On the upper broadcast domination number

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Abstract

A broadcast on a graph G is a function $f : V \longrightarrow \{0, \ldots, \text{diam}(G)\}$ such that for every vertex $v \in V(G)$, $f(v) \leq e(v)$, where diam(G) denotes the diameter of G and $e(v)$ denotes the eccentricity of vertex v. The upper broadcast domination number of a graph is the maximum value of $\sum_{v \in V} f(v)$ among all minimal broadcasts f for which each vertex of the graph is within distance $f(v)$ from some vertex v having $f(v) \geq 1$. We give a new upper bound on the upper broadcast domination number which improves a previous result of Dunbar et al. in [Broadcasts in graphs, Discrete Applied Mathematics 154 (2006) 59-75]. We also prove that the upper broadcast domination number of any grid graph $G_{m,n} = P_m \square P_n$ equals $m(n-1)$.

Keywords: Grid graph, Broadcast, Dominating broadcast, Upper broadcast domination number.

1 Introduction

Let $G = (V, E)$ be a graph of *order* $n = |V|$ and *size* $m = |E|$. The *eccentricity* $e(v)$ of a vertex v of G is the maximum distance from v to any other vertex of G. The minimum eccentricity in G is the *radius* rad(G) of G, while the maximum eccentricity in G is its *diameter* diam(G). For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree of* v in the graph G, denoted d(v) (or d$_G(v)$ if there is a risk of confusion), is the size of the open neighborhood of v. For a set $S \subseteq V$, its *open neighborhood* is $N(S) = \bigcup_{v \in S} N(v)$ and its *closed neighborhood* is $N[S] = N(S) \cup S$. For any $v \in S$, the *private neighborhood* $pn[v, S]$ of v with respect to S is the set

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of all vertices in $N[v]$ that are not contained in the closed neighborhood of any other vertex in S, i.e., $pn[v, S] = N[v] - N[S - v]$. S is an *irredundant set* if for every vertex $v \in S$, $pn[v, S] \neq \emptyset$. Let $ir(G)$ (resp. $IR(G)$) equal the minimum (resp. maximum) cardinalities of a maximal irredundant set in G.

A function $f : V \longrightarrow \{0, \ldots, \text{diam}(G)\}$ is a *broadcast* of G if for every vertex $v \in V$, $f(v) \leq e(v)$.

Given a broadcast f, a *broadcast vertex* (or *f-dominating vertex*) is a vertex v for which $f(v) > 0$. The *broadcast neighborhood* of a vertex u is the set $N_f[u] = \{v : d(u, v) \leq f(u)\}$. The set of all broadcast vertices is denoted $V_f^+(G)$, or briefly V^+ if there is no potential ambiguity. The *broadcast neighborhood* of f is $N_f[V^+] = \cup_{v \in V^+} N_f[v]$. If $u \in V^+$ is a broadcast vertex, $v \in V$ and $d(u, v) \leq f(u)$, then the vertex v *hears* a broadcast from u and u *broadcasts to* (or *f-dominates*) v. The set of vertices that a vertex $v \in V$ can hear is defined as $H(v) = \{u \in V^+ : d(u, v) \leq f(u)\}$. For a vertex $v \in V^+$, the *private f-neighborhood* $pn_f[v]$ is the set $\{u \in V : H(u) = \{v\}\}$. A vertex v is its own private f-neighbor if $v \in pn_f[v]$. The *cost* of a broadcast is $f(V) = \sum_{v \in V^+} f(v)$.

A broadcast f of some type is said to be *minimal* (resp. *maximal*) if there does not exist a broadcast $g \neq f$ of the same type such that $g(u) \leq f(u)$ (resp. $g(u) \geq f(u)$), for all $u \in V$.

A broadcast f is a *dominating broadcast* if every vertex in $V - V^+$ is f-dominated by some vertex in V^+ or equivalently, if for every $v \in V$, $|H(v)| \geq 1$. The maximum (resp. minimum) cost of a minimal dominating broadcast of a graph G is the *upper broadcast domination* (resp. *broadcast domination*) *number* and is denoted $\Gamma_b(G)$ (resp. $\gamma_b(G)$). A minimal dominating broadcast of cost equal to $\Gamma_b(G)$ (resp. $\gamma_b(G)$) is a *Γ_b-broadcast* (resp. *γ_b-broadcast*). If f is a minimal dominating broadcast such that $f(v) = 1$ for each $v \in V^+$, then V^+ is a *minimal dominating set* of G, and the maximum (resp. minimum) cost of such a broadcast is the *upper domination number* $\Gamma(G)$ (resp. *domination number* $\gamma(G)$).

A broadcast f is an *independent broadcast* if for every vertex $v \in V^+$, $N_f[v] \cap V^+ = \{v\}$, or equivalently, $|H(v)| = 1$. The maximum (resp. minimum) cost of a maximal independent broadcast of G is the *broadcast independence* (resp. *lower broadcast independence*) *number* and is denoted $\beta_b(G)$ (resp. $i_b(G)$). A maximal independent broadcast of cost equal to $\beta_b(G)$ (resp. $i_b(G)$) is a *β_b-broadcast* (resp. *i_b-broadcast*). If f is a maximal independent broadcast such that $f(v) = 1$ for each $v \in V^+$, then V^+ is a *maximal independent set* of G, and the maximum (resp. minimum) cost of such a broadcast is the *independance number* $\beta_0(G)$ (resp. *lower independence number* $i(G)$).

In 1978, Cockayne, Hedetniemi, and Miller [9, Prop. 4.2] first established the following inequality chain. These inequalities are primarily based on two observations: (i) every maximal independent set in a graph G is a minimal dominating set, and (ii) every minimal dominating set in a graph G is a maximal irredundant set.

Theorem 1. *For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

In 1988, Favaron [15, Prop. 4] established the following result:

Theorem 2. *For any graph G of order n and minimum degree $\delta(G)$, $IR(G) \leq n - \delta(G)$.*

From Theorem 1 and Theorem 2, we deduce that $\Gamma(G) \leq n - \delta(G)$. In Section 2, we prove that $n - \delta(G)$ is also an upper bound of $\Gamma_b(G)$. Note that the difference between $\Gamma_b(G)$ and $IR(G)$ can be large since the paths P_n of order $n \geq 2$ satisfy $\Gamma_b(P_n) = n - 1$ and $IR(P_n) = \lceil \frac{n}{2} \rceil$.

Broadcast domination was introduced by Erwin [13] in his Ph.D. thesis, in which he discussed several types of broadcast parameters and relationships between them. Many of these results appeared later in Dunbar et al [12]. Since then several papers have been published on various aspects of broadcasts in graphs, including the polynomial complexity of computing the broadcast domination number of arbitrary graphs [17, 18], the determination of the broadcast domination number for several classes of graphs [2, 3, 6, 11, 24], and a characterization of the classes of trees for which the broadcast domination number equals the radius [19] or equals the domination number $\gamma(G)$ [10, 20, 22]. The exact values of β_b [5] and γ_b [12] have been determined for arbitrary grid graphs. Other work on broadcast domination includes [7, 8, 21, 23, 24, 25]. In this paper, we give a new upper bound for the upper broadcast domination number Γ_b for arbitrary graphs, which improves on the bound previously obtained by Dunbar et al. [12], and we determine the value of $\Gamma_b(G_{m,n})$ for arbitrary grid graphs $G_{m,n}$, $2 \leq m \leq n$, which answers a question raised in Dunbar et al [12].

2 New upper bound on the upper broadcast domination number.

It was shown in [12, Obs. 1] that for any graph G,

$$\gamma_b(G) \leq \min\{\gamma(G), \mathrm{rad}(G)\} \leq \max\{\Gamma(G), \mathrm{diam}(G)\} \leq \Gamma_b(G).$$

Concerning the upper bound on $\Gamma_b(G)$, the size of a graph constitutes the only known value.

Theorem 3. *[12, Thm. 5] If $G = (V, E)$ is a graph of size $m = |E|$, then $\Gamma_b(G) \leq m$ with equality if and only if G is a nontrivial star or path.*

In this section, we shall establish a much better upper bound on $\Gamma_b(G)$ than that given in Theorem 3. In order to do this, we will need some preliminary results.

We say that a vertex or edge of G *lies between* two vertices u and v if that vertex or edge is on some $u - v$ geodesic (shortest $u - v$ path).

Theorem 4. *[13, Thm. 2.1.2] Let f be a dominating broadcast on a graph G. Then f is minimal if and only if the following two conditions are satisfied:*

1. *Every vertex v with $f(v) \geq 2$ has a private f-neighbor that is at distance $f(v)$ from v, and*
2. *every vertex v with $f(v) = 1$ has a private f-neighbor in $N[v]$.*

Lemma 1. *[13, Lem. 3.2.1] Let f be a dominating broadcast on a graph G, $u, v \in V^+$ with $u \neq v$, and let u^p, v^p be private f-neighbors of (respectively) u and v. For every pair x, y of vertices of G, if x lies between u and u^p and y lies between v and v^p, then $x \neq y$.*

Lemma 2. *Let f be a minimal dominating broadcast on a graph $G = (V, E)$. If $\max(f) = \max_{v \in V_f^+}\{f(v)\} > 1$, then*

1. *the broadcast g, defined as $g(v) = f(v) - 1$ for every $v \in V_f^+$, and $g(v) = 0$ otherwise, is a minimal dominating broadcast on the induced subgraph $G[N_g[V_g^+]]$, and*
2. $|V_f^+| \leq |V \setminus N_g[V_g^+]|$.

Proof.

1. The function g is obviously a dominating broadcast on the subgraph $G[N_g[V_g^+]]$. We only have to prove that g is minimal. Let v be a broadcast vertex of $N_g[V_g^+]$. From Theorem 4, v has a private f-neighbor (denoted v^p) such that $d(v, v^p) = f(v) \geq 2$. Let u be an adjacent vertex to v^p lying between v and v^p. The vertex u is a private g-neighbor of v, for otherwise there would exist a broadcast vertex w, $w \neq v$, such that $d(v^p, w) = d(u, w) + 1 \leq g(w) + 1 = f(w)$ and then w would f-dominate v^p, a contradiction. From Theorem 4, we infer the minimality of g.

2. The inequality $|V_f^+| \leq |V \setminus N_g[V_g^+]|$ comes from the fact that the number of non g-dominated vertices is at least equal to $|V_f^+|$. □

In order to give an upper bound on $\Gamma_b(G)$, let us define an elimination process (on private neighborhoods) on a graph $G = (V, E)$. Its principle is to move from one iteration to the next one by deleting the set of private neighbors and subtracting one unit to the cost of each broadcast vertex whose the weight is different from 1. This procedure keeps the minimality of the current dominating broadcast.

Input : A minimal dominating broadcast f on a graph $G = (V, E)$ with $max(f) > 1$.
Output: A minimal dominating broadcast f_1 on G_1 with $max(f_1) = 1$.

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k = max(f) = max_{v∈V+} f(v) ;
G_k := G, V_k := V, f_k := f;
while k > 1 do
    k ← k − 1;
    for v ∈ V do
        if f_{k+1}(v) > 1 then
            f_k(v) = f_{k+1}(v) − 1;
        else
            f_k(v) = 0;
        end
    end
    V_k := N_{f_k}[V_{f_k}^+];
    G_k := G[V_k]
end
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Theorem 5. *If G is a graph of order n and minimum degree $\delta(G)$, then $\Gamma_b(G) \leq n - \delta(G)$ and this bound is sharp.*

Proof. Let G be a graph of order n and minimum degree $\delta(G)$ and let f be a Γ_b-broadcast on G. If $\max(f) = 1$, then $\Gamma_b(G) = \Gamma(G) \leq n - \delta(G)$, from Theorem 1 and Theorem 2. Let us now assume $\max(f) \geq 2$ and let $G_k, \ldots, G_2, G_1$ be the subgraphs obtained from the procedure above. By Lemma 2, f_k is a minimal dominating broadcast on G_k and

$$|V_k \setminus V_{k-1}| \geq |V_{f_k}^+| \qquad \forall k = 2, \ldots, \max(f).$$

For every $v \in V_{f_1}^+$, $f_1(v) = 1$, $d_G(v) = d_{G_1}(v)$, and v has a private neighbor outside $V_{f_1}^+$. Hence,

$$|V_1| - \delta(G) \geq |V_1| - d(v) \geq |V_{f_1}^+| \qquad \text{for every } v \in V_{f_1}^+.$$

It follows,

$$\begin{array}{rcl} n-\delta(G) & = & |V_1|-\delta(G)+(|V|-|V_1|) \\ & = & |V_1|-\delta(G)+\sum_{k=2}^{max(f)}|V_k\setminus V_{k-1}| \\ & \geq & \sum_{k=1}^{max(f)}|V_{f_k}^+|. \end{array}$$

Since $\sum_{k=1}^{max(f)}|V_{f_k}^+| = \Gamma_b(G)$, we infer that $\Gamma_b(G) \leq n-\delta(G)$.

The bound is achieved for some classes of graphs. We can cite paths, stars and complete graphs. □

3 Upper broadcast domination number of grid graphs.

The *Cartesian product* of two graphs G and H, denoted $G\Box H$, is a graph with vertex set $\{(u,v) : u \in V(G); v \in V(H)\}$. Two vertices (u_1,v_1) and (u_2,v_2) are adjacent in $G\Box H$ if either $u_1=u_2$ and v_1 is adjacent to v_2 in H or $v_1=v_2$ and u_1 is adjacent to u_2 in G. The Cartesian product $P_m\Box P_n$ is called *the* $m\times n$ *grid graph* and is denoted $G_{m,n}$. The vertices in $G_{m,n}$ will be denoted $v_{i,j}$, $1\leq i\leq m$, $1\leq j\leq n$, and there is an edge between $v_{i,j}$ and $v_{k,l}$ if and only if $|i-k|+|j-l|=1$. We will refer to the rows and columns of a grid graph by $R^i=\{v_{i,1},\ldots,v_{i,n}\}$ and $C^j=\{v_{1,j},\ldots,v_{m,j}\}$. Denote by $R(x)$ (resp. $C(x)$) the row (resp. column) to which a vertex x belongs.
For $m=1$, the grid graph is isomorphic to the path P_n, and thanks to Theorem 3, $\Gamma_b(P_n)=n-1$. Now suppose that $2\leq m\leq n$, and let $S_{i,j}$ denote the square
$\{\{v_{i,j},v_{i,j+1}\},\{v_{i,j},v_{i+1,j}\},\{v_{i+1,j},v_{i+1,j+1}\},\{v_{i,j+1},v_{i+1,j+1}\}\}$ in $G_{m,n}$. In order to prove Theorem 6, let us start by proving the following two claims:

Claim 1. *A geodesic path P_v in $G_{m,n}$ contains at most two edges from any square $S_{i,j}$.*

Proof. If P_v contained all four sides of $S_{i,j}$, then P_v would contain a cycle. Which is absurd. If P_v contained three sides of $S_{i,j}$, one could make P_v shorter by replacing those three sides with the other edge in the square, which contradicts the assumption that P_v is a geodesic.

Claim 2. *The union of two pairwise disjoint paths P_u and P_v in $G_{m,n}$ contains at most two edges of any square $S_{i,j}$.*

Proof. If $P_u\cup P_v$ contained three of the four sides of a square $S_{i,j}$, then $P_u\cup P_v$ would necessarily have a common vertex. This contradicts that the paths P_u and P_v are pairwise disjoint.

Theorem 6. *For every pair of integers m and n, $2 \le m \le n$, $\Gamma_b(G_{m,n}) = m(n-1)$.*

Proof. Let f be the following broadcast on $G_{m,n}$, where $2 \le m \le n$,

$$f(v) = \begin{cases} n-1 & if \quad v \in C^1, \\ 0 & otherwise. \end{cases}$$

Since f is a minimal dominating broadcast, of cost $f(V) = m(n-1)$, it follows that $\Gamma_b(G_{m,n}) \ge m(n-1)$. Combining this inequality with Theorem 5, we already infer $\Gamma_b(G_{2,n}) = 2(n-1)$.
We now prove the inequality in the other direction for all integers m, n, $3 \le m \le n$ (this proof is also valid for $m = 2$).
Let g be any minimal dominating broadcast on $G_{m,n}$ and $V_g^+ = \{v_1, \dots, v_k\}$ be the set of all broadcast vertices of $G_{m,n}$. In view of Theorem 4, each $v \in V^+$ has a private g-neighbor (denoted v^p) such that either (i) $g(v) = d(v, v^p)$, or (ii) $g(v) = 1$ and $v = v^p$. For every broadcast vertex v, let P_v be any $v - v^p$ geodesic if $g(v) > 1$ and $\{e_v\}$, where e_v is any edge incident with v if $g(v) = 1$. Dunbar et al [12] proved (see Proof of Theorem 3) that if $\epsilon(v)$ equals the set of all edges lying on $v - v^p$ geodesic, then $\epsilon(u) \cap \epsilon(v) = \emptyset$ for any two broadcast vertices, u and v. From this, we deduce that $P_u \cap P_v = \emptyset$ for any two geodesic paths P_u and P_v, that is $\cup_{v \in V^+} P_v$ is a collection of pairwise disjoint geodesic paths in $G_{m,n}$ for any minimal dominating broadcast.
From Claims 1 and 2, we can infer that $|S_{i,j} \cap E(\cup_{v \in V^+} P_v)| \le 2$ for every square $S_{i,j}$ in $G_{m,n}$.
Now suppose that $|E(\cup_{v \in V_g^+} P_v)| > m(n-1)$ for some minimal dominating broadcast g. Then, there are more edges in $E(\cup_{v \in V_g^+} P_v)$ than horizontal edges in $G_{m,n}$. By the pigeonhole principle, there is at least one square $S_{i,j}$ in $G_{m,n}$ that contains at least three edges from $E(\cup_{v \in V_g^+} P_v)$, i.e., $|S_{i,j} \cap E(\cup_{v \in V_g^+} P_v)| > 2$. This contradicts the fact $|S_{i,j} \cap E(\cup_{v \in V^+} P_v)| \le 2$ for every minimal dominating broadcast.
It follows, $|E(\cup_{v \in V^+} P_v)| \le m(n-1)$ for any minimal dominating broadcast on $G_{m,n}$, and consequently, $\Gamma_b(G_{m,n}) \le m(n-1)$. □

Remark 1. *From the proof of Theorem 6, we can say:*

1. *If $m < n$, $G_{m,n}$ has only two distinct Γ_b-broadcasts f and g defined by $f(v) = n-1$ (resp. $g(v) = n-1$) if $v \in C^1$ (resp. $v \in C^n$) and $f(v) = 0$ (resp. $g(v) = 0$) otherwise.*

2. *If $m = n$, $G_{m,n}$ has only four distinct Γ_b-broadcasts f, g, h and i defined by $f(v) = n-1$ (resp. $g(v) = n-1$, $h(v) = n-1$, $i(v) = n-1$) if*

$v \in C^1$ (resp. $v \in C^n$, $v \in R^1$, $v \in R^n$), and $f(v) = 0$ (resp. $g(v) = 0$, $h(v) = 0$, $i(v) = 0$) otherwise.

4 Conclusion.

We presented a new upper bound for the upper broadcast domination number Γ_b for arbitrary graphs, which improves the bound established in Dunbar et al. [12]. Among other broadcasting invariants, there is the upper broadcast efficiency number $\Gamma_{eb}(G_{m,n})$. For a graph G, $\Gamma_{eb}(G)$ is defined as the maximum cost of a broadcast satisfying, for every vertex v of G, $|H(v)| = 1$. For a grid graph, we proved that $\Gamma_b(G_{m,n}) = m(n-1)$ for every pair of integers m and n with $1 \leq m \leq n$. From [12, Prop. 16] and [12, Cor. 19], we infer $\text{diam}(G_{m,n}) \leq \Gamma_{eb}(G_{m,n}) \leq \min\{\Gamma_b(G_{m,n}), \beta_b(G_{m,n})\}$. Although $\Gamma_b(G_{m,n})$ represents an upper bound for $\Gamma_{eb}(G_{m,n})$, $\beta_b(G_{m,n})$ constitutes a better bound, since for every integers m and n, $m \leq n$, $\beta_b(G_{m,n}) = 2(\text{diam}(G_{m,n}) - 1) = 2(m+n-3)$ if $m \leq 4$, and $\beta_b(G_{m,n}) = \lceil \frac{mn}{2} \rceil$ if $5 \leq m$, $(m,n) \neq (5,5), (5,6)$ [5]. Therefore, $m+n-2 \leq \Gamma_{eb}(G_{m,n}) \leq \beta_b(G_{m,n})$, and in particular, for $m \leq 4$, $m+n-2 \leq \Gamma_{eb}(G_{m,n}) \leq 2(m+n-3)$. This does not allow us to deduce the exact value of $\Gamma_{eb}(G_{m,n})$, but just bounds. In fact, we shall prove that $\Gamma_{eb}(G_{m,n}) = m+n-2$, for every pair of integers m and n with $1 \leq m \leq n$ and $m \leq 9$ [1].

In [12], Dunbar et al. raised thirteen problems. Some of them are now solved. For the fourth problem, Herke and Mynhardt in [19] were interested in the problem that concerns the characterization of trees satisfying $\gamma_b(T) = \text{rad}(T)$, while Cockayne, Herke and Mynhardt in [10], Mynhardt and Wodlinger in [22] and, Lunney and Mynhardt in [20] defined a large class of trees satisfying $\gamma_b(T) = \gamma(T)$. Regarding the sixth problem, Bouchemakh and Salhi in [3] determined the number of distinct efficient broadcasts in paths. Concerning the grid graph, Dunbar et al. in [12] have already determined the exact value of γ_b (see Th. 28) and i_b (see Cor. 12). The ninth problem is the determination of the the values of each of the broadcasting invariants for a grid graph. Bouchemakh and Zemir in [5] solved this problem for β_b, in this paper Bouchemakh and Fergani determined the exact values of Γ_b, and for the upper broadcast efficiency number, the paper is in preparation [1]. To conclude, we would like to present some open problems. We are resuming the unsolved problems of Dunbar et al. in [12] and state some new.

1. Can you characterize the class of graphs G of order n and minimum degree $\delta(G)$ with $\Gamma_b(G) = n - \delta(G)$?

2. Under what conditions is $\Gamma_b(G) = \text{diam}(G)$?

3. Under what conditions is $\Gamma_b(G) = \Gamma_{eb}(G)$?

4. Under what conditions is $\gamma_b(G) = i_b(G)$? [12]

5. For which graphs G does $\gamma_b(G) = \gamma(G)$? [12]

6. For which graphs G does $\gamma_b(G) = \text{rad}(G)$? [12]

7. For which graphs G is $\gamma_b(G) < \min\{\gamma(G), \text{rad}(G)\}$? [12]

8. What can you say about the class of minimum cost dominating broadcasts, where the number of broadcast vertices is a minimum (or a maximum)? [12]

9. In a grid graph, what is the number of dominating broadcasts? independent broadcasts ?

10. Can you construct linear algorithms for computing the values of each of the broadcasting invariants for trees? [12]

11. Can you settle the complexity of the decision problems associated with each of the broadcasting invariants? [12]

12. What are the values of each of the broadcasting invariants (not yet determined) for an $m \times n$ grid graph? [12]

13. Can you develop Nordhaus-Gaddum bounds for the broadcasting invariants? [12]

14. Suppose you are allowed to assign only broadcast powers of 0, 1 or 2 to the vertices of a graph. This suggests the concept of the broadcast domination number with limited broadcast power, say indexed by k, which could give rise to the k-limited broadcast domination number $\gamma_{kb}(G)$. What can you say about this invariant? [12]

15. Define and study irredundant broadcasts. [12]

16. Investigate graphs G and H of order n_1 and n_2 respectively, such that
    (a) $\Gamma_b(G \Box H) = \min\{n_1, n_2\} \max\{\Gamma_b(G), \Gamma_b(H)\}$. (The Cartesian product of two graphs satisfies this equality)
    (b) $\Gamma_b(G \Box H) < \min\{n_1, n_2\} \max\{\Gamma_b(G), \Gamma_b(H)\}$.
    (c) $\Gamma_b(G \Box H) > \min\{n_1, n_2\} \max\{\Gamma_b(G), \Gamma_b(H)\}$.

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