a_1$. If T cannot be covered and if $R_1 \subseteq S_2$, then by Lemma 1 we have

$$\sum A(S_n) < a_1^2 + a_2^2 + 3(1-a_1)^2 \leq 5a_1^2 - 6a_1 + 3.$$

This upper bound does not exceed 2, which is a contradiction.

If T cannot be covered and if S_2 does not contain R_1, then, by Lemma 1 and Lemma 2, we get

$$\sum A(S_n) < a_1^2 + a_2^2 + 6(1-a_1)^2 \leq 8a_1^2 - 12a_1 + 6.$$

This upper bound does not exceed 2, which is a contradiction.

Case 2: $0 < a_1 < \frac{1}{2}$.

In this case the covering method is the same as that of case 2 in Theorem 1, we omit it here.

We first consider the case that there is a finite number of created layers. By Figure 4 we know that

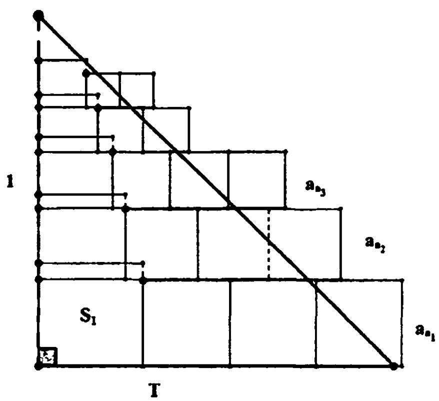

Figure 4: $0 < a_1 < \frac{1}{2}$

$$a_{n_1} + a_{n_2} + \cdots + a_{n_k} < 1. \tag{2}$$

The total area of the squares lying in the first layer is less than $1 \cdot a_1 + a_{n_1}^2$. The total area of the squares in the j-th layer, for $j \in \{2, 3, \ldots, k\}$, is less than

$$(1 - a_{n_1} - a_{n_2} - \cdots - a_{n_{j-1}})a_{n_{j-1}} + a_{n_j}^2.$$

Moreover, the total area of the squares lying in the $(k+1)$-th layer, if it has been created, is less than

$$(1 - a_{n_1} - a_{n_2} - \cdots - a_{n_k})a_{n_k}.$$

By (2), we know that the sum

$$(1 - a_{n_1})a_{n_1} + \frac{1}{2}a_{n_1}^2 + (1 - a_{n_1} - a_{n_2})a_{n_2} + \frac{1}{2}a_{n_2}^2 + \cdots$$
$$+(1 - a_{n_1} - a_{n_2} - \cdots - a_{n_k})a_{n_k} + \frac{1}{2}a_{n_k}^2$$

does not exceed the area of a right triangle of leg length 1, that is, does not exceed $\frac{1}{2}$.

Hence,

$$\begin{aligned}\sum A(S_n) &< a_1 + \frac{1}{2} + \frac{1}{2}(a_{n_1}^2 + a_{n_2}^2 + \cdots + a_{n_k}^2) \\ &\le a_1 + \frac{1}{2} + \frac{1}{2}a_{n_1}(a_{n_1} + a_{n_2} + \cdots + a_{n_k}) \\ &\le a_1 + \frac{1}{2} + \frac{1}{2}a_{n_1} \le a_1 + \frac{1}{2} + \frac{1}{2}a_1 < \frac{5}{4} < 2,\end{aligned}$$

which is a contradiction.

Now consider the case that there are infinitely many created layers. Also in this case the sum

$$(1 - a_{n_1})a_{n_1} + \frac{1}{2}a_{n_1}^2 + (1 - a_{n_1} - a_{n_2})a_{n_2} + \frac{1}{2}a_{n_2}^2 + \cdots$$

does not exceed the area of a right triangle of leg length 1, that is, does not exceed $\frac{1}{2}$.

Hence,

$$\sum A(S_n) \le a_1 + \frac{1}{2} + \frac{1}{2}a_1 < \frac{5}{4} < 2,$$

which is a contradiction. □

Remark 1. Without loss of generality we may assume that the hypotenuse length of T is 2. When the sides of the squares are parallel to the hypotenuse of T, observe that no square with side length less than 2 can parallel cover T. Therefore, from Theorem 1 we have $f(T, S) \cdot A(T) = 4$. Since $A(T) = 1$, we obtain $f(T, S) = 4$. Similarly, we may assume that the leg length of T is 1. When the sides of the squares are parallel to the leg of T, observe that two squares of side lengths less than 1 cannot parallel cover T. Hence, by Theorem 2 we get $f(T, S) \cdot A(T) = 2$. Since $A(T) = \frac{1}{2}$, we get $f(T, S) = 4$.

Corollary 1. *If a side of a square S is parallel to a side of T, then $f(T, S) = 4$.*

3 Parallel packing of an isosceles right triangle with squares

In this section we consider the parallel packing of T with squares.

3.1 The square S has a side parallel to the hypotenuse of T

Lemma 3. *Let $T(b)$ be an isosceles right triangle with hypotenuse length b and let S be a square with a side parallel to the hypotenuse of $T(b)$. Moreover, let $\{S_n\}$ be a sequence of homothetic copies of S such that $a_1 \geq a_2 \geq \ldots$, where a_n denotes the side length of S_n, for $n = 1, 2, \ldots$. If*

$$\sum A(S_n) \leq a_1^2 + \frac{1}{4}(b - 3a_1)^2$$

and if $a_1 \leq \frac{b}{3}$, then the squares can be parallel packed into $T(b)$.

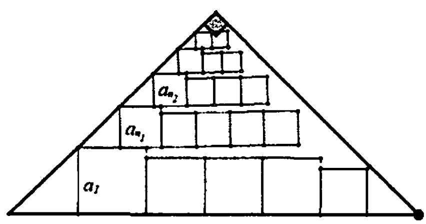

Figure 5: $T(b)$ with hypotenuse length b

Proof. We describe a method of packing $S_1, S_2, \ldots$ into $T(b)$. This method is based on the method from [8] and [10]. The hypotenuse of $T(b)$ that is parallel to a side of S is called the *base*. The other sides of $T(b)$ are called the *left* and the *right* side, respectively. We pack the squares from the sequence in layers, whose bases are parallel to the base of $T(b)$ (see Figure 5). We pack the squares into $T(b)$, side by side, beginning from the left side of $T(b)$ along the base of the layer. The base of the first layer is the base of $T(b)$. The assumption $a_1 \leq \frac{b}{3}$ guarantees that S_1 can be parallel packed in the first layer. If a square cannot be packed into $T(b)$ in a layer, then the new layer is created directly above the first square lying in the preceding layer. This packing process is terminated either as long as all squares have been packed or as long as there is a square which cannot be packed into $T(b)$ by this method.

We are going to prove the lemma indirectly. Assume on the contrary that the sequences $\{S_n\}$ does not permit a packing into $T(b)$ by the method described in the preceding paragraph.

Denote by k the number of created layers in which at least one square is packed. Denote by $\sigma_{n_j} S_{n_j}$ the first square lying in the $(j+1)$-th layer for $j = 1, 2, \dots, k-1$. Furthermore, let S_{n_k} be the square which stops the packing process.

We get

$$a_1 + a_{n_1} + \cdots + a_{n_k} + \frac{1}{2} a_{n_k} > \frac{b}{2}.$$

Consequently,

$$a_{n_1} + \cdots + a_{n_k} > \frac{1}{2}(b - a_{n_k}) - a_1 \geq \frac{1}{2}(b - a_1) - a_1 = \frac{1}{2}(b - 3a_1). \quad (3)$$

It is easy to see that

$$\sum_{n=2}^{n_1} A(S_n) > (b - 3a_1) a_{n_1},$$

$$\sum_{n=n_1+1}^{n_2} A(S_n) > (b - 2a_1 - 3a_{n_1}) a_{n_2} \geq (b - 3a_1 - 2a_{n_1}) a_{n_2},$$

and so on.

By (3), we know that the total area of $S_2, S_3, \dots, S_{n_k}$ is greater than the area of a right triangle of leg lengths $b - 3a_1$ and $\frac{1}{2}(b - 3a_1)$, hence

$$\sum_{n=1}^{n_k} A(S_n) > a_1^2 + \frac{1}{4}(b - 3a_1)^2,$$

which is contradiction. □

Without loss of generality we suppose that the hypotenuse length of T is 2.

Theorem 3. Any (finite or infinite) sequence of squares can be parallel packed into T provided the total area of the squares does not exceed $\frac{4}{9}$.

Proof. Let $\{S_n\}$ be a sequence of squares, let $\sum A(S_n) \leq s = \frac{4}{9}$, and let S be a square with a side parallel to the hypotenuse of T (where the hypotenuse length of T is 2).

Denote by a_n the side length of S_n for $n = 1, 2, \dots$. Without loss of generality we assume that $a_1 \geq a_2 \geq \dots$ and that each S_n is a homothetic copy of S.

Obviously, $a_1^2 \leq s$, i.e., $a_1 \leq \frac{2}{3}$.

Assume that it is impossible to pack $S_1, S_2, \ldots$ into T by the method described in the proof of Lemma 3. Let S_z be the square which stops the packing process.

Denote by k the number of created layers in which at least one square is packed. Moreover, denote by $\sigma_{n_j} S_{n_j}$ the first square lying in the $(j+1)$-th layer for $j \in \{1, \ldots, k-1\}$ and let $z = n_k$.

If S_1 cannot be packed into T, then $3a_1 > 2$, and thus $a_1^2 > \frac{4}{9}$, which is a contradiction. Otherwise, we place S_2 in this layer. If S_1 and S_2 cannot be packed into T (see Figure 6), then $2(a_1 + a_2) > 2$, that is, $a_2 > 1 - a_1$. Hence,

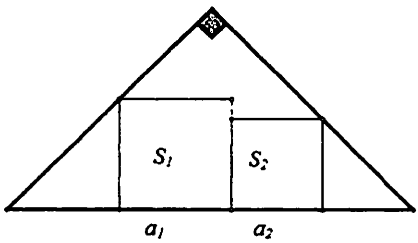

Figure 6: S_1 and S_2 cannot be packed into T

$$a_1^2 + a_2^2 \geq a_1^2 + (1 - a_1^2) = 2a_1^2 - 2a_1 + 1 \geq \frac{1}{2} > \frac{4}{9},$$

which is a contradiction.

If S_1 and S_2 can be packed into T, then $k \geq 2$. Otherwise, we have $k = 1$, and thus $a_1 + a_{n_1} + \frac{1}{2}a_{n_1} > 1$, i.e., $\frac{3}{2}a_{n_1} > 1 - a_1$.

Consequently,

$$a_1^2 + a_2^2 + a_3^2 \geq a_1^2 + 2a_{n_1}^2 \geq a_1^2 + 2 \cdot \frac{4}{9}(1 - a_1)^2$$
$$= \frac{17}{9}a_1^2 - \frac{16}{9}a_1 + \frac{8}{9} \geq \frac{8}{17} > \frac{4}{9},$$

which is a contradiction.

Therefore there are at least two layers in which at least one square has been packed.

It is easy to see that

$$\sum_{n=2}^{n_1} A(S_n) > (2 - 2a_1 - a_{n_1}) \cdot a_{n_1}.$$

By Lemma 3 we know that

$$\sum_{n=n_1+1}^{n_k} A(S_n) > \frac{1}{4}(2 - 2a_1 - 3a_{n_1})^2.$$

Thus,

$$\sum_{n=1}^{n_k} A(S_n) > a_1^2 + (2 - 2a_1 - a_{n_1}) \cdot a_{n_1} + \frac{1}{4}(2 - 2a_1 - 3a_{n_1})^2$$

$$= 1 + 2a_1^2 + \frac{5}{4}a_{n_1}^2 - 2a_1 - a_{n_1} + a_1 a_{n_1}.$$

Using standard methods of calculus, one can verify that the total area is greater than $\frac{4}{9}$ provided that $0 < a_{n_1} \le a_1 < \frac{2}{3}$, which is a contradiction. □

3.2 The square S has a side parallel to some leg of T

We may assume that the leg length of T is 1. By a similar discussion as that of Lemma 3 we can get the following lemma.

Lemma 4. *Let $T(b)$ be an isosceles right triangle with leg length b and let S be a square with a side parallel to some leg of $T(b)$. Moreover, let $\{S_n\}$ be a sequence of homothetic copies of S such that $a_1 \ge a_2 \ge \ldots$, where a_n denotes the side length of S_n, for $n = 1, 2, \ldots$. If*

$$\sum A(S_n) \le a_1^2 + \frac{1}{2}(b - 2a_1)^2$$

and if $a_1 \le \frac{b}{2}$, then the squares can be parallel packed into $T(b)$.

Theorem 4. Any (finite or infinite) sequence of squares can be parallel packed into T provided the total area of the squares does not exceed $\frac{2}{9}$.

Proof. Let $\{S_n\}$ be a sequence of squares, let $\sum A(S_n) \le s = \frac{2}{9}$, and let S be a square with a side parallel to some leg of T (where the leg length of T is 1).

Denote by a_n the side length of S_n for $n = 1, 2, \ldots$. Without loss of generality we assume that $a_1 \ge a_2 \ge \ldots$ and that each S_n is a homothetic copy of S.

Obviously, $a_1^2 \le s$, i.e., $a_1 \le \frac{\sqrt{2}}{3}$.

Assume that it is impossible to pack $S_1, S_2, \ldots$ into T by the method described in the proof of Lemma 3. Let S_z be the square which stops the packing process.

Denote by k the number of created layers in which at least one square is packed. Moreover, denote by $\sigma_{n_j} S_{n_j}$ the first square lying in the $(j+1)$-th layer for $j \in \{1, \dots, k-1\}$ and let $z = n_k$.

If S_1 cannot be packed into T, then $a_1 > \frac{1}{2}$, and thus $a_1^2 > \frac{1}{4} > \frac{2}{9}$. Otherwise, we place S_2 in this layer. If S_1 and S_2 cannot be packed into T, then $a_1 + 2a_2 > 1$, that is, $a_2 > \frac{1}{2}(1 - a_1)$ (see Figure 7). Hence,

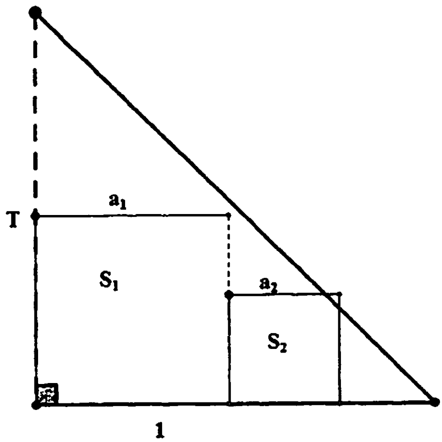

Figure 7: S_1 and S_2 cannot be packed into T

$$a_1^2 + a_2^2 \geq a_1^2 + \frac{1}{4}(1 - a_1)^2 = \frac{5}{4}a_1^2 - \frac{1}{2}a_1 + \frac{1}{4}.$$

Since $3a_1 > 1$, we get that $a_1^2 + a_2^2 \geq \frac{5}{4}a_1^2 - \frac{1}{2}a_1 + \frac{1}{4} > \frac{5}{4} \cdot \frac{1}{9} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} = \frac{2}{9}$, which is a contradiction.

If S_1 and S_2 can be packed into T, then $k \geq 3$.

Otherwise, when $k = 1$, we have $a_1 + 2a_{n_1} > 1$, i.e., $a_{n_1} > \frac{1}{2}(1 - a_1)$.

Consequently,

$$a_1^2 + a_2^2 + a_3^2 \geq a_1^2 + 2a_{n_1}^2 \geq a_1^2 + \frac{1}{2}(1 - a_1)^2$$
$$= \frac{3}{2}a_1^2 - a_1 + \frac{1}{2} \geq \frac{1}{3} > \frac{2}{9},$$

which is a contradiction.

When $k = 2$, we get $a_1 + a_{n_1} + 2a_{n_2} > 1$, and thus $a_{n_2} > \frac{1}{2}(1 - a_1 - a_{n_1})$.

It is easy to see that

$$\sum_{n=1}^{n_1} A(S_n) > a_1^2 + (1 - a_1 - a_{n_1}) \cdot a_{n_1}.$$

Hence,

$$\sum_{n=1}^{n_2} A(S_n) > a_1^2 + (1 - a_1 - a_{n_1}) \cdot a_{n_1} + a_{n_2}^2$$
$$> a_1^2 + (1 - a_1 - a_{n_1}) \cdot a_{n_1} + \frac{1}{4}(1 - a_1 - a_{n_1})^2$$
$$= \frac{1}{4} + \frac{5}{4}a_1^2 - \frac{3}{4}a_{n_1}^2 - \frac{1}{2}a_1 + \frac{1}{2}a_{n_1} - \frac{1}{2}a_1 a_{n_1}.$$

Using standard methods of calculus, one can verify that the total area is greater than $\frac{1}{4} > \frac{2}{9}$, which is a contradiction.

Therefore there are at least three layers in which at least one square has been packed.

It is easy to see that

$$\sum_{n=2}^{n_1} A(S_n) > (1 - a_1 - a_{n_1}) \cdot a_{n_1}$$

and that

$$\sum_{n=n_1+1}^{n_2} A(S_n) > (1 - a_1 - a_{n_1} - a_{n_2}) \cdot a_{n_2}.$$

By Lemma 4 we know that

$$\sum_{n=n_2+1}^{n_k} A(S_n) > \frac{1}{2}(1 - a_1 - a_{n_1} - 2a_{n_2})^2.$$

Thus,

$$\sum_{n=1}^{n_k} A(S_n) > a_1^2 + (1 - a_1 - a_{n_1}) \cdot a_{n_1}$$
$$+(1 - a_1 - a_{n_1} - a_{n_2}) \cdot a_{n_2} + \frac{1}{2}(1 - a_1 - a_{n_1} - 2a_{n_2})^2$$
$$= \frac{1}{2} + \frac{3}{2}a_1^2 - \frac{1}{2}a_{n_1}^2 + a_{n_2}^2 - a_1 - a_{n_2} + a_1 a_{n_2} + a_{n_1} a_{n_2}.$$

Using standard methods of calculus, one can verify that the total area is greater than $\frac{1}{4} > \frac{2}{9}$, which is a contradiction. □

Remark 2. We may assume that the hypotenuse length of T is 2. When the sides of the squares are parallel to the hypotenuse of T, observe that no square of side length greater than $\frac{2}{3}$ can be parallel packed into T. Therefore, by Theorem 3 we have $p(T, S) \cdot A(T) = \frac{4}{9}$. Since $A(T) = 1$, we get $p(T,S)=\frac{4}{9}$. Similarly, we may assume that the leg length of T is 1. When the sides of the squares are parallel to the leg of T, observe that two squares of side lengths greater than $\frac{1}{3}$ cannot be parallel packed into T. As a consequence, by theorem 4 we have $p(T,S)\cdot A(T)=\frac{2}{9}$. Since $A(T)=\frac{1}{2}$, we get $p(T,S)=\frac{4}{9}$.

Corollary 2. If a side of a square S is parallel to a side of T, then $p(T,S)=\frac{4}{9}$.

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