ON THE UNSPLITTABLE MINIMAL ZERO-SUM SEQUENCES OVER FINITE CYCLIC GROUPS OF PRIME ORDER

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ABSTRACT. Let $p > 165$ be a prime and let G be a cyclic group of order p. Let S be a minimal zero-sum sequence with elements over G, i.e., the sum of elements in S is zero, but no proper nontrivial subsequence of S has sum zero. We call S unsplittable, if there do not exist g in S and $x, y \in G$ such that $g = x + y$ and $Sg^{-1}xy$ is also a minimal zero-sum sequence. In this paper we determine the structure of S which is an unsplittable minimal zero-sum sequence of length $\frac{p-1}{2}$ or $\frac{p-3}{2}$. Furthermore, if S is a minimal zero-sum sequence with $|S| \geq \frac{p-3}{2}$, then $\mathrm{ind}(S) \leq 2$.

1. Introduction and Main Results

Our notation and terminology are consistent with [6] and [11]. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for real numbers a, b let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

Let G be an additive finite abelian group. Every sequence S over G can be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \in \mathbb{N}_0$ denote the *multiplicity* of g in S. We call

$\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\}$ the *support* of S;

$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) \mid g \in G\}$ the *maximum of the multiplicities* of g in S;

$|S| = \ell = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S;

$\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S.

A sequence T is called a *subsequence* of S and denoted by $T \mid S$ if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for all $g \in G$. Whenever $T \mid S$, let ST^{-1} denote the subsequence with T deleted from S. If S_1, S_2 are two disjoint subsequences of S, let

$$S_1 S_2$$

denote the subsequence of S satisfying that $\mathsf{v}_g(S_1 S_2) = \mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)$ for all $g \in G$. Let

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$$\Sigma(S) = \{\sigma(T) \mid T \text{ is a subsequence of } S \text{ with } 1 \le |T| \le |S|\}.$$

The sequence S is called *zero-sum* if $\sigma(S) = 0 \in G$ and *zero-sum free* if $0 \notin \Sigma(S)$. If $\sigma(S) = 0$ and $\sigma(T) \neq 0$ for every $T \mid S$ with $1 \le |T| < |S|$, then S is called *minimal zero-sum*.

Let G be a finite abelian group. The Davenport constant $\mathsf{D}(G)$ is the smallest integer $\ell \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge \ell$ has a zero-sum subsequence. The studies of the Davenport constant – together with the famous Erdős-Ginzburg-Ziv Theorem – is considered as a starting point in zero-sum theory, and it has initiated a huge variety of further research (more information can be found in the surveys [2, 6, 11], for recent progress see [8, 12, 14, 28]).

The associated inverse problem of Davenport constant studies the structure of sequences of length strictly smaller than $\mathsf{D}(G)$ which do not have a zero-sum subsequence. The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. Recall that the index of a sequence S over G is defined as follows.

Definition 1.1. [11, Definition 5.1.1]

1. Let $g \in G$ be a non-zero element with $\mathrm{ord}(g) < \infty$. For a sequence $S = (x_1 g) \cdot \ldots \cdot (x_l g)$ over G, where $l \in \mathbb{N}_0$ and $1 \le x_1, \ldots, x_l \le \mathrm{ord}(g)$, we define $\|S\|_g = \frac{x_1 + \cdots + x_l}{\mathrm{ord}(g)}$ to be the *g-norm* of S.
2. Let S be a sequence for which $\langle \mathrm{supp}(S) \rangle \subset G$ is a nontrivial finite cyclic group. Then we call $\mathrm{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } \langle \mathrm{supp}(S) \rangle = \langle g \rangle\}$ the *index* of S.
3. Let G be a finite cyclic group. $\mathsf{I}(G)$ denotes the smallest integer $l \in \mathbb{N}$ such that every minimal zero-sum sequence S of length $|S| \ge l$ has $\mathrm{ind}(S) = 1$.

Clearly, S has sum zero if and only if ind(S) is an integer. There are also slightly different definitions of the index in the literature, but they are all equivalent (see Lemma 5.1.2 in [11]).

The index of a sequence was named by Chapman, Freeze and Smith ([3]). It was first addressed by Kleitman-Lemke (in the Conjecture [15, page 344]), used as a key tool by Geroldinger ([10, page 736]), and then investigated by Gao ([5]) in a systematical way. Since then it has received a great deal of attention (see for examples [4, 7, 9, 16, 17, 18, 19, 21, 22, 23, 24, 25, 29]).

To investigate the index of long minimal zero-sum sequences, Gao ([5]) introduced the invariant $\mathsf{I}(G)$ for a cyclic group of G. The precise value of $\mathsf{I}(G)$ has been determined independently by Savchev and Chen ([20]), and by Yuan ([27]) in 2007.

Theorem 1.2. [27] *Let G be a finite cyclic group of order n. Then $\mathsf{I}(G) = 1$ if $n \in \{1, 2, 3, 4, 5, 7\}$, $\mathsf{I}(G) = 5$ if $n = 6$, and $\mathsf{I}(G) = \lfloor \frac{n}{2} \rfloor + 2$ if $n \ge 8$.*

Let S be a minimal zero-sum (resp. zero-sum free) sequence of elements over an abelian group G. We say that S is *splittable* if there exists an element $g \in \text{supp}(S)$ and two elements $x, y \in G$ such that $x + y = g$ and $Sg^{-1}xy$ is a minimal zero-sum (resp. zero-sum free) sequence as well; otherwise we say that S is *unsplittable.*

Let S be a minimal zero-sum sequence of length $\mathsf{l}(G) - 1$ over a finite cyclic group G. If S is splittable, it is easy to check that $\text{ind}(S) = 1$. If S is unsplittable, Gao ([5]) conjectured that $\text{ind}(S) = 2$. In 2010, Xia and Yuan ([26]) showed that Gao's conjecture is true when n is odd, and false when n is even.

Theorem 1.3. [26, Theorem 3.1] *Let G be a finite cyclic group of order n. Let S be an unsplittable minimal zero-sum sequence of length $|S| = \mathsf{l}(G) - 1$ over G. We have:*

1. *If $n > 9$ is odd, then there exists $g \in G$ such that $S = g^{\frac{n-5}{2}}(\frac{n+3}{2}g)^2(\frac{n-1}{2}g)$; if $n = 9$, then there exists $g \in G$ such that $S = g^2 \cdot (6g)^2 \cdot (4g)$ or $S = g \cdot (3g)^2 \cdot (4g) \cdot (7g)$. Moreover* $\text{ind}(S) = 2$.
2. *If n is even, then there exists $g \in G$ such that either $S = (2g)^{\frac{n}{2}-1}(x_1 g)((n-2-x_1)g)$, where $2 \nmid x_1, 1 < x_1 < n$, $x_1 \neq n + 2 - x_1$ or $S = g^t(\frac{n}{2}g)((1 + \frac{n}{2})g)^{2\ell}$, where t, l are positive integers with $t + 2\ell = \frac{n}{2}$. Moreover* $\text{ind}(S) \geq 2$.

In this paper, we characterize the unsplittable minimal zero-sum sequences of length $|S| = \mathsf{l}(G) - 2$ and $\mathsf{l}(G) - 3$ over a cyclic group G of prime order.

Theorem 1.4. *Let $p > 165$ be a prime and let G be a cyclic group of order p. Let S be an unsplittable minimal zero-sum sequence over G. We have:*

1. *If $|S| = \frac{p-1}{2}$, then there exists $g \in G$ such that $S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g)$ or $S = g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$.*
2. *If $|S| = \frac{p-3}{2}$, then there exists $g \in G$ such that $S = g^{\frac{p-17}{2}}(\frac{p+3}{2}g)^6(\frac{p-1}{2}g)$ or $S = g^{\frac{p-9}{2}}(\frac{p+7}{2}g)^2(\frac{p-5}{2}g)$.*

Moreover $\text{ind}(S) = 2$.

Theorem 1.5. *Let $p > 165$ be a prime and let G be a cyclic group of order p. Let T be a minimal zero-sum sequence of length $|T| \geq \mathsf{l}(G) - 3 = \frac{p-3}{2}$ over G. We have* $\text{ind}(T) \leq 2$.

We remark that it would be not hard and very interesting to prove the main theorems for all primes $p \in [19, 165]$.

The paper is organized as follows. In the next section, we provide some preliminary results. In Section 3, we prove Theorem 1.4. In the last section, we will prove Theorem 1.5 and give some further remarks.

2. Preliminaries

To prove the main results we need some preliminaries, beginning with the following lemma which will be be used frequently in the sequel.

Lemma 2.1. [13, Theorem 5.3.1] *Let G be an abelian group. Let S be a zero-sum free sequence over G. Suppose $S = S_1S_2\cdots S_t$, then* $|\Sigma(S)| \geq \sum_{i=1}^{t}|\Sigma(S_i)|$.

Lemma 2.2. [1] *Let p be a prime and let G be a cyclic group of order p. Suppose $A \subset G$ and $A \cap (-A) = \emptyset$. Then $|\Sigma(A)| \geq \min\{p, \frac{|A|(|A|+1)}{2}\}$.*

Lemma 2.3. *Let p be a prime and let G be a cyclic group of order p. Let A be a zero-sum free subset of G, then $|\Sigma(A)| \geq \min\{p, \frac{|A|(|A|+1)}{2}\}$.*

Proof. Since A is a zero-sum free subset, we have $A \cap (-A) = \emptyset$. Hence the result follows from Lemma 2.2. □

Lemma 2.4. [26, Lemma 2.14] *Let p be a prime and let G be a cyclic group of order p. Suppose S is a minimal zero-sum sequence of elements over G. Then S is unsplittable if and only if $|\Sigma(Sg^{-1})| = p-1$ for every $g \in \mathrm{supp}(S)$.*

Lemma 2.5. [26, Lemma 2.15] *Let p be a prime and let G be a cyclic group of order p. Let S be a minimal zero-sum sequence consisting of two distinct elements over G. Then S is splittable.*

For convenience, from Lemma 2.6 till Lemma 2.11 we always assume that p is a prime and G is a cyclic group of order p. Let S be an unsplittable minimal zero-sum sequence of elements over G.

Lemma 2.6. [26, Lemma 2.5] *Suppose $g, tg \in \mathrm{supp}(S)$ with $t \in [2, p-1]$. Then $t \geq \mathsf{v}_g(S) + 2$. Moreover $t \neq \frac{p+1}{2}$.*

Lemma 2.7. [26, Lemma 2.6] *Suppose $g, h \in \mathrm{supp}(S)$ with $g \neq h$. Then*
1. *If $k \in [0, \mathsf{v}_g(S)]$, then $|\Sigma(g^k h)| = 2k+1$.*
2. *If $\mathsf{v}_g(S) \geq 2$ and $\mathsf{v}_h(S) \geq 2$, then $|\Sigma(g^2h^2)| = 8$.*

Lemma 2.8. *Let $T = g^k(xg)^2$ be a subsequence of S, where $k \geq 3$. Then $|\Sigma(T)| \geq 2|T|$. Moreover apart from the case $T = g^k(\frac{p+3}{2}g)^2$, $|\Sigma(T)| \geq 2|T|+1$.*

Proof. Since S is unsplittable, by Lemma 2.6, we have $x \geq k+2$ and $x \neq \frac{p+1}{2}$.

Firstly, we assume that $2x < p$. Since S is minimal zero-sum, we obtain that $2x + k < p$. Then $g, 2g, \ldots, kg, xg, (x+1)g, \ldots, (x+k)g, 2xg, (2x+1)g, \ldots, (2x+k)g$ are pairwise distinct and hence $|\Sigma(T)| \geq 3k+2 \geq 2|T|+1$.

Next assume that $2x > p$. Then $x \geq \frac{p+3}{2}$. Since S is a minimal zero-sum sequence, we have $x + k < p$, and hence $x > 2x - p + k$.

If $2x-p>k$, then $g,2g,\dots,kg,(2x-p)g,(2x-p+1)g,\dots,(2x-p+k)g,xg,(x+1)g,\dots,(x+k)g$ are pairwise distinct and hence $|\Sigma(T)|\geq 3k+2\geq 2|T|+1$.

If $2x-p\leq k$, then $g,2g,\dots,kg,(k+1)g,\dots,(2x-p+k)g,xg,(x+1)g,\dots,(x+k)g$ are pairwise distinct. Hence $|\Sigma(T)|\geq(2x-p+k)+(k+1)\geq 2|T|$, and the equality holds if and only if $x=\frac{p+3}{2}$. □

Lemma 2.9. [26, Lemma 2.11] *Let $T=g_1^k g_2 g_3$ be a subsequence of S, where $k\geq 2$. Then $|\Sigma(T)|\geq 2|T|$, moreover apart from the case $T=g_1^k(\frac{p-1}{2}g_1)(\frac{p+3}{2}g_1)$, $|\Sigma(T)|\geq 2|T|+1$.*

Lemma 2.10. *Let T be a subsequence of S. If $|\operatorname{supp}(T)|\geq 2$, then there exists $g\in\operatorname{supp}(T)$ such that $|\Sigma(g^{-1}T)|\geq 2|g^{-1}T|-1$.*

Proof. Since $|\operatorname{supp}(T)|\geq 2$, T has a factorization

$$T=U_1\cdot\ldots\cdot U_t V_1\cdot\ldots\cdot V_r W,$$

where $U_1,U_2,\dots,U_t$ are 3-subsets of G, $V_1,V_2,\dots,V_r$ are of form $h_1^2h_2^2$ with $h_1,h_2\in\operatorname{supp}(T)$ and $W=h_1^x h_2^y$ with $y\leq 1$. By Lemma 2.3 we have $|\Sigma(U_i)|\geq 6=2|U_i|$ for $i=1,2,\dots,t$. By Lemma 2.7.2 we have $|\Sigma(V_j)|=8=2|V_j|$ for $j=1,2,\dots,r$.

If $y=1$, then by Lemma 2.7.1 we have $|\Sigma(h_1^{-1}W)|\geq 2|h_1^{-1}W|-1$. Take $g=h_1$. Now by Lemma 2.1, we infer that

$$\begin{aligned}|\Sigma(Tg^{-1})|&\geq\sum_{i=1}^{t}|\Sigma(U_i)|+\sum_{j=1}^{r}|\Sigma(V_j)|+|\Sigma(g^{-1}W)|\\&\geq 2\sum_{i=1}^{t}|U_i|+2\sum_{j=1}^{r}|V_j|+2|Wg^{-1}|-1=2|Tg^{-1}|-1,\end{aligned}$$

and we are done.

If $y=0$, we have that either $t\geq 1$ or $r\geq 1$. If $t\geq 1$, in view of Lemmas 2.3 and 2.9, there exists $g\in\operatorname{supp}(T)$ such that $|\Sigma(WU_tg^{-1})|\geq 2|WU_tg^{-1}|-1$. Therefore, by Lemma 2.1, we infer that $|\Sigma(Tg^{-1})|\geq 2|Tg^{-1}|-1$, and we are done. If $r\geq 1$, then by Lemma 2.7.1, we have $|\Sigma(WV_rh_2^{-1})|\geq 2|WV_rh_2^{-1}|-1$. Take $g=h_2$. By Lemma 2.1, we infer that $|\Sigma(Tg^{-1})|\geq 2|Tg^{-1}|-1$, and we are done.

This completes the proof. □

Lemma 2.11. *Suppose $S=g^{\mathsf{h}(S)}(t_1g)\cdot\ldots\cdot(t_kg)$, where $k\geq 3$ and $2\leq t_1\leq t_2\leq\cdots\leq t_k\leq p-1$. If $\mathsf{h}(S)>\frac{p}{3}-1$, we have*

(1) *either $\mathsf{h}(S)+2\leq t_1\leq\frac{p-1}{2}<\frac{p+3}{2}\leq t_2\leq\cdots\leq t_k\leq p-\mathsf{h}(S)-1$ or $\frac{p+3}{2}\leq t_1\leq\cdots\leq t_k\leq p-\mathsf{h}(S)-1$; moreover, if $\mathsf{v}_{t_ig}(S)\geq 2$ for some $i\in[1,k]$, then $t_i\geq\frac{p+3}{2}$;*

(2) *for every subset* $I \subset [1,k]$ *with* $|I| \equiv 1 \pmod 2$*, if* $|I| \ge 3$ *then* $\frac{|I|p}{2} < \sum_{i\in I} t_i \le \frac{(|I|+1)p}{2} - \mathsf{h}(S)$.

Moreover $\mathrm{ind}(S) \le 2$.

Proof. Since S is unsplittable, by Lemma 2.6, we have that $t_k \ge \cdots \ge t_1 \ge \mathsf{h}(S)+2 > \frac{p}{3}+1$. Since S is minimal zero-sum and $t_k \le p-1$, we infer that $t_k + \mathsf{h}(S) < p$. Therefore,

$$t_k \le p - \mathsf{h}(S) - 1 < \frac{2p}{3} \quad .$$

and thus $\mathsf{h}(S) \le \frac{p-3}{2}$.

We assert that

$$t_i + t_j > p \text{ for every } i,j \in [1,k].$$

Assume to the contrary that $t_i + t_j \le p$ for some $i,j \in [1,k]$. Since S is a minimal zero-sum sequence, we infer that $t_i + t_j + \mathsf{h}(S) < p$, which is impossible.

By the assertion we infer that either $t_1 < \frac{p}{2} < t_2$ or $t_1 > \frac{p}{2}$. By Lemma 2.6, we obtain that $t_i \ne \frac{p+1}{2}$ for $i = 1,2,\dots,k$. Hence

$$\text{either } t_1 \le \tfrac{p-1}{2} < \tfrac{p+3}{2} \le t_2 \text{ or } t_1 \ge \tfrac{p+3}{2},$$

and thus (1) holds.

The proof of (2) will be given by using mathematical induction on $|I|$. Suppose $I \subset [1,k]$ with $|I| = 3$. Since $t_i + t_j > p$ for every $i,j \in [1,k]$ and $\frac{p+3}{2} \le t_2 \le \cdots \le t_k < \frac{2p}{3}$, we have that $\frac{3p}{2} < \sum_{i\in I} t_i < 2p$. Since S is a minimal zero-sum sequence, we infer that $\sum_{i\in I} t_i + \mathsf{h}(S) \le 2p$. Therefore, $\frac{3p}{2} < \sum_{i\in I} t_i \le 2p - \mathsf{h}(S)$. Suppose that (2) holds for every subset $I \subset [1,k]$ with $|I| = 2\ell+1$, where $\ell \ge 1$. If $I \subset [1,k]$ with $|I| = 2(\ell+1)+1$, then we may choose $I_1 \subset I$ with $|I_1| = 2\ell+1$. By the hypothesis we have $\frac{|I_1|p}{2} < \sum_{i\in I_1} t_i \le \frac{(|I_1|+1)p}{2} - \mathsf{h}(S)$. This together with $t_1 \le t_2 \le \cdots \le t_k < \frac{2p}{3}$, $t_i + t_j > p$ for every $i,j \in [1,k]$ and $\frac{p}{3} - 1 < \mathsf{h}(S) < \frac{p-3}{2}$ give that $\frac{|I|p}{2} < \sum_{i\in I} t_i \le \frac{(|I|+1)p}{2}$. Since S is minimal zero-sum, we infer that $\frac{|I|p}{2} < \sum_{i\in I} t_i \le \frac{(|I|+1)p}{2} - \mathsf{h}(S)$ and we are done.

Note that $t_i \le p - \mathsf{h}(S) - 1$ for every $i \in [1,k]$. If $k \equiv 0 \pmod 2$, then $\sum_{i=1}^{k} t_i = \sum_{i=1}^{k-1} t_i + t_k \le \frac{kp}{2} - \mathsf{h}(S) + (p - \mathsf{h}(S) - 1) = \frac{(k+2)p}{2} - 2\mathsf{h}(S) - 1$. If $k \equiv 1 \pmod 2$, then $\sum_{i=1}^{k} t_i \le \frac{(k+1)p}{2} - \mathsf{h}(S) < \frac{(k+2)p}{2} - 2\mathsf{h}(S) - 1$. Let $h \in G \setminus \{0\}$ such that $g = 2h$. Then $S = (2h)^{\mathsf{h}(S)}((2t_1)h) \cdot ((2t_2 - p)h) \cdot \ldots \cdot$

$((2t_k-p)h)$ or $(2h)^{\mathsf{h}(S)}((2t_1-p)h)\cdot((2t_2-p)h)\cdot\ldots\cdot((2t_k-p)h)$. Hence

$$\begin{aligned}\|S\|_h \leq & \frac{2\mathsf{h}(S)+2\sum_{i=1}^{k}t_i-(k-1)p}{p}\\ \leq & \frac{2\mathsf{h}(S)+2(\frac{(k+2)p}{2}-2\mathsf{h}(S)-1)-(k-1)p}{p}\\ = & \frac{3p-2\mathsf{h}(S)-2}{p}<3.\end{aligned}$$

Therefore, $\operatorname{ind}(S)\leq 2$.

This completes the proof. □

3. Proof of Theorem 1.4

Throughout this section, we always assume that $p>165$ is a prime and G is a cyclic group of order p and S is an unsplittable minimal zero-sum sequence of length $\frac{p-1}{2}$ or $\frac{p-3}{2}$ over G.

Lemma 3.1. $3\leq|\operatorname{supp}(S)|\leq 5$.

Proof. Since S is unsplittable, by Lemma 2.5 we have that $|\operatorname{supp}(S)|\geq 3$. It remains to show that $|\operatorname{supp}(S)|\leq 5$.

Assume to the contrary that $|\operatorname{supp}(S)|\geq 6$. Suppose $S=g_1^{r_1}g_2^{r_2}\cdot\ldots\cdot g_k^{r_k}$, where $r_1\geq r_2\geq\cdots\geq r_k\geq 1$ and $k\geq 6$. Now S has a factorization

$$S=TU,$$

where $T=g_1g_2\cdot\ldots\cdot g_6$ and $|U|\geq\frac{p-3}{2}-6=\frac{p-15}{2}$. By Lemma 2.3 we have $|\Sigma(T)|\geq 21$.

If $|\operatorname{supp}(U)|\geq 2$, by Lemma 2.10, there exists $a\in[1,k]$ such that $|\Sigma(Ug_a^{-1})|\geq 2|Ug_a^{-1}|-1$. By Lemma 2.1, we infer that $|\Sigma(Sg_a^{-1})|\geq|\Sigma(T)|+|\Sigma(Ug_a^{-1})|\geq 21+2|Ug_a^{-1}|-1\geq 21+2(\frac{p-15}{2}-1)-1>p$, yielding a contradiction to Lemma 2.4.

Next assume that $|\operatorname{supp}(U)|=1$. Then $k=6$ and $U=g_1^{r_1-1}$. Therefore,

$$S=g^{r_1}(t_1g)\cdot\ldots\cdot(t_5g)$$

with $g=g_1$ and $2\leq t_1<\cdots<t_5\leq p-1$. Then $\mathsf{h}(S)=r_1=|S|-5\geq\frac{p-13}{2}>\frac{p}{3}-1$. By Lemma 2.11.1 we may assume that $\frac{p+3}{2}\leq t_2<t_3<t_4<t_5\leq p-\mathsf{h}(S)-1$. Then $t_3+t_4+t_5\geq(\frac{p+3}{2}+1)+(\frac{p+3}{2}+2)+(\frac{p+3}{2}+3)=\frac{3p+21}{2}$. But by Lemma 2.11.2 we infer that

$$t_3+t_4+t_5\leq\frac{(3+1)p}{2}-\mathsf{h}(S)\leq\frac{p+13}{2},$$

yielding a contradiction.

Therefore, $|\operatorname{supp}(S)|\leq 5$. This completes the proof. □

Lemma 3.2. *Suppose* $S = g^{\mathsf{h}(S)}(t_1g)^{r_1}(t_2g)^{r_2}(t_3g)^{r_3}(t_4g)^{r_4}$, *where* $r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0$ *and* $2 \leq t_1, t_2, t_3, t_4 \leq p-1$. *Then* $r_1 \leq 17$ *and* $r_3 + r_4 \leq 2$.

Proof. We first show that $r_1 \leq 17$. Assume to the contrary that $r_1 \geq 18$. Then $\mathsf{h}(S) \geq r_1 \geq 18$. By Lemma 2.8, we have that either $|\Sigma(g^3(t_1g)^2)| \geq 11$ or $|\Sigma(g^2(t_1g)^3)| \geq 11$. Now S has a factorization

$$S = T_1 \cdot \ldots \cdot T_6 \cdot U,$$

where $T_1 = \cdots = T_6 = g^3(t_1g)^2$ or $g^2(t_1g)^3$ such that $|\Sigma(T_i)| \geq 11$ for $i = 1, 2, \ldots, 6$ and $|U| \geq |S| - \sum_{i=1}^{6} |T_i| \geq \frac{p-3}{2} - 6 \times 5 = \frac{p-63}{2}$. Since $|\operatorname{supp}(S)| \geq 3$, we have that $r_2 \geq 1$ and thus $|\operatorname{supp}(U)| \geq 2$. By Lemma 2.10, there exists $h \in \operatorname{supp}(U)$ such that $|\Sigma(Uh^{-1})| \geq 2|Uh^{-1}| - 1 \geq p - 66$. By Lemma 2.1, we infer that $|\Sigma(Sh^{-1})| \geq \sum_{i=1}^{6} |\Sigma(T_i)| + |\Sigma(Uh^{-1})| \geq 6 \times 11 + p - 66 = p$, yielding a contradiction to Lemma 2.4. Therefore, $r_1 \leq 17$.

Claim: $r_3 \leq 2$.

Assume to the contrary that $r_3 \geq 3$. Then S has a factorization

$$S = T_1 \cdot T_2 \cdot T_3 \cdot U,$$

where $T_1 = T_2 = T_3 = g(t_1g)(t_2g)(t_3g)$. By Lemma 2.3 we have $|\Sigma(T_i)| \geq 10$ for $i = 1, 2, 3$.

If $|\operatorname{supp}(U)| \geq 2$, by Lemma 2.10, there exists $h \in \operatorname{supp}(U)$ such that $|\Sigma(Uh^{-1})| \geq 2|Uh^{-1}| - 1 \geq p - 30$. By Lemma 2.1, we infer that $|\Sigma(Sh^{-1})| \geq \sum_{i=1}^{3} |\Sigma(T_i)| + |\Sigma(Uh^{-1})| \geq 3 \times 10 + p - 30 = p$, yielding a contradiction to Lemma 2.4.

If $|\operatorname{supp}(U)| = 1$, then $r_4 = 0$ and $U = g^{\mathsf{h}(S)-3}$. Hence

$$S = g^{\mathsf{h}(S)}(t_1g)^3 \cdot (t_2g)^3 \cdot (t_3g)^3.$$

Then $\mathsf{h}(S) = |S| - 9 \geq \frac{p-19}{2} > \frac{p}{3}$. By Lemma 2.11.1 we have that $\frac{p+3}{2} \leq t_1, t_2, t_3 \leq p - \mathsf{h}(S) - 1$. Then $3t_1 + 3t_2 + t_3 \geq 3(\frac{p+3}{2}) + 3(\frac{p+3}{2} + 1) + (\frac{p+3}{2} + 2) = \frac{7p+31}{2}$. Applying Lemma 2.11.2 to $(t_1g)^3 \cdot (t_2g)^3 \cdot (t_3g)$, we have that

$$3t_1 + 3t_2 + t_3 \leq \frac{(7+1)p}{2} - \mathsf{h}(S) \leq \frac{7p+19}{2},$$

yielding a contradiction. This proves the claim.

In order to prove $r_3 + r_4 \leq 2$, we only need to show that if $r_3 = 2$ then $r_4 = 0$. Next assume that $r_3 = 2$ and $r_4 \geq 1$. By Lemma 2.9, there exist $a, b \in \{1, 2, 3\}$ such that $|\Sigma(g^2(t_ag)(t_bg))| \geq 9$. Now S has a factorization

$$S = T_1 \cdot T_2 \cdot U,$$

where $T_1 = g(t_1g)(t_2g)(t_3g)(t_4g)$, $T_2 = g^2(t_ag)(t_bg)$. Then $|U| \geq |S| - 9 \geq \frac{p-21}{2}$ and $|\operatorname{supp}(U)| \geq 2$. By Lemma 2.10, there exists $h \in \operatorname{supp}(U)$ such that $|\Sigma(Uh^{-1})| \geq 2|Uh^{-1}| - 1 \geq p - 24$. It follows from Lemmas 2.1 and

2.3 that $|\Sigma(Sh^{-1})| \geq |\Sigma(T_1)| + |\Sigma(T_2)| + |\Sigma(Uh^{-1})| \geq 15+9+p-24 = p$, yielding a contradiction to Lemma 2.4.

Therefore, $r_3 + r_4 \leq 2$. This completes the proof. □

Lemma 3.3. *Suppose* $S = g^{\mathsf{h}(S)}(t_1g)^{r_1}(t_2g)^{r_2}(t_3g)^{r_3}(t_4g)^{r_4}$, *where* $r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0$ *and* $2 \leq t_1, t_2, t_3, t_4 \leq p-1$. *Then* $r_2 = 1$.

Proof. By Lemma 3.3, we have $3 \leq |\operatorname{supp}(S)| \leq 5$ and thus $r_2 \geq 1$. Assume to the contrary that $r_2 \geq 2$. Then $r_1 \geq r_2 \geq 2$.

We first show that $r_2 \leq 8$. Suppose that $r_1 \geq r_2 \geq 9$. By Lemma 3.2 we have that $r_3 + r_4 \leq 2$ and $r_2 \leq r_1 \leq 17$. Therefore, $\mathsf{h}(S) = |S| - r_1 - r_2 - (r_3 + r_4) \geq \frac{p-3}{2} - 17 - 17 - 2 = \frac{p-75}{2} > 18$. By Lemma 2.8 we have either $|\Sigma(g^3(t_1g)^2)| \geq 11$ or $|\Sigma(g^2(t_1g)^3)| \geq 11$ and either $|\Sigma(g^3(t_2g)^2)| \geq 11$ or $|\Sigma(g^2(t_2g)^3)| \geq 11$. Let

$$S = T_1 \cdot \ldots \cdot T_6 \cdot U,$$

where $T_1 = T_2 = T_3 = g^3(t_1g)^2$ or $g^2(t_1g)^3$ such that $|\Sigma(T_i)| \geq 11$ for $i = 1,2,3$ and $T_4 = T_5 = T_6 = g^3(t_2g)^2$ or $g^2(t_2g)^3$ such that $|\Sigma(T_i)| \geq 11$ for $i = 4,5,6$. Then $|V| \geq |S| - 30 \geq \frac{p-63}{2}$. Note that $|\operatorname{supp}(V)| \geq 2$. By Lemma 2.10, there exists $h \in \operatorname{supp}(V)$ such that $|\Sigma(Vh^{-1})| \geq 2|Vh^{-1}| - 1 \geq p - 66$. By Lemma 2.1, we infer that $|\Sigma(Sh^{-1})| \geq \sum_{i=1}^{6} |\Sigma(T_i)| + |\Sigma(Vh^{-1})| \geq 66 + p - 66 = p$, yielding a contradiction to Lemma 2.4. Therefore, $r_2 \leq 8$.

Now $\mathsf{h}(S) = |S| - r_1 - r_2 - (r_3 + r_4) \geq \frac{p-3}{2} - 17 - 8 - 2 = \frac{p-57}{2} > \max\{\frac{p}{3} - 1, 30\}$. By Lemma 2.11.1, we have that

$$t_1, t_2 \geq \frac{p+3}{2}.$$

By Lemma 2.9, we have $|\Sigma(g^2(t_1g)(t_2g))| \geq 9$. Now S has a factorization

$$S = T_1 \cdot \ldots \cdot T_x \cdot U_1 \cdot \ldots \cdot U_y \cdot V,$$

where $T_1 = \cdots = T_x = g^2(t_1g)(t_2g)$ and $2 \leq x = r_2 \leq 8$, $U_1 = \cdots = U_y = g^3(t_1g)^2$ or $g^2(t_1g)^3$ such that $|\Sigma(U_j)| \geq 11$ for $j = 1, 2, \ldots, y$. Then $|V| \geq |S| - 4x - 5y \geq \frac{p-3}{2} - 4x - 5y$.

Claim: $x + y \leq 6$.

Suppose that $x + y \geq 7$. Since $|\operatorname{supp}(T_xV)| \geq 2$, by Lemma 2.10, there exists $h \in \operatorname{supp}(T_xV)$ such that $|\Sigma(T_xVh^{-1})| \geq 2|T_xVh^{-1}| - 1 \geq p - 8x - 10y + 2$. By Lemma 2.1, we infer that $|\Sigma(Sh^{-1})| \geq \sum_{i=1}^{x-1} |\Sigma(T_i)| + \sum_{j=1}^{y} |\Sigma(U_j)| + |\Sigma(T_xVh^{-1})| \geq 9(x-1) + 11y + p - 8x - 10y + 2 \geq p$, yielding a contradiction to Lemma 2.4. This proves the claim.

By the claim we infer that $r_2 \leq 6$. On the other hand, $y \leq 6 - x = 6 - r_2$. By the choice of $U_1, \ldots, U_y$, we infer that $r_1 - r_2 - 3y \leq 2$. It follows that

$$r_1 \leq 20 - 2r_2.$$

We will distinguish four cases according to the value of r_2.

Case 1. $5 \le r_2 \le 6$.

Note that $r_1 \le 20 - 2r_2 \le 10$. Then $\mathsf{h}(S) = |S| - r_1 - r_2 - r_3 - r_4 \ge \frac{p-3}{2} - 10 - 6 - 2 = \frac{p-39}{2}$. If $r_1 \ge 6$, applying Lemma 2.11.2 to $(t_1 g)^6 \cdot (t_2 g)^5$, we have that

$$6t_1 + 5t_2 \le \frac{(11+1)p}{2} - \mathsf{h}(S) \le \frac{11p+39}{2}.$$

This yields a contradiction to that $6t_1 + 5t_2 \ge 6(\frac{p+3}{2}) + 5(\frac{p+3}{2} + 1) = \frac{11p+43}{2}$. So $r_1 = r_2 = 5$. Then $\mathsf{h}(S) \ge \frac{p-27}{2}$. Applying Lemma 2.11.2 to $(t_1 g)^5 \cdot (t_2 g)^4$, we have that $5t_1 + 4t_2 \le \frac{(9+1)p}{2} - \mathsf{h}(S) \le \frac{9p+27}{2}$. But $5t_1 + 4t_2 \ge 5(\frac{p+3}{2}) + 4(\frac{p+3}{2} + 1) = \frac{9p+35}{2}$, yielding a contradiction.

Case 2. $r_2 = 4$.

Note that $r_1 \le 20 - 2r_2 = 12$. Then $\mathsf{h}(S) = |S| - r_1 - r_2 - r_3 - r_4 \ge \frac{p-3}{2} - 12 - 4 - 2 = \frac{p-39}{2}$. Similar to the proof of Case 1 we can show that $r_1 \ge 7$ is impossible. So $r_1 \le 6$. Then $\mathsf{h}(S) \ge \frac{p-27}{2}$. Since $t_1, t_2 \ge \frac{p+3}{2}$, we have either $3t_1 + 4t_2 \ge \frac{7p+29}{2}$ or $4t_1 + 3t_2 \ge \frac{7p+29}{2}$. In the former case applying Lemma 2.11.2 to $(t_1 g)^3 \cdot (t_2 g)^4$ we will get a contradiction; in the later case applying Lemma 2.11.2 to $(t_1 g)^4 \cdot (t_2 g)^3$ we will get a contradiction, too.

Case 3. $r_2 = 3$.

Since $r_1 \le 20 - 2r_2$, we have $r_1 \le 14$. Similar to the proof of Case 1, if $10 \le r_1 \le 14$ or $6 \le r_1 \le 9$ or $4 \le r_1 \le 5$, we will always have some contradictions. So $r_1 = r_2 = 3$ and then $\mathsf{h}(S) = |S| - r_1 - r_2 - r_3 - r_4 \ge \frac{p-3}{2} - 3 - 3 - 2 = \frac{p-19}{2}$. Without loss of generality we assume that $t_1 > t_2$. Then $3t_1 + 2t_2 \ge 3(\frac{p+5}{2}) + 2(\frac{p+3}{2}) = \frac{p+21}{2}$. Applying Lemma 2.11.2 to $(t_1 g)^3 \cdot (t_2 g)^2$ we will get a contradiction.

Case 4. $r_2 = 2$.

Since $r_1 \le 20 - 2r_2$, we have $r_1 \le 16$. Similar to the proof of Case 1, we will always have some contradictions if $13 \le r_1 \le 16$ or $9 \le r_1 \le 12$ or $7 \le r_1 \le 8$ or $5 \le r_1 \le 6$. So

$$r_1 \le 4.$$

If $r_1 \ge 3$, then $\mathsf{h}(S) = |S| - r_1 - r_2 - r_3 - r_4 \ge \frac{p-3}{2} - 4 - 2 - 2 = \frac{p-19}{2}$. Now applying Lemma 2.11.2 to $(t_1 g)^3 \cdot (t_2 g)^2$, we have that $3t_1 + 2t_2 \le \frac{(5+1)p}{2} - \mathsf{h}(S) \le \frac{5p+19}{2}$. Since $3t_1 + 2t_2 \ge 3(\frac{p+3}{2}) + 2(\frac{p+3}{2} + 1) = \frac{5p+19}{2}$, we have $3t_1 + 2t_2 = \frac{5p+19}{2}$. Since S is minimal zero-sum, we infer that $S = g^{\frac{p-19}{2}} \cdot (t_1 g)^3 \cdot (t_2 g)^2$. Then $|S| = \frac{p-19}{2} + 3 + 2 = \frac{p-9}{2}$, and this is a contradiction to that $|S| \ge \frac{p-3}{2}$.

Finally, we consider the case $r_1 = 2$. Then $\mathsf{h}(S) \ge \frac{p-15}{2}$. Without loss of generality we assume that $t_1 > t_2 \ge \frac{p-3}{2}$. Applying Lemma 2.11.2

to $(t_1g)^2 \cdot (t_2g)$, we obtain that $2t_1 + t_2 \le \frac{(5+1)p}{2} - \mathsf{h}(S) \le \frac{3p+15}{2}$. This forces that $t_1 = \frac{p+5}{2}$ and $t_2 = \frac{p+3}{2}$. Since S is minimal zero-sum, we have $\mathsf{h}(S) = \frac{p-15}{2}$ and therefore $r_3 = r_4 = 1$. By Lemma 2.11.1, we may assume that $t_3 \ge \frac{p+3}{2}$. In view of $t_1 = \frac{p+5}{2}$ and $t_2 = \frac{p+3}{2}$, we have $t_3 \ge \frac{p+7}{2}$. Applying Lemma 2.11.2 to $(t_1g)^2 \cdot (t_3g)$, we obtain that $2t_1 + t_3 \le \frac{(5+1)p}{2} - \mathsf{h}(S) \le \frac{3p+15}{2}$. This is a contradiction to that $2t_1 + t_3 \ge \frac{3p+17}{2}$.

All in all, we have shown that $r_2 \ge 2$ is impossible. This completes the proof. □

Lemma 3.4. *Suppose $S = g^{\mathsf{h}(S)}(t_1g)^{r_1}(t_2g)^{r_2}(t_3g)^{r_3}(t_4g)^{r_4}$, where $r_1 \ge r_2 \ge r_3 \ge r_4 \ge 0$ and $2 \le t_1, t_2, t_3, t_4 \le p-1$. Then $r_1 \ge 2$.*

Proof. Assume to the contrary that $r_1 = 1$. Then $r_2 = 1$ and $r_3 + r_4 \le 2$. Since S is an unsplittable, by Lemma 2.6, we have that $t_i \ge \mathsf{h}(S) + 2$ for $i = 1, 2, 3, 4$. Since S is minimal zero-sum, we have that $t_i \le p - \mathsf{h}(S) - 1$ for $i = 1, 2, 3, 4$ and

$$\mathsf{h}(S) + t_1 + t_2 + r_3t_3 + r_4t_4 \equiv 0 \pmod{p}.$$

An easy calculation shows that both $r_3 = r_4 = 1$ and $r_3 = r_4 = 0$ are impossible. Hence we may assume that $r_3 = 1$ and $r_4 = 0$.

Since $\mathsf{h}(S) \ge \frac{p-3}{2} - 3 = \frac{p-9}{2} > \frac{p}{3} - 1$, by Lemma 2.11.1, we may assume that $\frac{p+3}{2} \le t_2 < t_3 \le p - \mathsf{h}(S) - 1 = \frac{p+7}{2}$.

Firstly we consider that case that $|S| = \frac{p-3}{2}$. If $t_2 = \frac{p+3}{2}$ and $t_3 = \frac{p+5}{2}$, then $t_1 = \frac{p+1}{2}$, yielding a contradiction to Lemma 2.6. If $t_2 = \frac{p+3}{2}$ and $t_3 = \frac{p+7}{2}$, then $t_1 = \frac{p-1}{2}$ and thus $S = g^{\frac{p-9}{2}}(\frac{p-1}{2}g)(\frac{p+3}{2}g)(\frac{p+7}{2}g)$. It is easy to check that $\Sigma(S(\frac{p+7}{2}g)^{-1}) = G \setminus \{\frac{p-5}{2}g, \frac{p-3}{2}g, (p-2)g, (p-1)g, 0\}$, yielding a contradiction to Lemma 2.4. If $t_2 = \frac{p+5}{2}$ and $t_3 = \frac{p+7}{2}$, then $t_1 = \frac{p-3}{2}$ and thus $S = g^{\frac{p-9}{2}}(\frac{p-3}{2}g)(\frac{p+5}{2}g)(\frac{p+7}{2}g)$. It is easy to check that $\Sigma(S(\frac{p+7}{2}g)^{-1}) = G \setminus \{\frac{p-5}{2}g, (p-1)g, 0\}$, yielding a contradiction to Lemma 2.4.

Similar to the case $|S| = \frac{p-3}{2}$, we can always get contradictions if $|S| = \frac{p-1}{2}$.

This completes the proof. □

Lemma 3.5. *If S is of one of the following forms:*

$$g^{\frac{p-11}{2}}(\tfrac{p+3}{2}g)^4(\tfrac{p-1}{2}g) \text{ or } g^{\frac{p-7}{2}}(\tfrac{p+5}{2}g)^2(\tfrac{p-3}{2}g) \text{ or } g^{\frac{p-17}{2}}(\tfrac{p+3}{2}g)^6(\tfrac{p-1}{2}g) \text{ or } g^{\frac{p-9}{2}}(\tfrac{p+7}{2}g)^2(\tfrac{p-5}{2}g),$$

then S is unsplittable and $\mathrm{ind}(S) = 2$.

Proof. Suppose $S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g)$. Using Lemma 2.4, we can easily show that S is unsplittable. For every $h \in G \setminus \{0\}$, there exists $m \in [1, p-1]$

such that $g = mh$. It is easy to check that $\|S\|_h \geq 2$ for every $h \in G \setminus \{0\}$. If $g = 2h$, then $\|S\|_h = 2$. Hence $\mathrm{ind}(S) = 2$.

Similarly, if $S = g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$ or $S = g^{\frac{p-17}{2}}(\frac{p+3}{2}g)^6(\frac{p-1}{2}g)$ or $S = g^{\frac{p-9}{2}}(\frac{p+7}{2}g)^2(\frac{p-5}{2}g)$, S is unsplittable and $\mathrm{ind}(S) = 2$. □

Now we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4: Suppose S is an unsplittable minimal zero-sum sequence of length $\frac{p-1}{2}$ or $\frac{p-3}{2}$. By Lemma 3.5, it remains to show that S is of one of the forms mentioned in this theorem. By Lemma 3.3, we have $3 \leq |\mathrm{supp}(S)| \leq 5$. Suppose

$$S = g^{\mathsf{h}(S)}(t_1g)^{r_1}(t_2g)^{r_2}(t_3g)^{r_3}(t_4g)^{r_4},$$

where $r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0$ and $2 \leq t_1, t_2, t_3, t_4 \leq p-1$. By Lemmas 3.2, 3.3 and 3.4 we have $2 \leq r_1 \leq 17$, $r_2 = 1$ and $r_3 + r_4 \leq 2$. It follows that $\mathsf{h}(S) = |S| - r_1 - r_2 - r_3 - r_4 \geq \frac{p-3}{2} - 17 - 1 - 2 = \frac{p-43}{2} > \frac{p}{3} - 1$. By Lemma 2.11.1 we have that $t_1 \geq \frac{p+3}{2}$.

Claim 1: $r_1 \leq 12$.

If $r_1 \geq 15$, then applying Lemma 2.11.2 to $(t_1g)^{15}$, we have that $15t_1 \leq \frac{(15+1)p}{2} - \mathsf{h}(S) \leq \frac{15p+43}{2}$, which yields a contradiction to that $15t_1 \geq \frac{15p+45}{2}$. Hence $r_1 \leq 14$ and thus $\mathsf{h}(S) \geq |S| - r_1 - r_2 - r_3 - r_4 \geq \frac{p-3}{2} - 14 - 1 - 2 = \frac{p-37}{2}$. Similarly if $r_2 \geq 13$, we can also obtain a contradiction. This proves Claim 1.

Claim 2: $r_4 = 0$.

Assume to the contrary that $r_4 \geq 1$. Since $r_2 = 1$ and $r_3 + r_4 \leq 2$, we infer that $r_3 = r_4 = 1$. Since $\mathsf{h}(S) \geq |S| - r_1 - r_2 - r_3 - r_4 \geq \frac{p-3}{2} - 12 - 1 - 2 = \frac{p-33}{2} > \frac{p}{3} - 1$, by Lemma 2.11.1 we may assume that $t_2, t_3 \geq \frac{p+3}{2}$. If $r_1 \geq 9$, then $9t_1 + t_2 + t_3 \geq 9(\frac{p+3}{2}) + (\frac{p+3}{2} + 1) + (\frac{p+3}{2} + 2) = \frac{11p+39}{2}$. Applying Lemma 2.11.2 to $(t_1g)^9 \cdot (t_2g) \cdot (t_3g)$, we have that $9t_1 + t_2 + t_3 \leq \frac{(11+1)p}{2} - \mathsf{h}(S) \leq \frac{11p+33}{2}$, yielding a contradiction. So $r_1 \leq 8$. Similarly if $5 \leq r_1 \leq 8$ or $3 \leq r_1 \leq 4$ or $r_1 = 2$, we can always obtain some contradictions. This proves Claim 2.

Claim 3: $r_3 = 0$.

Assume to the contrary that $r_3 = 1$. By Lemma 2.11.1 we may assume that $t_2 \geq \frac{p+3}{2}$. Similar to the proof of Claim 2, if $10 \leq r_1 \leq 12$ or $8 \leq r_1 \leq 9$ or $6 \leq r_1 \leq 7$ or $4 \leq r_1 \leq 5$, we will always obtain some contradictions. So we have that $r_1 \leq 3$. Then $\mathsf{h}(S) = |S| - r_1 - r_2 - r_3 \geq \frac{p-3}{2} - 3 - 1 - 1 = \frac{p-13}{2}$. Applying Lemma 2.11.2 to $(t_1g)^2 \cdot (t_2g)$, we have that $2t_1 + t_2 \leq \frac{(2+1)p}{2} - \mathsf{h}(S) \leq \frac{3p+13}{2}$. Note that $2t_1 + t_2 \geq 2(\frac{p+3}{2}) + (\frac{p+3}{2} + 1) = \frac{3p+11}{2}$. So $2t_1 + t_2 = \frac{3p+13}{2}$ or $2t_1 + t_2 = \frac{3p+11}{2}$. In the former case, $g^{\frac{p-13}{2}} \cdot (t_1g)^2 \cdot (t_2g)$ is a proper zero-sum subsequence of S, yielding a contradiction to that S is minimal zero-sum. So $2t_1 + t_2 = \frac{3p+11}{2}$ and thus $t_1 = \frac{p+3}{2}$, $t_2 = \frac{p+5}{2}$, $\mathsf{h}(S) = \frac{p-13}{2}$. Since S is minimal zero-sum, we have $t_3 = \frac{p-1}{2}$ and $S = g^{\frac{p-13}{2}}(\frac{p-1}{2}g)(\frac{p+3}{2}g)^3(\frac{p+5}{2}g)$. It is easy to check that $\Sigma(S(\frac{p+5}{2}g)^{-1}) = G \setminus \{\frac{p-3}{2}g, (p-1)g, 0\}$, yielding a contradiction to Lemma 2.4. This proves Claim 3.

Now $\mathsf{h}(S) = |S| - r_1 - r_2 \geq \frac{p-3}{2} - 12 - 1 = \frac{p-29}{2}$. Similar to the proof of Claim 1, if $11 \leq r_1 \leq 12$ or $9 \leq r_1 \leq 10$ or $7 \leq r_1 \leq 8$, we will always obtain some contradictions. So we have that

$$r_1 \leq 6.$$

Then we will distinguish five cases according to the value of r_1.

Case 1. $r_1 = 6$. Note that $\mathsf{h}(S) = |S| - r_1 - r_2 \geq \frac{p-3}{2} - 6 - 1 = \frac{p-17}{2}$. Applying Lemma 2.11.2 to $(t_1g)^5$, we have that $5t_1 \leq \frac{(5+1)p}{2} - \mathsf{h}(S) \leq \frac{5p+17}{2}$. Note that $5t_1 \geq 5(\frac{p+3}{2}) = \frac{5p+15}{2}$. So $5t_1 = \frac{5p+15}{2}$. If $|S| = \frac{p-1}{2}$, then $\mathsf{h}(S) = \frac{p-15}{2}$. Hence $g^{\frac{p-15}{2}} \cdot (t_1g)^5$ is a proper zero-sum subsequence of S, yielding a contradiction to that S is minimal zero-sum. If $|S| = \frac{p-3}{2}$, then $\mathsf{h}(S) = \frac{p-17}{2}$ and $t_2 = \frac{p-1}{2}$. Therefore,

$$S = g^{\frac{p-17}{2}}(\frac{p+3}{2}g)^6(\frac{p-1}{2}g),$$

and we are done.

Case 2. $r_1 = 5$. Then $\mathsf{h}(S) \geq \frac{p-15}{2}$. Applying Lemma 2.11.2 to $(t_1g)^5$, we have that $5t_1 \leq \frac{(5+1)p}{2} - \mathsf{h}(S) \leq \frac{5p+15}{2}$. Note that $5t_1 \geq 5(\frac{p+3}{2}) = \frac{5p+15}{2}$. So $5t_1 = \frac{5p+15}{2}$. Hence $g^{\frac{p-15}{2}} \cdot (t_1g)^5$ is a proper zero-sum subsequence of S, yielding a contradiction to that S is minimal zero-sum.

Case 3. $r_1 = 4$. Then $\mathsf{h}(S) \geq \frac{p-13}{2}$. Applying Lemma 2.11.2 to $(t_1g)^3$, we have that $3t_1 \leq \frac{(3+1)p}{2} - \mathsf{h}(S) \leq \frac{3p+13}{2}$. Note that $3t_1 \geq 3(\frac{p+3}{2}) = \frac{3p+9}{2}$. So $3t_1 = \frac{3p+9}{2}$. If $|S| = \frac{p-1}{2}$, then $\mathsf{h}(S) = \frac{p-11}{2}$ and $t_2 = \frac{p-1}{2}$. Therefore,

$$S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g),$$

and we are done. If $|S| = \frac{p-3}{2}$, then $\mathsf{h}(S) = \frac{p-13}{2}$ and $t_2 = \frac{p+1}{2}$, yielding a contradiction to Lemma 2.6.

Case 4. $r_1 = 3$. Then $\mathsf{h}(S) \geq \frac{p-11}{2}$. Note that $\mathsf{h}(S) + 3t_1 + t_2 \geq \mathsf{h}(S) + 3(\mathsf{h}(S)+2) + (\mathsf{h}(S)+3) = 5\mathsf{h}(S)+9 = 5(|S|-r_1-r_2)+9 \geq 5\frac{p-3}{2} - 11 > 2p$ and $\mathsf{h}(S)+3t_1+t_2 \leq \mathsf{h}(S)+3(p-\mathsf{h}(S)-1)+(p-\mathsf{h}(S)-2) = 4p-3\mathsf{h}(S)-5 = 4p - 3(|S| - r_1 - r_2) - 5 \leq 4p - 3\frac{p-1}{2} + 7 < 3p$. This is impossible since S is zero-sum.

Case 5. $r_1 = 2$. Assume that $|S| = \frac{p-1}{2}$, then $\mathsf{h}(S) = \frac{p-7}{2}$. So $\frac{p+3}{2} \leq t_1 \leq p - \mathsf{h}(S) - 1 = \frac{p+5}{2}$. If $t_1 = \frac{p+3}{2}$, then $t_2 = \frac{p+1}{2}$, yielding a contradiction to Lemma 2.6. If $t_1 = \frac{p+5}{2}$, then $t_2 = \frac{p-3}{2}$. Therefore,

$$S = g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g),$$

and we are done. Next assume that $|S| = \frac{p-3}{2}$, then $\mathsf{h}(S) = \frac{p-9}{2}$. So $\frac{p+3}{2} \le t_2 \le p - \mathsf{h}(S) - 1 = \frac{p+7}{2}$. If $t_1 = \frac{p+3}{2}$, then $t_2 = \frac{p+3}{2}$, yielding a contradiction. If $t_1 = \frac{p+5}{2}$, then $t_1 = \frac{p-1}{2}$ and thus $S = g^{\frac{p-9}{2}}(\frac{p+5}{2}g)^2(\frac{p-1}{2}g)$. But $\Sigma(S(\frac{p+5}{2}g)^{-1}) = G \setminus \{\frac{p-3}{2}g, (p-1)g, 0\}$, yielding a contradiction to Lemma 2.4. If $t_1 = \frac{p+7}{2}$, then $t_2 = \frac{p-5}{2}$. Therefore,

$$S = g^{\frac{p-9}{2}}(\frac{p+7}{2}g)^2(\frac{p-5}{2}g),$$

and we are done.

This completes the proof. □

4. Proof of Theorem 1.5 and Concluding remarks

Proof of Theorem 1.5: If $|T| \ge \frac{p+3}{2}$, by Theorem 1.2, $\mathrm{ind}(T) = 1$.

Next assume that $|T| = \frac{p+1}{2}$. If T is unsplittable, by Theorem 1.3.1, we have $\mathrm{ind}(T) = 2$. If T is splittable, there exists $h \in \mathrm{supp}(T)$ and $x, y \in G$ such that $h = x + y$ and $T' = xyTh^{-1}$ is a minimal zero-sum sequence of length $\frac{p+3}{2}$. Then by Theorem 1.2, $\mathrm{ind}(T') = 1$. Clearly $\|T\|_g \le \|T'\|_g$ for every $g \in G \setminus \{0\}$. Hence $\mathrm{ind}(T) \le \mathrm{ind}(T') = 1$.

If $|T| = \frac{p-1}{2}$ or $\frac{p-3}{2}$, similar to above we can show that $\mathrm{ind}(T) \le 2$. This completes the proof. □

Definition 4.1.

1. Let n be an integer. $\mathsf{I}(n)$ denotes the maximal value of index of minimal zero-sum sequences S over a cyclic group G of order n.
2. Let G be a finite cyclic group and $k \ge 1$ be an integer. $\mathsf{I}_k(G)$ denotes the smallest integer $l \in \mathbb{N}$ such that every minimal zero-sum sequence S of length $|S| \ge l$ has $\mathrm{ind}(S) \le k$.

To determine $\mathsf{I}(n)$ was proposed by Gao ([5]), and he conjectured that $\mathsf{I}(n) \le c \ln n$ for some absolute constant c ([5, Conjecture 4.2]). If $n \equiv 0 \pmod 8$, let G be a cyclic group of order n. Suppose

$$S = g^{\frac{n}{4}}(\tfrac{n}{2}g)((1+\tfrac{n}{2})g)^{\frac{n}{4}}.$$

Then $\mathrm{ind}(S) = \frac{n}{8} + 1$. Hence the conjecture of Gao is not true for $n \equiv 0 \pmod 8$. In fact, the conjecture is also not true for every even n (see Theorem 1.3.2).

Let G be a finite cyclic group of order n. Clearly, if $k \ge \mathsf{I}(n)$, then $\mathsf{I}_k(G) = 1$. If $k = 1$, then $\mathsf{I}_1(G) = \mathsf{I}(G)$. By Theorem 1.5, we infer that $\mathsf{I}_2(G) \le \frac{p-3}{2}$, provided that $n = p$ is prime.

Problem. Determine $\mathsf{I}(n)$ for all integers n and determine $\mathsf{I}_k(G)$ for all the cyclic groups G.

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