Partition theoretic interpretations of the A_2 Rogers–Ramanujan identities

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Abstract

Using partition theoretic methods we combinatorially interpret the four A_2 Rogers–Ramanujan identities of Andrews, Schilling and Warnaar.

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1 Introduction

The Rogers–Ramanujan identities [14,15,16] are given by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} (1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{1.2}$$

They were first discovered by Rogers [14] in 1894 and were rediscovered by Ramanujan in 1913. Ramanujan [15] published a paper in 1919 which contains two proofs (one by Ramanujan and the other by Rogers) and a note by Hardy. After the publication of this paper the Identities (1.1) and (1.2) became known as the Rogers–Ramanujan identities. The fame of these identities lies not only in their beauty and fascinating history [4,12], but also in their relevance to the theory of partitions and many other branches of mathematics and physics. In particular, MacMahon [13] and Schur [16] independently noted the number theoretic interpretations of (1.1) and (1.2) as

Theorem 1.1 *For all integers n, the number of partitions λ of n where*

$$\lambda_i - \lambda_{i+1} \geq 2 \ \ \textit{for} \ \ 1 \leq i \leq \ell(\lambda) - 1$$

equals the number of partitions of n into parts which are congruent to $\pm 1 \pmod 5$.

Theorem 1.2 *For all integers n, the number of partitions λ of n where*

$$\lambda_i - \lambda_{i+1} \geq 2 \text{ for } 1 \leq i \leq \ell(\lambda) - 1$$

and the smallest part at least 2 equals the number of partitions of n into parts which are congruent to $\pm 2 \pmod 5$.

Gordon gave the following generalization of Theorems 1.1 and 1.2 [11]:

Theorem 1.3 (B. Gordon) *For $k \geq 2$ and $1 \leq i \leq k$, let $B_{k,i}(n)$ denote the number of partitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$, where $\lambda_j - \lambda_{j+k-1} \geq 2$, and at most $i-1$ of the λ_j equal 1. Let $A_{k,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$. Then*

$$A_{k,i}(n) = B_{k,i}(n) \text{ for all } n.$$

Obviously, Theorem 1.1 is the particular case $k = i = 2$ of Theorem 1.3 and Theorem 1.2 is the particular case $k = i + 1 = 2$ of Theorem 1.3.
The analytic counterpart of Theorem 1.3 was found by Andrews [5]:

Theorem 1.4 (G.E. Andrews) *For $1 \leq i \leq k, k \geq 2$,*

$$\sum_{n_1, n_2, \ldots, n_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + N_{i+1} + \cdots + N_{k-1}}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_{k-1}}}$$

$$= \prod_{n=1}^{\infty} (1 - q^n)^{-1}, \tag{1.3}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

It can be easily seen that Identities (1.1) and (1.2) are the particular cases, $k = i = 2$ and $k = i + 1 = 2$, of Theorem 1.4.

1.1 Notation

The Gaussian polynomial or q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{for } 0 \leq m \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

where $(a;q)_\infty = (a)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)$ and

$$(a;q)_n = (a)_n = \frac{(a)_\infty}{(aq^n)_\infty} \text{ for all integers } n.$$

In particular, for nonnegative n

$$(q)_0 = 1, \quad \text{and} \quad (q)_n = (q;q)_n = (1-q)(1-q^2)\cdots(1-q^n).$$

Also note that $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix}$ and the degree of $\begin{bmatrix} n \\ m \end{bmatrix}$ is $m(n-m)$.

A partition λ of n is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots)$ such that only finitely many λ_i are positive and such that $\sum_i \lambda_i = n$. The positive λ_i are known as the parts of λ and the number of parts is known as the length $\ell(\lambda)$. The sum of the parts of λ, denoted by $|\lambda|$, is called the weight of λ. The unique partition of weight zero is denoted by 0, and the multiplicity of the part i in the partition λ is denoted by $m_i(\lambda)$.

Fix a non-negative integer t. A partition with "$n+t$ copies of n", $t < \infty$, is a partition in which a part of size n, $n \geq 0$, can come in $n+t$ different colours denoted by subscripts: $n_1, n_2, \cdots, n_{n+t}$.

An analogue of Theorem 1.3 for partition with "$n+t$ copies of n" was obtained in [2].

1.2 The A_2 Rogers–Ramanujan identities

Over the years many generalizations of both the analytic and the combinatorial statement of the Rogers–Ramanujan identities have been found, see e.g., [3,7,8,9]. All the cited analytic generalizations are accessible through the classical, or A_1 Bailey lemma and can thus be classified as "A_1 Rogers–Ramnujan-type identities". Rogers–Ramnujan-type identities and Andrews–Gordon are all identities for the Lie algebra A_1 and they were generalized to A_2 in [6] (for details of A_1 and A_2 Rogers–Ramanujan identities the reader is referred to [6]). The following four identities are A_2 Rogers–Ramanujan identities,

$$\sum_{l,k=0}^{\infty} \frac{q^{k^2-lk+k^2}}{(q)_k} \begin{bmatrix} 2k \\ l \end{bmatrix} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})^2(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-6})^2}, \tag{1.4}$$

$$\sum_{l,k=0}^{\infty} \frac{q^{k^2-lk+l^2+l}}{(q)_k} \begin{bmatrix} 2k \\ l \end{bmatrix} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})(1-q^{7n-2})(1-q^{7n-3})(1-q^{7n-4})(1-q^{7n-5})(1-q^{7n-6})}, \tag{1.5}$$

$$\sum_{l,k=0}^{\infty} \frac{q^{k^2-lk+l^2+k+l}}{(q)_k} \begin{bmatrix} 2k \\ l \end{bmatrix} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-2})(1-q^{7n-3})^2(1-q^{7n-4})^2(1-q^{7n-5})}, \tag{1.6}$$

$$\sum_{l,k=0}^{\infty} \frac{q^{k^2-lk+l^2+l}}{(q)_k} \begin{bmatrix} 2k+1 \\ l \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{7n-1})(1-q^{7n-2})^2(1-q^{7n-5})^2(1-q^{7n-6})}, \tag{1.7}$$

(1.4)–(1.6) are due to Andrews et al. [6] and (1.7) is a conjectured identity due to Feigin et al. [10]. The Identity (1.4) was also proved by Warnaar [17]. But in [6,10,17] no combinatorial interpretation was provided in the spirit of "MacMahon–Schur–Gordon" and this is the purpose of our paper. Following the technique of [1], in this paper we provide combinatorial interpretations of identities "(1.4)–(1.7)" using partition theoretic methods.

2 Main Results

Let $B_1(k,n)$ count the number of partitions of n such that no part exceeds k. For integers k and l such that $0 \le l \le 2k$, we define the counting function $C_i(l,k,n)$ for $i = 1,2,3,4$ by distinguishing the cases $0 \le l < k$, $l = k$ and $k < l \le 2k$ with certain conditions given in Theorem 2.1–2.4 respectively, and we now state our main results which are combinatorial interpretations of (1.4)–(1.7) respectively.

Theorem 2.1 *For $0 \le l < k$, let $C_1(l,k,n)$ count the number of partitions λ of n such that $\lambda_1 \le 2k-1$, $\ell(\lambda) \le 2k-l$ and*

$$\lambda_i - \lambda_{i+1} \ge \begin{cases} 2 & \text{for } i = 1,\ldots,k-l-1, \\ 1 & \text{for } i = k-l, \\ l & \text{for } i = k, \end{cases}$$

for $l = k$, let $C_1(l,l,n)$ count the number of partitions λ of n such that $\ell(\lambda) = l$, with parts not exceeding $3l-1$ and the parts differ by at least 2,
for $k < l \le 2k$, let $C_1(l,k,n)$ count the number of partitions λ of n such that $\ell(\lambda) = l$, $k \le \lambda_i \le k+l-1$ and

$$\lambda_i - \lambda_{i+1} \ge \begin{cases} 2 & \text{for } i = 1,\ldots,l-k-1, \\ 1 & \text{for } i = l-k. \end{cases}$$

Let

$$C_1(k,n) = \sum_{l=0}^{2k} C_1(l,k,n)$$

and let $A_1(n) = \sum_{k=0}^{\infty} \sum_{j=0}^{n} B_1(k,j) C_1(k,n-j)$ and $D_1(n)$ denote the number of partitions of n with "$n+t$ - copies of n" such that parts are congruent *to* $\pm 1, \pm 3 \pmod 7$, *first two copies of parts* $\pm 1 \pmod 7$ *and only first copy of parts* $\pm 3 \pmod 7$ *are used. Then*

$$A_1(n) = D_1(n) \textit{ for all } n.$$

Example. This example demonstrates the theorem for $n = 5$, by showing that $A_1(5) = D_1(5) = 11$.
First, $D_1(5) = 11$ with the following coloured partitions of 5 contributing:

$$4_1 + 1_1,\ 4_1 + 1_2,\ 3_1 + 1_1 + 1_1,\ 3_1 + 1_2 + 1_1,\ 3_1 + 1_2 + 1_2,$$

$$1_1 + 1_1 + 1_1 + 1_1 + 1_1,\ 1_2 + 1_1 + 1_1 + 1_1 + 1_1,\ 1_2 + 1_2 + 1_1 + 1_1 + 1_1,$$

$$1_2 + 1_2 + 1_2 + 1_1 + 1_1,\ 1_2 + 1_2 + 1_2 + 1_2 + 1_1,\ 1_2 + 1_2 + 1_2 + 1_2 + 1_2.$$

To compute $A_1(5)$ we first note that $C_1(l, 2, 5) = 0$ for $l = 0, 3, 4$ and $C_1(1, 2, 5) = 1$ with $3 + 2 + 0$ contributing and $C_1(2, 2, 5) = 1$ with $4 + 1$ contributing. Hence

$$C_1(2, 5) = \sum_{l=0}^{4} C_1(l, 2, 5) = 2$$

In a similar fashion we find that

$$C_1(0, 0) = 1,$$

$$C_1(1, 1) = 2, C_1(1, 2) = 1, C_1(1, 3) = 1,$$

$$C_1(2, 3) = 1, C_1(2, 4) = 3, C_1(2, 5) = 2$$

and the rest of the C_1's are zero.
The corresponding $B_1(k, j)$ are

$$B_1(0, 0) = 1,$$

$$B_1(1, 0) = 1, B_1(1, 1) = 1, B_1(1, 2) = 1,$$

$$B_1(1, 3) = 1, B_1(1, 4) = 1, B_1(1, 5) = 1,$$

$$B_1(2, 0) = 1, B_1(2, 1) = 1, B_1(2, 2) = 2,$$

$$B_1(2, 3) = 2, B_1(2, 4) = 3, B_1(2, 5) = 3,$$

and the rest of the B_1's are zero, hence

$$\begin{aligned} A_1(5) &= \sum_{k=0}^{2} \sum_{j=0}^{5} B_1(k, j) C_1(k, 5 - j) \\ &= (0 + 0 + 0 + 0 + 0 + 0) + (0 + 0 + 1 + 1 + 2 + 0) \\ &+ (2 + 3 + 2 + 0 + 0 + 0) \\ &= 11. \end{aligned}$$

Theorem 2.2 *For $0 \le l < k$, let $C_2(l,k,n)$ count the number of partitions λ of n such that $\lambda_1 \le 2k$, $\ell(\lambda) \le 2k-l$ and*

$$\lambda_i - \lambda_{i+1} \ge \begin{cases} 2 & \text{for } i = 1,\dots,k-l-1, \\ 1 & \text{for } i = k-l, \\ l & \text{for } i = k, \\ 1 & \text{for } i = l, \end{cases}$$

for $l = k$, let $C_2(l,l,n)$ count the number of partitions λ of n such that $\ell(\lambda) = l$, $2 \le \lambda_i \le 3l$ and the parts differ by at least 2,
for $k < l \le 2k$, let $C_2(l,k,n)$ count the number of partitions λ of n such that $\ell(\lambda) = l$, $k+1 \le \lambda_i \le k+l$ and

$$\lambda_i - \lambda_{i+1} \ge \begin{cases} 2 & \text{for } i = 1,\dots,l-k-1, \\ 1 & \text{for } i = l-k. \end{cases}$$

Let

$$C_2(k,n) = \sum_{l=0}^{2k} C_2(l,k,n)$$

and let $A_2(n) = \sum_{k=0}^{\infty}\sum_{j=0}^{n} B_1(k,j)C_2(k,n-j)$ and $D_2(n)$ count the number of ordinary partitions of n such that parts are congruent to $\pm1, \pm3, \pm2 \pmod 7$. Then

$$A_2(n) = D_2(n) \text{ for all } n.$$

Theorem 2.3 *For $0 \le l < k$, let $C_3(l,k,n)$ count the number of partitions λ of n such that $\lambda_1 \le 2k+1$, $\ell(\lambda) \le 2k-l$ and*

$$\lambda_i - \lambda_{i+1} \ge \begin{cases} 2 & \text{for } i = 1,\dots,k-l-1, \\ 1 & \text{for } i = k-l, \\ l+1 & \text{for } i = k, \\ 1 & \text{for } i = l, \end{cases}$$

for $l = k$, let $C_3(l,l,n)$ count the number of partitions λ of n such that $\ell(\lambda) = l$, $3 \le \lambda_i \le 3l+1$ and the parts differ by at least 2,
for $k < l \le 2k$, let $C_3(l,k,n)$ count the number of partitions λ of n such that $\ell(\lambda) = l$, $k+1 \le \lambda_i \le k+l+1$ and

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 & \text{for } i = 1, \ldots, l-k-1, \\ 1 & \text{for } i = l-k, \\ 1 & \text{for } i = k. \end{cases}$$

Let

$$C_3(k,n) = \sum_{l=0}^{2k} C_3(l,k,n)$$

and let $A_3(n) = \sum_{k=0}^{\infty} \sum_{j=0}^{n} B_1(k,j) C_3(k,n-j)$ *and* $D_3(n)$ *count the number of ordinary partitions of* n *such that parts are congruent to* $\pm 2, \pm 3 \pmod 7$*, first two copies of the part congruent to* $\pm 3 \pmod 7$ *and only first copy of parts* $\pm 2 \pmod 7$ *are used. Then*

$$A_3(n) = D_3(n) \text{ for all } n.$$

Theorem 2.4 *For* $0 \leq l < k$*, let* $C_4(l,k,n)$ *count the number of partitions* λ *of* n *such that* $\lambda_1 \leq 2k$*,* $\ell(\lambda) \leq 2k - l + 1$ *and*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 & \text{for } i = 1, \ldots, k-l-1, \\ 1 & \text{for } i = k-l, \\ l & \text{for } i = k, \\ 1 & \text{for } i = l, \end{cases}$$

for $l = k$*, let* $C_4(l,l,n)$ *count the number of partitions* λ *of* n *such that* $\ell(\lambda) = l$*,* $2 \leq \lambda_i \leq 3l + 1$ *and the parts differ by at least* 2*,*
for $k < l \leq 2k$*, let* $C_4(l,k,n)$ *count the number of partitions* λ *of* n *such that* $\ell(\lambda) = l$*,* $k + 1 \leq \lambda_i \leq k + l + 1$ *and*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 & \text{for } i = 1, \ldots, l-k-1, \\ 1 & \text{for } i = l-k. \end{cases}$$

Let

$$C_4(k,n) = \sum_{l=0}^{2k} C_4(l,k,n)$$

and let $A_4(n) = \sum_{k=0}^{\infty} \sum_{j=0}^{n} B_1(k,j) C_4(k,n-j)$ *and* $D_4(n)$ *count the number of ordinary partitions of* n *such that parts are congruent to* $\pm 1, \pm 2 \pmod 7$*, first two copies of the part congruent to* $\pm 2 \pmod 7$ *and only first copy of parts* $\pm 1 \pmod 7$ *are used. Then*

$$A_4(n) = D_4(n) \text{ for all } n.$$

In the next section we give the detailed proof of Theorem 2.1 and in Section 4 we sketch the proofs of the remaining theorems.

3 Proof of Theorem 2.1

We will prove this theorem in three steps as follows:

Step 1. It is clear that

$$\sum_{n=0}^{\infty} B_1(k,n)q^n = \frac{1}{(q;q)_k}. \tag{3.1}$$

Step 2. Note that

$$\sum_{l=0}^{2k} q^{k^2+l^2-lk} \begin{bmatrix} 2k \\ l \end{bmatrix} = \sum_{l=0}^{k-1} q^{(k-l)^2+lk} \begin{bmatrix} 2k \\ l \end{bmatrix} + q^{k^2} \begin{bmatrix} 2k \\ l \end{bmatrix} + \sum_{l=k+1}^{2k} q^{(l-k)^2+lk} \begin{bmatrix} 2k \\ l \end{bmatrix}.$$

Now for $0 \le l < k$, by the definition of the Gaussian polynomial, $\begin{bmatrix} 2k \\ l \end{bmatrix} = \begin{bmatrix} 2k \\ 2k-l \end{bmatrix}$ generates partitions into at most $2k-l$ parts such that no part exceeds l. Multiplying $\begin{bmatrix} 2k \\ l \end{bmatrix}$ by q^{lk} may be interpreted by adding l to each of the first k parts of the partitions being generated. In other words, $q^{lk}\begin{bmatrix} 2k \\ l \end{bmatrix}$ is the generating function of partitions into at most $2k-l$ parts, no part exceeding $2l$ and the difference between the k^{th} and $(k+1)^{th}$ part is at least l. Multiplying $q^{lk}\begin{bmatrix} 2k \\ l \end{bmatrix}$ by $q^{(k-l)^2}$ means that we are adding $2(k-l)-1$ to the first part, $2(k-l)-3$ to the second part,$\cdots$, 1 to the $(k-l)^{th}$ part. So, $q^{(k-l)^2+lk}\begin{bmatrix} 2k \\ l \end{bmatrix}$ generates partitions into at most $2k-l$ parts where parts do not exceed $2k-1$, difference between k^{th} and $(k+1)^th$ parts is $\ge l$, difference between $(k-l)^{th}$ and $(k-l+1)^{th}$ parts is ≥ 1 and the first $k-l$ parts differing by at least 2.
For $l = k$, by definition of Gaussian Polynomial, $\begin{bmatrix} 2k \\ k \end{bmatrix}$ generates partitions into at most k parts such that no part exceeds k. Multiplying $\begin{bmatrix} 2k \\ l \end{bmatrix}$ by q^{k^2} means that we are adding $2k-1$ to the first part, $2k-3$ to the second part,$\cdots$, 1 to the last part. In other words, $q^{k^2}\begin{bmatrix} 2k \\ l \end{bmatrix}$ generates partitions into exactly k parts, parts not exceeding $3k-1$ and differing by at least 2, or it is equivalent to say that this generates partitions into exactly l parts, parts not exceeding $3l-1$ and differing by at least 2.
Finally, for $k < l \le 2k$, by definition of Gaussian Polynomial, $\begin{bmatrix} 2k \\ l \end{bmatrix}$ generates partitions into at most l parts such that no part exceeds $2k-l$. Multiplying $\begin{bmatrix} 2k \\ l \end{bmatrix}$ by q^{lk} may be interpreted by adding k to each of the first l parts of the partitions being generated. In other words, $q^{lk}\begin{bmatrix} 2k \\ l \end{bmatrix}$ generates partitions into exactly l parts such that parts are $\ge k$ and not exceeding $3k-l$. Further multiplication of $q^{lk}\begin{bmatrix} 2k \\ l \end{bmatrix}$

by $q^{(l-k)^2}$ means that we are adding $2(l-k)-1$ to the first part, $2(l-k)-3$ to the second part, $\cdots$, 1 to the $(l-k)$ part. So, $q^{(l-k)^2+lk}\begin{bmatrix}2k\\l\end{bmatrix}$ generates partitions into exactly l parts such that smallest part being at least k, not exceeding $k+l-1$ and the first $l-k$ parts differing by at least 2, hence,

$$\sum_{n=0}^{n_1} C_1(l,k,n)q^n = q^{k^2-lk+l^2}\begin{bmatrix}2k\\l\end{bmatrix}, \tag{3.2}$$

where $n_1 = k(k+l)$, now (3.2) implies

$$\sum_{n=0}^{n_2} C_1(k,n)q^n = \sum_{n=0}^{n_1}\sum_{l=0}^{\infty} C_1(l,k,n)q^n = \sum_{l=0}^{\infty} q^{k^2-lk+l^2}\begin{bmatrix}2k\\l\end{bmatrix},$$

where $n_2 = 3k^2$ and noting $C_1(l,k,n) = 0$ for $l > 2k$, equivalently we may write above as

$$\sum_{n=0}^{\infty} C_1(k,n)q^n = \sum_{l=0}^{\infty} q^{k^2-lk+l^2}\begin{bmatrix}2k\\l\end{bmatrix}, \tag{3.3}$$

since $C_1(k,n) = 0$ for $n > n_2$.

Step 3. As,

$$A_1(n) = \sum_{k,j} B_1(k,j)C_1(k,n-j)$$

So,

$$\begin{aligned}
\sum_{n=0}^{\infty} A_1(n)q^n &= \sum_{k,j,n} B_1(k,j)C_1(k,n-j)q^n \\
&= \sum_{k,j,n} B_1(k,j)C_1(k,n)q^{n+j} \\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} B_1(k,j)q^j\right)\left(\sum_{n=0}^{\infty} C_1(k,n)q^n\right) \\
&= \sum_{k=0}^{\infty}\frac{1}{(q;q)_k}\sum_{l=0}^{\infty} q^{k^2-lk+l^2}\begin{bmatrix}2k\\l\end{bmatrix} \\
&= \sum_{l,k=0}^{\infty}\frac{1}{(q)_k}q^{k^2-lk+l^2}\begin{bmatrix}2k\\l\end{bmatrix} \\
&= \frac{1}{(q,q,q^3,q^4,q^6,q^6,q^7)_\infty} \\
&= \sum_{n=0}^{\infty} D_1(n)q^n
\end{aligned}$$

Coefficient comparison in the extremes of the above leads to Theorem 2.1.

4 Sketch of proofs of Theorems 2.2–2.4

Since, the proofs of Theorems 2.2–2.4 are similar to that of Theorem 2.1, we omit the details and give only a brief sketch.

Following the proof of Theorem 2.1, the extra factor q^l in the left hand side of (1.5) may be interpreted by adding 1 to each of the first l parts which proves Theorem 2.2. Similarly, the extra factor q^{k+l} in the left hand side of (1.6) may be interpreted by adding 1 to each of the first l parts and then adding 1 to each of the first k parts which leads to Theorem 2.3.

Now, replacing $\begin{bmatrix}2k\\l\end{bmatrix}$ by $\begin{bmatrix}2k+1\\l\end{bmatrix}$ and then again following the steps of Theorem 2.1, extra factor q^l in the left hand side of (1.7) may be interpreted by adding 1 to each of the first l parts which proves Theorem 2.4.

5 Conclusion

In this paper we have given partition theoretic interpretations of Identities (1.4)–(1.7). The analytic generalization of (1.4)–(1.7) is available in [6]. The obvious questions which arise from this work are;
(a) Is it possible to find a combinatorial generalization of Theorems 2.1–2.4.
(b) Can we find the combinatorial counterpart of the generalized analytic identity available in [6].

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