The signed cycle domatic number of a graph *

Wei Meng[a], † and Ruixia Wang[a]

[a]School of Mathematical Sciences, Shanxi University, Taiyuan, P. R. China

Abstract: For a nonempty graph $G = (V(G), E(G))$, a *signed cycle dominating function* on G is introduced by Xu in 2009 as a function $f : E(G) \to \{1, -1\}$ such that $\sum_{e \in E(C)} f(e) \geq 1$ for any induced cycle C of G. A set $\{f_1, f_2, ..., f_d\}$ of distinct signed cycle dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq 1$ for each $e \in E(G)$, is called a *signed cycle dominating family* (of functions) on G. The maximum number of functions in a signed cycle dominating family on G is the *signed cycle domatic number* of G, denoted by $d'_{sc}(G)$. In this paper we study the signed cycle domatic numbers in graphs and present sharp bounds for $d'_{sc}(G)$. In addition, we determine the signed cycle domatic number of some special graphs.

Keywords: Induced cycle; Signed cycle dominating function; Signed cycle domatic number

1 Introduction and Terminology

In this paper we continue the study of signed cycle dominating functions in graphs, which was first introduced by Xu in [7]. According to different kinds of dominating functions, one proposed corresponding domatic numbers. In some sense, domatic number is a dual concept to the domination number. Up to now many kinds of domatic number have been investigated, such as signed domatic number [6], signed star domatic number [1], signed edge-domatic number[3], signed Roman k-domatic number [5], etc. In this paper, we consider the signed cycle domatic number.

For graph-theoretical notation and terminology not defined here we follow [2]. Specially, $G = (V(G), E(G))$ is a simple finite graph with vertex

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†Corresponding author. *E-mail address:* mengwei@sxu.edu.cn

set $V(G)$ and edge set $E(G)$. The number of vertices (edges, respectively) of G is called the *order* (*size*, respectively) of G. If $S \subseteq V(G)$, then $G[S]$ will denote the subgraph of G induced by S. A cycle C of G is said to be an *induced cycle* if $G[V(C)] = C$. A cycle C is called an *odd (even,* respectively) cycle if its length is odd (even, respectively). We write K_n for the *complete graph* of order n, $K_{p,q}$ for the *complete bipartite graph*, C_n (P_n, respectively) for the cycle (path, respectively) of order n. Finally, $\overline{G}$ will denote the complement of G.

A signed cycle dominating function (SCDF) on a nonempty graph G is defined in [7] as a function $f : E(G) \to \{1, -1\}$ such that $\sum_{e \in E(C)} f(e) \geq 1$ for each induced cycle C of G. The *weight* of an SCDF f is the value $w(f) = \Sigma_{e \in E(G)} f(e)$. For any graph G, if $E(G) \neq \emptyset$, then the *signed cycle domination number* of G, denoted by $\gamma'_{sc}(G)$, equals the minimum weight of an SCDF on G; if $E(G) = \emptyset$, then we define $\gamma'_{sc}(G) = 0$. A $\gamma'_{sc}(G)$*-function* is an SCDF on G with weight $\gamma'_{sc}(G)$. The signed cycle domination number of graphs and digraphs were investigated in [7] and [4], respectively.

A set $\{f_1, f_2, ..., f_d\}$ of distinct signed cycle dominating functions on G with the property that $\sum_{i=1}^{d} f_i(e) \leq 1$ for each $e \in E(G)$, is called a *signed cycle dominating family* (of functions) on G. The maximum number of functions in a signed cycle dominating family (SCD family) on G is the *signed cycle domatic number* of G, denoted by $d'_{sc}(G)$. A $d'_{sc}(G)$-SCD family is an SCD family on G consisting of $d'_{sc}(G)$ distinct signed cycle dominating functions on G. In order to investigate the signed cycle domatic number of G, it is reasonable to claim that $E(G) \neq \emptyset$. Thus we assume throughout this paper that the graphs are all nonempty. The signed cycle domatic number is well-defined and $d'_{sc}(G) \geq 1$ for all graphs G with $E(G) \neq \emptyset$, since the set consisting of any SCDF forms an SCD family on G.

For an arbitrary graph G, it is very hard to determine the value of $d'_{sc}(G)$. Thus, determine $d'_{sc}(G)$ for some special graphs is of interest. Our purpose in this paper is to give an original study on the signed cycle domatic number in graphs. We first derive basic properties and bounds for the signed cycle domatic number of a graph, and then, we determine the signed cycle domatic number of some special graphs, such as paths, stars, cycles, fans, wheels, complete graphs and complete bipartite graphs.

We make use of the following results in this paper.

Theorem A ([7]) *$\gamma'_{sc}(G) = -|E(G)|$ if and only if G has no cycles.*

Theorem B ([7]) *$\gamma'_{sc}(G) = |E(G)|$ if and only if $G = \overline{K}_n$ for some positive integer n.*

Theorem C ([7]) *Let G be a graph which is not a tree. Then $\gamma'_{sc}(G) \geq |E(G)| - 2|V(G)| + 4$.*

2 Bounds on the signed cycle domatic number

In this section we present basic properties and sharp bounds on the signed cycle domatic number of a graph.

Theorem 2.1 *Let G be a graph without any cycle and of size m. Then $d'_{sc}(G) = 2^m$.*

Proof. Since G has no cycles, for any SCDF f on G and arbitrary edge $e \in E(G)$, $f(e)$ may equal 1 or -1. So there are 2^m distinct signed cycle dominating functions on G in all, denoted by $f_1, f_2, ..., f_{2^m}$. It is clear that $\sum_{i=1}^{2^m} f_i(e) = 0 \leq 1$ for each $e \in E(G)$. Therefore, $\{f_1, f_2, ..., f_{2^m}\}$ forms an SCD family on G, and then, $d'_{sc}(G) = 2^m$. □

Corollary 2.2 $d'_{sc}(P_n) = 2^{n-1}$, $d'_{sc}(K_{1,n}) = 2^n$.

In view of Theorem 2.1, we only need to consider the graphs containing at least one cycle in the following.

Theorem 2.3 *Let G be a graph with a shortest cycle of length ℓ. Then*

$$d'_{sc}(G) \leq \begin{cases} \ell & \ell \equiv 1 \ (mod\ \ 2); \\ \frac{\ell}{2} & \ell \equiv 0 \ (mod\ \ 2). \end{cases}$$

Moreover, if $d'_{sc}(G) = \ell$, then ℓ is odd and for each SCD family $\{f_1, f_2, ..., f_d\}$ on G with $d = d'_{sc}(G)$ and each shortest cycle C, $\sum_{e\in E(C)} f_i(e) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(e) = 1$ for all $e \in E(C)$.

Proof. Let C be a shortest cycle of G and $\{f_1, f_2, ..., f_d\}$ be an SCD family on G with $d = d'_{sc}(G)$. Then C is an induced cycle and we deduce that

$$d \leq \sum_{i=1}^{d} \sum_{e\in E(C)} f_i(e) = \sum_{e\in E(C)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e\in E(C)} 1 = \ell. \tag{1}$$

Specially, if ℓ is even, then $\sum_{e\in E(C)} f_i(e) \geq 2$ for each $i \in \{1, 2, ..., d\}$. Thus, we have

$$2d \leq \sum_{i=1}^{d} \sum_{e\in E(C)} f_i(e) = \sum_{e\in E(C)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e\in E(C)} 1 = \ell.$$

Therefore, when ℓ is odd, $d'_{sc}(G) \leq \ell$; when ℓ is even, $d'_{sc}(G) \leq \frac{\ell}{2}$.

If $d'_{sc}(G) = \ell$, then ℓ is odd and the two inequalities occurring in (1) become equalities. Hence for the SCD family $\{f_1, f_2, ..., f_d\}$ on G with $d = d'_{sc}(G)$ and each shortest cycle C, $\sum_{e\in E(C)} f_i(e) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(e) = 1$ for all $e \in E(C)$. □

Theorem 2.3 immediately implies the following Nordhaus-Gaddum type result.

Corollary 2.4 *If both G and $\overline{G}$ have cycles, then*

$$d'_{sc}(G) + d'_{sc}(\overline{G}) \leq 2|V(G)|.$$

The upper bound in Corollary 2.4 is sharp for $G \cong C_5$ (see Theorem 3.1).

Theorem 2.5 *If G is a graph of size m, then*

$$\gamma'_{sc}(G) \cdot d'_{sc}(G) \leq m.$$

Moreover, if $\gamma'_{sc}(G) \cdot d'_{sc}(G) = m$, then G has at least one cycle and for each SCD family $\{f_1, f_2, ..., f_d\}$ on G with $d = d'_{sc}(G)$, every function f_i is a $\gamma'_{sc}(G)$-function and $\Sigma_{i=1}^{d} f_i(e) = 1$ for all $e \in E(G)$.

Proof. Let $\{f_1, f_2, ..., f_d\}$ be an SCD family on G with $d = d'_{sc}(G)$. Then

$$d \cdot \gamma'_{sc}(G) = \sum_{i=1}^{d} \gamma'_{sc}(G) \leq \sum_{i=1}^{d} \sum_{e\in E(G)} f_i(e) = \sum_{e\in E(G)} \sum_{i=1}^{d} f_i(e) \leq m. \quad (2)$$

If $\gamma'_{sc}(G) \cdot d'_{sc}(G) = m$, then by Theorem A, the graph G has at least one cycle and the two inequalities occurring in (2) become equalities. Hence, for the SCD family $\{f_1, f_2, ..., f_d\}$ on G and for each i, $\sum_{e\in E(G)} f_i(e) = \gamma'_{sc}(G)$. Thus each function f_i is a $\gamma'_{sc}(G)$-function and $\sum_{i=1}^{d} f_i(e) = 1$ for all $e \in E(G)$. □

Corollary 2.6 *Let G be a graph of size m and order n. If G has at least one cycle and $2n - m \leq 3$, then*

$$d'_{sc}(G) \leq \lfloor \frac{m}{m - 2n + 4} \rfloor.$$

Proof. From the assumption of this corollary we know that G is not a tree. Combining Theorem C with Theorem 2.5 we deduce that

$$(m-2n+4)d'_{sc}(G) \leq \gamma'_{sc}(G) \cdot d'_{sc}(G) \leq m.$$

It follows from $2n-m \leq 3$ that $m-2n+4 \geq 1$ and thus, $d'_{sc}(G) \leq \lfloor \frac{m}{m-2n+4} \rfloor$. □

Corollary 2.6 directly implies the following result.

Corollary 2.7 *$d'_{sc}(K_5) \leq 2$, $d'_{sc}(K_6) \leq 2$, $d'_{sc}(K_n) = 1$ for $n \geq 7$.*

From Corollary 2.7 we can see that for any complete graph K_n $(n \geq 5)$ the upper bound in Corollary 2.6 improves this one in Theorem 2.3.

Theorem 2.8 *Let G be a graph containing at least one cycle and of size m. Then*

$$\gamma'_{sc}(G) + d'_{sc}(G) \leq m+1.$$

If $\gamma'_{sc}(G) + d'_{sc}(G) = m+1$, then $\gamma'_{sc}(G) = 1$, $d'_{sc}(G) = m$, and hence, G is an odd cycle.

Proof. It follows from Theorem 2.5 that

$$\gamma'_{sc}(G) + d'_{sc}(G) \leq \frac{m}{d'_{sc}(G)} + d'_{sc}(G).$$

According to Theorem 2.3, we have $1 \leq d'_{sc}(G) \leq m$. Using these bounds and the fact that the function $g(x) = m/x + x$ is decreasing for $1 \leq x \leq \sqrt{m}$ and increasing for $\sqrt{m} \leq x \leq m$, we obtain

$$\gamma'_{sc}(G) + d'_{sc}(G) \leq \max\{1+m, m+1\} = m+1,$$

and the desired bound is proved.

Now assume that $\gamma'_{sc}(G) + d'_{sc}(G) = m+1$. The above inequality leads to

$$m+1 = \gamma'_{sc}(G) + d'_{sc}(G) \leq \frac{m}{d'_{sc}(G)} + d'_{sc}(G) \leq m+1.$$

Since $E(G) \neq \emptyset$, then $\gamma'_{sc}(G) \neq m$ by Theorem B. This implies that $d'_{sc}(G) = m$ and $\gamma'_{sc}(G) = 1$. Let ℓ be the length of the shortest cycle of G. It follows from $m = d'_{sc}(G) \leq \ell \leq m$ that $\ell = m$ is odd and then $G \cong C_m$. □

3 Value of d'_{sc} for some graphs

In this section we investigate the value of d'_{sc} for cycles, complete graphs, fans, complete bipartite graphs and wheels. We first consider odd cycles.

Theorem 3.1 $d'_{sc}(C_{2p+1}) = 2p + 1$ for $p \in Z^+$.

Proof. Let $E(C_{2p+1}) = \{e_0, e_1, ..., e_{2p}\}$ and define

$$f_j(e_i) = \begin{cases} -1 & i = (j-1)p, (j-1)p+1, ..., (j-1)p+p-1; \\ 1 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, ..., 2p+1$, where the subscripts of e_i are taken module $2p + 1$. It is not difficult to check that each f_j is an SCDF on C_{2p+1} and $\{f_1, f_2, ..., f_{2p+1}\}$ is an SCD family on C_{2p+1}. This yields $d'_{sc}(C_{2p+1}) = 2p + 1$ by Theorem 2.3. □

Theorem 3.1 shows that the upper bound in Theorem 2.5 is sharp since $\gamma'_{sc}(C_p) = 1$ for any odd positive integer p and the upper bound in Theorem 2.3 is also sharp when ℓ is odd.

Theorem 3.2 $d'_{sc}(C_{2p}) = \max\{d \mid d \leq p \text{ and } d \text{ is odd}\}$ for $p \geq 2$.

Proof. Let $E(C_{2p}) = \{e_0, e_1, ..., e_{2p-1}\}$ and denote $d_0 = \max\{d \mid d \leq p \text{ and } d \text{ is odd}\}$. Define

$$f_j(e_i) = \begin{cases} -1 & i = (j-1)(p-1), (j-1)(p-1)+1, ..., \\ & \quad (j-1)(p-1)+p-2; \\ 1 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, ..., d_0$, where the subscripts of e_i are taken module $2p$. It is not difficult to check that each f_j is an SCDF on C_{2p} and $\{f_1, f_2, ..., f_{d_0}\}$ is an SCD family on C_{2p}. Hence, $d'_{sc}(C_{2p}) \geq d_0$.

Suppose there exists an SCD family $\{f_1, f_2, ..., f_d\}$ on C_{2p} with $d > d_0$. Then we construct a matrix $A = (a_{ij})_{2p \times d}$, where $a_{ij} = f_j(e_i)$, for $i = 0, 1, ..., 2p-1$ and $j = 1, 2, ..., d$. Since each f_j is an SCDF on C_{2p}, every column of A contains at most $p - 1$ elements -1, and then A contains at most $(p-1)d$ elements -1. On the other hand, since $\{f_1, f_2, ..., f_d\}$ is an SCD family on C_{2p}, every row of A contains at least $\lfloor \frac{d}{2} \rfloor$ elements -1, and thus, A contains at least $\lfloor \frac{d}{2} \rfloor 2p$ elements -1. If d is even, then

$$(p-1)d < pd = \frac{d}{2}2p = \lfloor \frac{d}{2} \rfloor 2p,$$

which yields a contradiction. So d is odd. Since $d > d_0$ and d_0 is the largest odd number which is no more than p, we have $d > p$. However, Theorem 2.3 implies $d \leq p$, a contradiction. So, $d'_{sc}(C_{2p}) \leq d_0$.

From the discussion above we obtain $d'_{sc}(C_{2p}) = d_0 = \max\{d \mid d \leq p \text{ and } d \text{ is odd}\}$. □

According to Theorem 3.2 we have $d'_{sc}(C_{4p+2}) = 2p+1$ for $p \geq 1$, which shows that the upper bound in Theorem 2.3 is sharp when ℓ is even.

Theorem 3.3 *If $n \geq 2$, then*

$$d'_{sc}(K_n) = \begin{cases} 2 & n = 2; \\ 3 & n = 3, 4; \\ 1 & n \geq 5. \end{cases}$$

Proof. Theorem 2.1, Theorem 3.1 and Corollary 2.7 imply $d'_{sc}(K_2) = 2$, $d'_{sc}(K_3) = 3$ and $d'_{sc}(K_n) = 1$ for $n \geq 7$, respectively. So we only need to consider the case $n = 4, 5, 6$.

First we consider the case $n = 4$. Let $V(K_4) = \{v_0, v_1, v_2, v_3\}$ and define

$$f_1(e) = \begin{cases} -1 & e = v_0v_1, v_2v_3; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_2(e) = \begin{cases} -1 & e = v_1v_2, v_0v_3; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_3(e) = \begin{cases} -1 & e = v_0v_2, v_1v_3; \\ 1 & \text{otherwise}. \end{cases}$$

It is not difficult to check that f_1, f_2, f_3 are signed cycle dominating functions on K_4 and $\{f_1, f_2, f_3\}$ forms an SCD family on K_4. Thus $d'_{sc}(K_4) \geq 3$. Combining with Theorem 2.3 we have $d'_{sc}(K_4) = 3$.

Next we consider the case $n = 5$. Suppose to the contrary that $d'_{sc}(K_5) \geq 2$. Then by Theorem 2.3, we have $2 \leq d'_{sc}(K_5) \leq 3$. For any SCDF on K_5, at most two edges can be assigned to -1, otherwise there must exist a 3-cycle containing at least two -1 edges, which contradicts the definition of an SCDF. Also we note that there are altogether 10 edges in K_5. So at least four edges are assigned to 1 by all functions in any $d'_{sc}(K_5)$-SCD family, which contradicts the definition of an SCD family. Therefore, we have $d'_{sc}(K_5) = 1$.

Finally we consider the case $n = 6$. By a similar argument as in case $n = 5$, we deduce that $d'_{sc}(K_6) = 1$, where we need to note that for any SCDF on K_6, at most 3 edges can be assigned to -1 and there are altogether 15 edges in K_6. □

A *fan* of order n is a graph obtained from a path P_{n-1} and a vertex v by joining it to each vertex of P_{n-1}, denoted by $P_{n-1} \vee K_1$.

Theorem 3.4 *Let $G = P_{n-1} \vee K_1$ be a fan of order $n \geq 3$, then $d'_{sc}(G) = 3$.*

Proof. Let $V(P_{n-1}) = \{v_1, v_2, ..., v_{n-1}\}$ and $V(K_1) = \{v\}$. Assume, without loss of generality, that n is odd. Then define

$$f_1(e) = \begin{cases} -1 & e = v_1v_2, v_2v_3, ..., v_{n-2}v_{n-1}; \\ 1 & \text{otherwise,} \end{cases}$$

$$f_2(e) = \begin{cases} -1 & e = vv_1, vv_3, ..., vv_{n-2}; \\ 1 & \text{otherwise,} \end{cases}$$

$$f_3(e) = \begin{cases} -1 & e = vv_2, vv_4, ..., vv_{n-1}; \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that $\{f_1, f_2, f_3\}$ is an SCD family on G. Combining this with Theorem 2.3 we deduce that $d'_{sc}(G) = 3$. □

Theorem 3.5 $d'_{sc}(K_{p,q}) = 1$ for $p, q \geq 2$.

Proof. Suppose to the contrary that $d'_{sc}(K_{p,q}) \geq 2$. Since the length of the shortest cycle of $K_{p,q}$ is 4, it follows from Theorem 2.3 that $d'_{sc}(K_{p,q}) \leq 2$. Therefore, $d'_{sc}(K_{p,q}) = 2$. For any SCDF on $K_{p,q}$, at most one edge can be assigned to -1, otherwise there must exist an induced 4-cycle containing at least two -1 edges, which contradicts the definition of an SCDF. Also we note that there are altogether pq (≥ 4) edges in $K_{p,q}$. So at least two edges are assigned to 1 by all functions in any $d'_{sc}(K_{p,q})$-SCD family, which contradicts the definition of an SCD family. Therefore, $d'_{sc}(K_{p,q}) = 1$ for $p, q \geq 2$. □

A *wheel* $W_n = C_n \vee K_1$ of order $n+1$ is a graph obtained from a cycle C_n and a vertex v by joining it to each vertex of C_n.

Theorem 3.6 Let $W_n = C_n \vee K_1$ be a wheel with $n \geq 3$, then

$$d'_{sc}(W_n) = \begin{cases} 1 & n = 4, 5, 8; \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Note that W_n have $n+1$ induced cycles altogether, where one induced n-cycle and n induced 3-cycles. Since the length of the shortest cycle is 3, we have $d'_{sc}(W_n) \leq 3$.

Firstly, we consider the case $n = 4$. Suppose to the contrary that $d'_{sc}(W_4) \geq 2$. Then $2 \leq d'_{sc}(W_4) \leq 3$. For any SCDF on W_4, at most two edges can be assigned to -1, otherwise there exists either a 3-cycle or a 4-cycle containing at least two -1 edges, which contradicts the definition of an SCDF. Also we note that there are altogether 8 edges in W_4. So at least two edges are assigned to 1 by all functions in any $d'_{sc}(W_4)$-SCD family, which contradicts the definition of an SCD family. Therefore, we have $d'_{sc}(W_4) = 1$.

Secondly, we consider the case $n = 5$. By a similar argument as in case $n = 4$, we deduce that $d'_{sc}(W_5) = 1$, where we need to note that for any SCDF on W_5, at most 3 edges can be assigned to -1 and there are altogether 10 edges in W_5.

Thirdly, we consider the case $n = 8$. Note that for any SCDF on W_8, at most 5 edges can be assigned to -1 and there are 16 edges in W_8 in all. So by a similar argument as in case $n = 4$, we deduce that $d'_{sc}(W_8) = 1$.

In the following we only need to consider the case $n \geq 3$ and $n \neq 4, 5, 8$. Let $V(C_n) = \{v_0, v_1, ..., v_{n-1}\}$ and $V(K_1) = \{v\}$. We distinguish three cases as follows.

Case 1. $n \equiv 0 \pmod 3$.

Define

$$f_1(e) = \begin{cases} -1 & e = vv_0, vv_3, ..., vv_{n-3}, v_1v_2, v_4v_5, ..., v_{n-2}v_{n-1}; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_2(e) = \begin{cases} -1 & e = vv_1, vv_4, ..., vv_{n-2}, v_2v_3, v_5v_6, ..., v_{n-1}v_0; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_3(e) = \begin{cases} -1 & e = vv_2, vv_5, ..., vv_{n-1}, v_0v_1, v_3v_4, ..., v_{n-3}v_{n-2}; \\ 1 & \text{otherwise}. \end{cases}$$

It is not difficult to check that $\{f_1, f_2, f_3\}$ is an SCD family on W_n for $n \equiv 0 \pmod 3$. Recall $d'_{sc}(W_n) \leq 3$. So $d'_{sc}(W_n) = 3$, where $n \equiv 0 \pmod 3$.

Case 2. $n \equiv 1 \pmod 3$ and $n \neq 4$.

Define

$$f_1(e) = \begin{cases} -1 & e = vv_0, vv_3, ..., vv_{n-4}, v_1v_2, v_4v_5, ..., \\ & \quad v_{n-3}v_{n-2}, v_{n-2}v_{n-1}; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_2(e) = \begin{cases} -1 & e = vv_1, vv_4, ..., vv_{n-3}, vv_{n-1}, v_2v_3, \\ & \quad v_5v_6, ..., v_{n-5}v_{n-4}; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_3(e) = \begin{cases} -1 & e = vv_2, vv_5, ..., vv_{n-2}, v_0v_1, v_3v_4, ..., \\ & \quad v_{n-4}v_{n-3}, v_{n-1}v_0; \\ 1 & \text{otherwise}. \end{cases}$$

It is easy to see that $\{f_1, f_2, f_3\}$ is an SCD family on W_n for $n \equiv 1 \pmod 3$ and $n \neq 4$. So $d'_{sc}(W_n) = 3$, where $n \equiv 1 \pmod 3$ and $n \neq 4$.

Case 3. $n \equiv 2 \pmod 3$ and $n \neq 5, 8$.

Define

$$f_1(e) = \begin{cases} -1 & e = vv_0, vv_3, ..., vv_{n-2}, v_1v_2, v_4v_5, ..., v_{n-4}v_{n-3}; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_2(e) = \begin{cases} -1 & e = vv_1, vv_4, ..., vv_{n-1}, v_2v_3, v_5v_6, ..., v_{n-3}v_{n-2}; \\ 1 & \text{otherwise}, \end{cases}$$

$$f_3(e) = \begin{cases} -1 & e = vv_2, vv_5, ..., vv_{n-3}, v_0v_1, v_3v_4, ..., v_{n-2}v_{n-1}, \\ & \quad v_{n-1}v_0; \\ 1 & \text{otherwise}. \end{cases}$$

It is not difficult to check that $\{f_1, f_2, f_3\}$ is an SCD family on W_n for $n \equiv 2 \pmod 3$ and $n \geq 11$. So $d'_{sc}(W_n) = 3$, where $n \equiv 2 \pmod 3$ and $n \neq 5, 8$. □

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