Implicit degree sum condition for long cycles*

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Abstract: For a vertex v of a graph G, Zhu, Li and Deng introduced the concept of implicit degree $id(v)$, according to the degrees of the neighbors of v and the vertices at distance 2 with v in G. For a subset S of $V(G)$, let $i\Delta_2(S)$ denote the maximum value of the implicit degree sum of two vertices of S. In this paper, we will prove: Let G be a 2-connected graph on $n \geq 3$ vertices and d be a nonnegative integer. If $i\Delta_2(S) \geq d$ for each independent set S of order $\kappa(G)+1$, then G has a cycle of length at least $\min\{d, n\}$.

Keywords: Implicit degree sum; Independent set; Circumference

1 Introduction

In this paper, we consider only finite, undirected and simple graphs. Notation and terminology not defined here can be found in [3]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and H be a subgraph of G. For a vertex $u \in V(G)$, $N_H(u)$ and $d_H(u)$ denote the set and the number of vertices adjacent to u in H, respectively. We call $N_H(u)$ and $d_H(u)$ the *neighborhood* and *degree* of u in H, respectively. If $H = G$, we use $N(u)$ and $d(u)$ in place of $N_G(u)$ and $d_G(u)$, respectively. We use $N_2(u)$ denote the set of vertices which are at distance 2 with u in G, $\alpha(G)$ and $\kappa(G)$ denote the independence number and the connectivity of G, respectively.

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A graph G is called hamiltonian if it contains a hamiltonian cycle, i.e a cycle containing all vertices of G. Degree condition is an important type of sufficient conditions for the existence of hamiltonian cycles in graphs. The following result due to Ore is classical.

Theorem 1 ([8]). *Let G be a graph on $n \geq 3$ vertices. If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x and y, then G is hamiltonian.*

By considering the relationship between the independence number and the connectivity of a graph, Chvátal give a sufficient condition for a graph to be hamiltonian.

Theorem 2 ([4]). *Let G be a 2-connected graph. If $\alpha(G) < \kappa(G)$, then G is hamiltonian.*

Next, we consider the length of a longest cycle, called circumference, denoted by $c(G)$. Many researchers have estimated the lower bound of the circumference of graphs. The following result is famous.

Theorem 3 ([2], [5]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $d(x) + d(y) \geq d$ for every pair of nonadjacent vertices x and y, then $c(G) \geq \min\{d, n\}$.*

For a nonempty subset S of $V(G)$, let $\Delta_k(S) = \max\{\sum_{x \in X} d(x) : X$ is a subset of S with k vertices$\}$. Yamashita [9] generalized Theorem 3 as follows.

Theorem 4 ([9]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $\Delta_2(S) \geq d$ for every independent set S of order $\kappa(G) + 1$, then $c(G) \geq \min\{d, n\}$.*

In order to generalize and improve the classic results of hamiltonian problem, Zhu, Li and Deng [10] gave the concept of implicit degree of a vertex.

Definition 1 ([10]). *Let v be a vertex of a graph G and $k = d(v) - 1$. Set $M_2 = \max\{d(u) : u \in N_2(v)\}$ and $m_2 = \min\{d(u) : u \in N_2(v)\}$. Suppose $d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_k \leq d_{k+1} \leq \ldots$ is the degree sequence of vertices in $N(v) \cup N_2(v)$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, define*

$$d^*(v) = \begin{cases} m_2, & \text{if} \quad d_k < m_2; \\ d_{k+1}, & \text{if} \quad d_{k+1} > M_2; \\ d_k, & \text{otherwise}, \end{cases}$$

then the implicit degree *of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id(v) = d(v)$.*

Clearly, $id(v) \geq d(v)$ for each vertex v from the definition of implicit degree. For $S \subseteq V(G)$ with $S \neq \emptyset$, let $i\Delta_k(G) = \max\{\sum_{x \in X} id(x) : X$ is a subset of S with k vertices$\}$. The authors [10] used implicit degree sum instead of degree sum in Theorem 3, and got a lower bound of the circumference of graphs.

Theorem 5 ([10]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $id(u) + id(v) \geq d$ for each pair of nonadjacent vertices u and v in G, then $c(G) \geq \min\{d, n\}$.*

In 2012, by considering the implicit degree sum of $k+1$ independent vertices, Li, Ning and Cai [7] gave a sufficient condition for a k-connected graph to be hamiltonian.

Theorem 6 ([7]). *Let G be a k-connected graph on $n \geq 3$ vertices. If the implicit degree sum of any $k+1$ independent vertices is more than $(k+1)(n-1)/2$, then G is hamiltonian.*

Motivated by the results of Theorem 3 and Theorem 5, we use $i\Delta_2(S)$ in place of $\Delta_2(S)$ and obtain the following main result.

Theorem 7. *Let G be a 2-connected graph on $n \geq 3$ vertices. If $i\Delta_2(S) \geq d$ for every independent set S of order $\kappa(G)+1$, then $c(G) \geq \min\{d, n\}$.*

Remark 1. *Theorem 6 is a corollary of Theorem 7. (Proof. Suppose that G satisfies the assumption of Theorem 6, and S is any independent set in G of order $\kappa(G)+1$. Then since the implicit degree sum of vertices in S is more than $(\kappa(G)+1)(n-1)/2$, $i\Delta_2(S) \geq n$. Hence G is hamiltonian by Theorem 7.)*

We postpone the proof of Theorem 7 in next section. Here we give an example to show that Theorem 7 is much stronger than Theorem 4 and Theorem 5.

Remark 2. *The graph in Fig.1 shows that Theorem 7 is much stronger than Theorem 4 and Theorem 5. It is easy to check that G is a 2-connected graph on $n = 14$ vertices. Choose $S = \{v_2, v_3, v_4\}$, then $\Delta_2(S) = 6$ and by using Theorem 4, we can only get that $c(G) \geq 5$. By the definition of implicit degree, we can get that $id(v_2) = 3$, $id(v_3) = id(v_4) = 6$, $id(v_1) = id(v_5) = id(v_6) = id(v_7) = 5$ and $id(v_j) = 6$ for $8 \leq j \leq 14$. So by using Theorem 7, we can get that $c(G) \geq 10$. But by using Theorem 5, we can only get that $c(G) \geq 8$.*

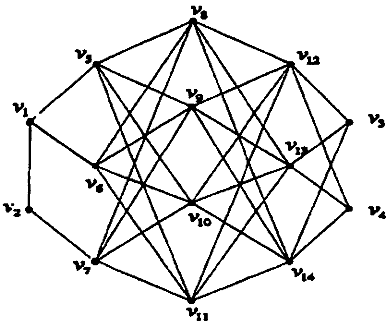

*Fig.*1

2 Proof of Theorem 7

A path P connecting x and y is called an xy-path. For a subgraph H of G, an xy-path P is called an H-path if $V(P) \cap V(H) = \{x, y\}$ and $E(P) \cap E(H) = \emptyset$. An xy-lollipop is a graph $C \cup P$ where C is a cycle and P is an xy-path such that $V(P) \cap V(C) = \{y\}$. A path P is called a maximal path of G if the order of each path in G containing P equals the order of P.

For a cycle C in a graph G with a given orientation and a vertex x in C, x^+ and x^- denote the successor and the predecessor of x in C, respectively. For any $I \subseteq V(C)$, let $I^- = \{x : x^+ \in I\}$ and $I^+ = \{x : x^- \in I\}$. For two vertices x and y in C, we define xCy to be the path of C from x to y. $y\bar{C}x$ denotes the path from y to x in the reversed direction of C. A similar notation is used for paths.

For a path $P = x_1x_2 \dots x_p$ of a graph G, let $l_P(x_1) = \max\{i : x_ix_1 \in E(G) \text{ and } x_i \in V(P)\}$ and $l_P(x_p) = \min\{i : x_ix_p \in E(G) \text{ and } x_i \in V(P)\}$. Set $L_P(x_1) = x_{l_P(x_1)}$ and $L_P(x_p) = x_{l_P(x_p)}$.

The proof of Theorem 7 is based on the following lemmas.

Lemma 1 ([6]). *Let G be a 2-connected graph and X be a subset of $V(G)$. If $|X| \leq \kappa(G)$, then G has a cycle that includes every vertex of X.*

Lemma 2 ([1]). *Let G be a 2-connected graph and C be a longest cycle of G with length at most $d - 1$. If P is an xy-path in G such that $|V(C)| < |V(P)|$, then $d(x) + d(y) < d$.*

Lemma 3 ([7]). *Let G be a 2-connected graph and $P = x_1x_2 \dots x_p$ be a path of G. If $x_1x_p \notin E(G)$, and $d(u) < id(x_1)$ for any $u \in N_{G-V(P)}(x_1) \cup \{x_1\}$, then either*

(1) there exists a vertex $x_j \in N_P(x_1)^-$ such that $d(x_j) \geq id(x_1)$; or

(2) $N_P(x_1)^- = N_P(x_1) \cup \{x_1\} - \{L_P(x_1)\}$, $d(x_j) < id(x_1)$ for any vertex $x_j \in N_P(x_1)^-$ and $id(x_1) = \min\{d(v) : v \in N_2(x_1)\}$.

By similar argument as in [10], we can get the following lemma.

Lemma 4. *Let G be a 2-connected graph and $P = u_1u_2 \ldots u_p$ (with $u_1 = a$ and $u_p = b$) be a maximal path such that $l_P(a) - l_P(b)$ is as large as possible. If $c(G) < p$, then $N_P(a)^- \neq N_P(a) \cup \{a\} - \{L_P(a)\}$.*

Proof. Suppose to the contrary that $N_P(a)^- = N_P(a) \cup \{a\} - \{L_P(a)\}$. Then $N_P(a) = \{u_2, u_3, \ldots, u_{l_P(a)}\}$. Since $c(G) < p$, $l_P(b) \geq l_P(a)$. Since G is 2-connected, $G - \{L_P(a)\}$ is connected. Therefore, there exist some $u_i \in N_P(a)^-$ and some $u_j \in V(u_{l_P(a)+1}Pb)$ such that $u_iu_j \in E(G)$. Then $P'(u_i, b) = u_iu_{i-1} \ldots u_1u_{i+1}u_{i+2} \ldots u_p$ is a maximal path such that $l_{P'}(u_i) - l_{P'}(b) \geq j - l_{P'}(b) > l_P(a) - l_P(b)$. This contradicts the choice of P. □

Proof of Theorem 7. Suppose to the contrary that C is a longest cycle of length at most $\min\{d, n\} - 1$. Clearly, $G - V(C) \neq \emptyset$. Without loss of generality, we give C a clockwise direction. Let $k = \kappa(G)$ and H be a component of $G - V(C)$. By Lemma 1, we have $|V(C)| \geq k$. Since G is k-connected, $|N(H) \cap V(C)| \geq k$. Let $\{v_1, v_2, \ldots, v_k\} \subseteq N(H) \cap V(C)$ and suppose $v_1, v_2, \ldots, v_k$ occur on C in this order.

For $1 \leq i < j \leq k$, let $Q_{i,j}$ be a maximal C-path connecting v_i and v_j such that $V(Q_{i,j}) \cap V(H) \neq \emptyset$. For $i = 1, 2, \ldots, k$, let $C \cup P_i$ be an $x_iv_i^+$-lollipop in G such that P_i is as long as possible. Without loss of generality, we orient P_i from x_i to v_i^+ for $i = 1, 2, \ldots, k$. Then for $1 \leq i < j \leq k$, by the choice of P_i and $Q_{i,j}$, $P_{i,j} = x_iP_iv_i^+Cv_j\bar{Q}_{i,j}v_i\bar{C}v_j^+\bar{P}_jx_j$ is a maximal path of G such that $|V(P_{i,j})| > |V(C)|$. For $1 \leq i < j \leq k$, we choose such $P_{i,j}$ such that $l_{P_{i,j}}(x_i) - l_{P_{i,j}}(x_j)$ is as large as possible. By similar argument as in [10], we can get the following claim.

Claim 1. $V(P_i) \cap V(H) = \emptyset$ and $V(P_i) \cap V(P_j) = \emptyset$ for $1 \leq i < j \leq k$.

Proof. Suppose there exists a vertex $x \in V(P_i) \cap V(H)$ for some i with $1 \leq i \leq k$. Since H is connected, there exists a path P' connecting x and v_i. Then $C' = v_i^+Cv_i\bar{P}'xP_iv_i^+$ is a cycle longer than C, this contradicts the choice of C. Thus $V(P_i) \cap V(H) = \emptyset$ for each i with $1 \leq i \leq k$.

Suppose there exists a vertex $y \in V(P_i) \cap V(P_j)$. Then $C'' = v_i^+Cv_j\bar{Q}_{i,j}v_i\bar{C}v_j^+\bar{P}_jyP_iv_i^+$ is a cycle longer than C, this contradicts the choice of C. Thus $V(P_i) \cap V(P_j) = \emptyset$ for each i, j with $1 \leq i < j \leq k$. □

Claim 2. For any $1 \leq i < j \leq k$, $id(x_i) + id(x_j) < d$.

Proof. Suppose to the contrary that there exist some i and j with $1 \leq i < j \leq k$ such that $id(x_i) + id(x_j) \geq d$. Since $P_{i,j}$ is a maximal path of G such that $|V(P_{i,j})| > |V(C)|$, by Lemma 2, we can assume without loss of generality, that $d(x_i) < id(x_i)$.

For convenience, set $P_{i,j} = y_1 y_2 \dots y_p$ with $y_1 = x_i$ and $y_p = x_j$. Since $l_{P_{i,j}}(x_i) - l_{P_{i,j}}(x_j)$ is as large as possible, $N_{P_{i,j}}(x_i)^- \neq N_{P_{i,j}}(x_i) \cup \{x_i\} - \{L_{P_{i,j}}(x_i)\}$ by Lemma 4. Then by Lemma 3, there exists a vertex $y_s \in N_{P_{i,j}}(x_i)^-$ such that $d(y_s) \geq id(x_i)$. Let

$$P' = y_s \bar{P}_{i,j} y_1 y_{s+1} P_{i,j} y_p,$$

which is a another maximal path of G such that $V(C) \subset V(P')$. If $d(y_p) = id(y_p)$, then $d(y_s) + d(y_p) \geq id(y_1) + id(y_p) \geq d$, this contradicts Lemma 2.

Next, we suppose $d(y_p) < id(y_p)$. If $s + 1 \leq l_{P_{i,j}}(x_j)$, similarly, there exists a vertex $y_t \in N_{P_{i,j}}(x_j)^+$ such that $d(y_t) \geq id(x_j)$. Then $P'' = y_s \bar{P}_{i,j} y_1 y_{s+1} P_{i,j} y_{t-1} y_p \bar{P}_{i,j} y_t$ is a maximal path of G with $V(C) \subset V(P'')$ and $d(y_s) + d(y_t) \geq id(x_i) + id(x_j) \geq d$. This contradicts Lemma 2.

So let $s + 1 > l_{P_{i,j}}(x_j)$. Set

$$A = \{y_l : y_{l+1} \in N_{P_{i,j}}(x_j) \text{ and } \ l < s\},$$

$$B = \{y_l : y_{l-1} \in N_{P_{i,j}}(x_j) \text{ and } \ l > s+1\} \text{ and}$$

$$C = \{y_l : y_{l+1} \in N_{P_{i,j}}(x_j) \ l \geq s+1 \text{ and } \ l \text{ is as small as possible}\}.$$

Clearly, $x_j \in B$ and $|C| = 1$. Then $|A| + |B \setminus \{x_j\}| + |C| = d(x_j)$, $y_{l_{P_{i,j}}(x_j)-1} \in A \cap N_2(x_j)$. Since $y_s \notin N(x_j)$, $C \subseteq N_2(x_j)$. By the definition of $id(x_j)$, there is a vertex $y_t \in (A \cup B) - \{x_j\}$ such that $d(y_t) \geq id(x_j)$. When $y_t \in B - \{x_j\}$, set

$$\tilde{P} = y_s \bar{P}_{i,j} x_i y_{s+1} P_{i,j} y_{t-1} x_j \bar{P}_{i,j} y_t.$$

When $u_l \in A$, set

$$\tilde{P} = y_s \bar{P}_{i,j} y_{t+1} x_j \bar{P}_{i,j} y_{s+1} x_i P_{i,j} y_t.$$

Then $\tilde{P}$ is a maximal path such that $V(C) \subset V(\tilde{P})$ and $d(y_s) + d(y_t) \geq id(x_i) + id(x_j) \geq d$. This contradicts Lemma 2. □

Hence by Claim 2, $id(x_i) + id(x_j) < d$ for any $1 \leq i < j \leq k$. Without loss of generality, we may assume $id(x_1) = \max\{id(x_i) : 1 \leq i \leq k\}$. Let $C \cup P_0$ be an $x_0 v_1$-lollipop in G, where P_0 is as long as possible and $x_0 \in V(H)$. Without loss of generality, we orient P_0 from x_0 to v_1. Then $P_{0,1} = x_0 P_0 v_1 \bar{C} v_1^+ \bar{P}_1 x_1$ is a maximal path of G such that $V(C) \subset V(P_{0,1})$. We choose such $P_{0,1}$ such that $l_{P_{0,1}}(x_0) - l_{P_{0,1}}(x_1)$ is as large as possible.

Hence by similar argument as in the proof of Claim 2, we can get $id(x_0) + id(x_1) < d$. Since $id(x_1) = \max\{id(x_i) : 1 \leq i \leq k\}$, $id(x_0) + id(x_i) < d$ for each $i = 1, 2, \ldots, k$. Therefore, by Claim 1, $S = \{x_0, x_1, \ldots, x_k\}$ is an independent set of G of order $k+1$ and $i\Delta_2(S) < d$, which contradicts the hypothesis of Theorem 7. □

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