CLASSIFYING PENTAVALENT SYMMETRIC GRAPHS OF ORDER $40p$

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ABSTRACT. A graph is said to be symmetric if its automorphism group is transitive on its arcs. A complete classification is given of pentavalent symmetric graphs of order $40p$ for each prime p. It is shown that a connected pentavalent symmetric graph of order $40p$ exists if and only if $p = 3$, and up to isomorphism, there are only two such graphs.

KEYWORDS. symmetric graph; normal quotient; automorphism group.

1. INTRODUCTION

In this paper, all graphs are assumed to be finite, simple, connected and undirected.

Let Γ be a graph. We denote by $V\Gamma$, $E\Gamma$, $A\Gamma$ and $\mathrm{Aut}\Gamma$ its vertex set, edge set, arc set and automorphism group, respectively. Then the order of Γ is the number of elements of $V\Gamma$, denoted by $|V\Gamma|$. Let s be a positive integer. An *s-arc* in a graph Γ is an $(s+1)$-tuple $(v_0, v_1, \cdots, v_s)$ of $s+1$ vertices such that $(v_{i-1}, v_i) \in A\Gamma$ for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$. Let X be a subgroup of $\mathrm{Aut}\Gamma$. We say Γ is (X,s)-*arc-transitive* if X is transitive on the s-arcs of Γ and we say Γ is (X,s)-*transitive* if it is (X,s)-arc-transitive but not $(X,s+1)$-arc-transitive. In the case where $X = \mathrm{Aut}\Gamma$, we say an (X,s)-arc-transitive or (X,s)-transitive graph is an *s-arc-transitive* or *s-transitive* graph. In particular, we say 0-arc-transitive graph is *vertex-transitive* graph, and 1-arc-transitive graph is *arc-transitive* graph or *symmetric* graph.

Characterizing symmetric graphs with small valency is a current topic in the literature. Since cubic and tetravalent graphs have been studied extensively, it would be natural toward considering pentavalent graphs. For

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example, a characterization of pentavalent graphs has been studied in [4–6, 9, 11, 14, 15, 17]. In this paper, we classify pentavalent symmetric graphs of order $40p$ with p a prime. By using the Magma codes in Appendices, determining graph in this paper is more simple than some relative papers.

For a given small permutation group X, we can determine all graphs which admit X as an arc-transitive automorphism group by using codes in Appendices. Then there is an unique pentavalent symmetric graph of order 120 admitting $A_5\times D_{10}\times \mathbb{Z}_2$ as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}^1_{120}$. There is an unique pentavalent symmetric graph of order 120 which admits $S_5\times D_{10}$ as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}^2_{120}$. The main result of this paper is the following theorem.

Theorem 1.1. *Let Γ be a pentavalent symmetric graph of order $40p$, where p is a prime. Then $p = 3$ and, up to isomorphism, there exist two such graphs Γ. Furthermore,* $\mathrm{Aut}\Gamma$, $(\mathrm{Aut}\Gamma)_v$ *and Γ are described in Table 1, where $v \in V\Gamma$.*

Γ	$\mathrm{Aut}\Gamma$	$(\mathrm{Aut}\Gamma)_v$	Girth	Diameter	Bipartite?	Cayley?
$\mathcal{C}^1_{120}$	$A_5\times D_{10}\times \mathbb{Z}_2$	D_{10}	6	6	Yes	Yes
$\mathcal{C}^2_{120}$	$S_5\times D_{10}$	D_{10}	4	6	Yes	Yes

TABLE 1. Pentavalent symmetric graphs of order $40p$

The properties in Table 1 are determined with the help of the Magma [1]. Furthermore, $\mathcal{C}^1_{120}$ is a Cayley graph on $A_5\times \mathbb{Z}_2$, $A_4\times \mathbb{Z}_{10}$ or $A_4\times D_{10}$ and $\mathcal{C}^2_{120}$ is a Cayley graph on S_5, $S_4\times \mathbb{Z}_5$ or $(A_4\times \mathbb{Z}_5){:}\mathbb{Z}_2$.

2. Preliminary Results

We give some necessary preliminary results in this section.

For a graph Γ and a vertex-transitive subgroup $X \leq \mathrm{Aut}\Gamma$. Let N be an intransitive normal subgroup of X on $V\Gamma$. Denote V_N the set of N-orbits in $V\Gamma$. The *normal quotient graph* Γ_N is the graph with vertex set V_N and two N-orbits $B, C \in V_N$ are adjacent in Γ_N if and only if some vertex of B is adjacent in Γ to some vertex of C. The following lemma ([10, Lemma 2.5]) provides a basic reduction method for studying our pentavalent symmetric graphs.

Lemma 2.1. *Let Γ be an X-arc-transitive graph of prime valency $p > 2$, where $X \leq$* $\mathrm{Aut}\Gamma$, *and let $N \trianglelefteq X$ have at least three orbits on $V\Gamma$. Then the following statements hold.*

(i) N *is semiregular on* $V\Gamma$, $X/N \leq \mathrm{Aut}\Gamma_N$, *and* Γ_N *is an* X/N*-arc-transitive graph of valency* p;
(ii) Γ *is* (X,s)*-transitive if and only if* Γ_N *is* $(X/N,s)$*-transitive, where* $1 \leq s \leq 5$ *or* $s=7$.

By [17, Theorem 4.1] and [4, Theorem 1.1], we have the following lemma.

Lemma 2.2. *Let* Γ *be a pentavalent* (G,s)*-transitive graph for some* $G \leq \mathrm{Aut}\Gamma$ *and* $s \geq 1$. *Let* $v \in V\Gamma$. *Then the order of* G_v *is a divisor of* $2^9 \cdot 3^2 \cdot 5$.

From [7, pp.12-14], we may obtain the following proposition by checking the 3-prime factor nonabelian simple groups.

Proposition 2.3. *Let* G *be a nonabelian simple group and* $|G| = 2^k \cdot 3^l \cdot 5$, *then* $G = \mathrm{A}_5$, A_6 *or* $\mathrm{PSU}(4,2)$.

By checking the orders of nonabelian simple groups, see [7, pp.134-136] for example, we have the following proposition.

Proposition 2.4. *Let* $p > 5$ *be a prime and let* G *be a* $\{2,3,5,p\}$*-nonabelian simple group such that* $|G|$ *divides* $2^{12} \cdot 3^2 \cdot 5^2 \cdot p$ *and* $2^2 \cdot 5^2 \cdot p$ *divides* $|G|$. *Then* $G = \mathrm{PSL}(2,25)$, $\mathrm{PSU}(3,4)$ *or* $\mathrm{PSp}(4,4)$.

By [14, Theorem 1.1] and [9, Theorem 4.2] and with the help of Magma [1], we give some information of pentavalent symmetric graphs of order $10p$ in the following lemma. The graph $\mathcal{C}_n$ denotes the corresponding pentavalent symmetric graph of order n in [9]. For the graph $\mathcal{CD}^l_{10p}$ we use the same symbols in [9, Theorem 4.2].

Lemma 2.5. *Let* Γ *be a pentavalent symmetric graph of order* $10p$, *where* p *is a prime. Then*

(1) $\Gamma \cong \mathcal{C}_{50}$ *with* $p=5$ *and* $\mathrm{Aut}\Gamma \cong G{:}(\mathbb{Z}_4^2{:}\mathbb{Z}_2)$ *is soluble, where* $G = \langle a,b,c \mid a^5 = b^5 = c^5 = [a,c] = [b,c] = 1, [a,b] = c\rangle$;
(2) $\Gamma \cong \mathcal{C}_{170}$ *with* $p = 17$ *and* $\mathrm{Aut}\Gamma \cong \mathrm{Aut}(\mathrm{PSp}(4,4))$;
(3) $\Gamma \cong \mathcal{CD}^l_{10p}$ *with* $\mathrm{Aut}\Gamma \cong \mathrm{D}_{10p}{:}\mathbb{Z}_5$.

By [8, Theorem 1] and with the help of Magma [1], we give some information of pentavalent symmetric graphs of order $8p$ in the following lemma. For the graph CL_{16} and the graph $\mathrm{I}^{(2)}$, we use the same symbols in [8, Theorem 1].

Lemma 2.6. *Let* Γ *be a pentavalent symmetric graph of order* $8p$, *where* p *is a prime. Then*

(1) $\Gamma \cong \mathrm{CL}_{16}$ *with* $p = 2$ *and* $\mathrm{Aut}\Gamma \cong \mathbb{Z}_2^4{:}\mathrm{S}_5$;
(2) $\Gamma \cong \mathrm{I}^{(2)}$ *with* $p=3$ *and* $\mathrm{Aut}\Gamma \cong (\mathrm{A}_5 \times \mathbb{Z}_2^2){:}\mathbb{Z}_2$;

(3) $\Gamma \cong C_{248}$ *with* $p = 31$ *and* $\mathrm{Aut}\Gamma \cong \mathrm{PSL}(2, 31)$.

In the following, we need to introduce the concept of Schur multiplier. Let G be a perfect group, that is, $G' = G$. A *central extension* of G is a group H satisfying $H/N \cong G$ for $N \leq Z(H)$. If H is perfect, we call H is a covering group of G. If N is the largest abelian group such that $M = N.G$ is perfect and the extension is a central extension, then M is called the *full covering group* of G and N is called the *Schur Multiplier* of G, written $\mathsf{Mult}(G)$. By [13, Lemma 2.11], we have the following lemma.

Lemma 2.7. *Let* $M = N.T^d$ *be a central extension, where* $d \geq 1$ *and* T *is a nonabelian simple group. Then* $M = NM'$ *and* $M' = Z.T^d$, *where* Z *is a factor group of* $\mathsf{Mult}(T)^d$ *and* $Z \leq N$.

The next lemma is about the solvability of a finite group of order $40p$.

Lemma 2.8. *Let* p *be a prime and let* G *be a finite group of order* $40p$. *If* $p \neq 3$, *then* G *is soluble.*

Proof. If $p \leq 19$, then we can check that G can not have an unsoluble composition factor, therefore G is soluble. If $p > 19$, then the Sylow p-subgroup of G is normal, it follows that G is soluble. ∎

3. The proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by giving some lemmas. Now let Γ be a pentavalent symmetric graph of order $40p$, where p is a prime. Let $A = \mathrm{Aut}\Gamma$. Denote by $\mathrm{SmallGroup(n, m)}$ the n-th group of order m in the SmallGroupDatabase in Magma [1].

The next two simple lemmas is helpful to our argument.

Lemma 3.1. *Let* $X \leq A$ *be a subgroup of* A *which is arc-transitive on* Γ. *Let* N *be an insoluble normal subgroup of* X. *Then* N *has at most two orbits on* $V\Gamma$. *Furthermore, if* $|N| \nmid 120$, *then the following statements hold.*

(1) *For each* $v \in V\Gamma$, $5 \mid |N_v^{\Gamma(v)}|$.
(2) $2^2 \cdot 5^2 \cdot p \mid |N|$.

Proof. Suppose that N has at least three orbits on $V\Gamma$. Lemma 2.1 implies that $N_v = 1$ for each $v \in V\Gamma$. Hence $|N| \mid 40p$. If $p \neq 3$, then by Lemma 2.8, a group of order $40p$ is soluble, which follows that N is soluble, a contradiction. If $p = 3$, then $|N| \mid 40 \cdot 3 = 120$. It implies that $|N| = 60$ or 120 as N is insoluble, a contradiction with N has at least three orbits on $V\Gamma$. Hence N has at most two orbits on $V\Gamma$.

(1) For each $v \in V\Gamma$, if $N_v = 1$, then, arguing as the above paragraph, a contradiction occurs. Thus, $N_v \neq 1$. Since X is transitive on $V\Gamma$, $N \trianglelefteq X$ and Γ is connected, so we can conclude that $|N_v^{\Gamma(v)}| \neq 1$. It follows that $5 \mid |N_v^{\Gamma(v)}|$ since $N_v^{\Gamma(v)} \trianglelefteq X_v^{\Gamma(v)}$ and $X_v^{\Gamma(v)}$ acts primitively on $\Gamma(v)$.

(2) Since N has at most two orbits on $V\Gamma$, that is, $2^2 \cdot 5 \cdot p$ divides $|N : N_v|$ and by (1), $5 \mid |N_v|$, which implies that $2^2 \cdot 5^2 \cdot p \mid |N|$, as required. ∎

Lemma 3.2. *If A has no soluble minimal normal subgroup, then for every minimal normal subgroup N of A, N is isomorphic to T, where T is nonablelian simple group.*

Proof. Let N be a minimal normal subgroup of A. Then $N = T^d$ with T a nonabelian simple group. We just need to prove that $d = 1$. By Lemma 3.1, N has at most two orbits on $V\Gamma$, and so $20p$ divides $|N|$. It implies that $p \mid |T|$. Suppose that $d \geq 2$. Then $N = T_1 \times T_2 \times \cdots \times T_d$ and $p^d \mid |N|$, where $T_1 \cong T_2 \cong \ldots \cong T_d \cong T$. By Lemma 2.2, $|A_v| \mid 2^9 \cdot 3^2 \cdot 5$, we have $|N| \mid |A| \mid 2^{12} \cdot 3^2 \cdot 5^2 \cdot p$. Since $p \mid |N|$, we have $p \mid |T|$. It follows that $p^d \mid |N|$. Then the only possible case is $d = 2$ and $p \leq 5$. It implies that T is a $\{2, 3, 5\}$-nonabelian simple group. By Proposition 2.3, T is isomorphic to one of the following groups: A_5, A_6 or $PSU(4, 2)$. If $T \cong PSU(4, 2)$, then $3^8 \mid |A|$ as $|PSU(4, 2)| = 2^6 \cdot 3^4 \cdot 5$, a contradiction with $|A| \mid 2^{12} \cdot 3^2 \cdot 5^2 \cdot p$. If $T \cong A_6$, then $3^4 \mid |A|$ as $|A_6| = 2^3 \cdot 3^2 \cdot 5$, a contradiction with $|A| \mid 2^{12} \cdot 3^2 \cdot 5^2 \cdot p$. Hence $T = A_5$ and $N = A_5^2$. Let $C = C_A(N)$. Then $C \lhd A$ and $CN = C \times N$. If $C \neq 1$, then C is insoluble because A has no soluble minimal normal subgroup. Therefore, $3^3 \cdot 5^3 \mid |CN| \mid |A| \mid 2^{12} \cdot 3^2 \cdot 5^2 \cdot p$, a contradiction. Thus, $C = 1$. Hence, by 'N/C' theorem, $N \leq A \leq \mathrm{Aut}(N) = \mathrm{Aut}(T) \wr S_2$. With the help of the Magma [1], see our Magma codes in Appendices, there is no pentavalent symmetric graph of order $40p$. Hence $d = 1$, as required. ∎

We first consider the special cases that $p = 2, 3$ and 5 in the following lemmas.

Lemma 3.3. *If $p = 2$, then there is no pentavalent symmetric graph of order 80.*

Proof. Let N be a minimal normal subgroup of A. Suppose first that N is soluble. Then N is isomorphic to $\mathbb{Z}_r^d$ for some prime r. On the other hand, for each $v \in V\Gamma$, $|v^N|$ is a prime power and a divisor of 80, N has at least three orbits on $V\Gamma$. By Lemma 2.1, N is semiregular on $V\Gamma$. It follows that $|N| \mid |V\Gamma| = 2^4 \cdot 5$ and so $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3$, $\mathbb{Z}_2^4$ or $\mathbb{Z}_5$. If $N \cong \mathbb{Z}_2^4$, then Lemma 2.1 implies that Γ_N is a pentavalent symmetric graph of odd order, a contradiction. If $N \cong \mathbb{Z}_2^2$ or $\mathbb{Z}_2$, then Lemma 2.1 implies that Γ_N is a pentavalent symmetric graph of order 20 or 40. However, by Lemma 2.5 and Lemma 2.6, there is no pentavalent symmetric graph of order 20 or 40.

If $N \cong \mathbb{Z}_5$, then Γ_N is a pentavalent symmetric graph of order 16. By Lemma 2.6, $\Gamma_N \cong \mathrm{CL}_{16}$ and $\mathrm{Aut}\Gamma \cong \mathbb{Z}_2^4{:}\mathrm{S}_5$. By Magma [1], every arc-transitive subgroups of $\mathrm{Aut}\Gamma_N$ contains $\mathbb{Z}_2^4{:}\mathbb{Z}_5$. By Magma [1], $\mathbb{Z}_2^4{:}\mathbb{Z}_5$ is arc-regular on Γ_N. Therefore, A/N contains $H/N \cong \mathbb{Z}_2^4{:}\mathbb{Z}_5$, that is, A contains an arc-transitive subgroup $H \cong \mathbb{Z}_5.(\mathbb{Z}_2^4{:}\mathbb{Z}_5)$. By Magma [1] (see our Magma codes in Appendices), $H \cong \mathrm{SmallGroup}(400, 52)$ or $\mathrm{SmallGroup}(400, 213)$ and there is no pentavalent symmetric graph of order 80 for each two cases.

If $N \cong \mathbb{Z}_2^3$, then Γ_N is a pentavalent symmetric graph of order 10. By [3], $\Gamma_N \cong \mathsf{K}_{5,5}$ and $\mathrm{Aut}\Gamma_N \cong \mathrm{S}_5 \wr \mathrm{S}_2$. By Magma [1], every arc-transitive subgroups of $\mathrm{Aut}\Gamma_N$ contains one of the following arc-transitive subgroups:

$$(\mathbb{Z}_5{\times}\mathbb{Z}_5){:}\mathbb{Z}_2 \cong \mathrm{D}_{10}{\times}\mathbb{Z}_5, (\mathbb{Z}_5{\times}\mathbb{Z}_5){:}\mathbb{Z}_4, (\mathbb{Z}_5{\times}\mathbb{Z}_5){:}\mathbb{Z}_8.$$

Therefore, A/N contains $H/N \cong (\mathbb{Z}_5{\times}\mathbb{Z}_5){:}\mathbb{Z}_2$, $(\mathbb{Z}_5{\times}\mathbb{Z}_5){:}\mathbb{Z}_4$ or $(\mathbb{Z}_5{\times}\mathbb{Z}_5){:}\mathbb{Z}_8$. By Magma [1], there is no pentavalent symmetric graph of order 80 for these three cases.

Now we suppose that A has no soluble minimal normal subgroup. Then, by Lemma 3.2, $N = T \trianglelefteq A$, where T is a $\{2,3,5\}$-nonabelian simple group. By Proposition 2.3, N is isomorphic to A_5, A_6 or $\mathrm{PSU}(4,2)$. If $N \cong \mathrm{A}_5$, then Lemma 3.1 implies that N has at most two orbits on $V\Gamma$, that is, $2^3{\cdot}5 \mid |N|$, a contradiction with $|N| = 2^2{\cdot}3{\cdot}5$. If $N \cong \mathrm{A}_6$ or $\mathrm{PSU}(4,2)$, then Lemma 3.1(2) implies that $2^3{\cdot}5^2 \mid |N|$, a contradiction with $|\mathrm{A}_6| = 2^3{\cdot}3^2{\cdot}5$ and $|\mathrm{PSU}(4,2)| = 2^6 \cdot 3^4 \cdot 5$. ∎

Lemma 3.4. *If $p = 3$, then Γ is isomorphic to $\mathcal{C}_{120}^1$ or $\mathcal{C}_{120}^2$ as in Table 1.*

Proof. Let N be a minimal normal subgroup of A. Then $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3$, $\mathbb{Z}_3$ or $\mathbb{Z}_5$. If $N \cong \mathbb{Z}_2^3$, then Lemma 2.1 implies that Γ_N is a pentavalent symmetric graph of odd order, a contradiction. If $N \cong \mathbb{Z}_2^2$ or $\mathbb{Z}_3$, then Lemma 2.1 implies that Γ_N is a pentavalent symmetric graph of order 30 or 40. However, by Lemma 2.5 and Lemma 2.6, there is no pentavalent symmetric graph of order 30 or 40.

If $N \cong \mathbb{Z}_2$, then Γ_N is pentavalent symmetric graph of order 60. By [6], Γ_N is isomorphic to $\mathcal{C}_{60}$ and $\mathrm{Aut}(\mathcal{C}_{60}) \cong \mathrm{A}_5{\times}\mathrm{D}_{10}$. By Magma [1], A/N contains an arc-regular subgroup $H/N \cong \mathrm{A}_5{\times}\mathbb{Z}_5$. Hence $H \cong \mathbb{Z}_5{\times}\mathrm{SL}(2,5)$ or $\mathbb{Z}_{10}{\times}\mathrm{A}_5$ is arc-transitive on Γ. By Magma [1], $\Gamma \cong \mathcal{C}_{120}^1$ in Table 1.

If $N \cong \mathbb{Z}_5$, then Γ_N is a pentavalent symmetric graph of order 24. By Lemma 2.6, Γ_N is isomorphic to $\mathrm{I}^{(2)}$ with $\mathrm{Aut}\Gamma_N \cong (\mathrm{A}_5{\times}\mathbb{Z}_2^2){:}\mathbb{Z}_2$. By Magma [1], the arc-transitive subgroups of $\mathrm{Aut}\Gamma_N$ are one of the following groups:

$$S_5, A_5\times\mathbb{Z}_2, \mathbb{Z}_2\times S_5, \mathbb{Z}_2^2\times A_5, (A_5\times\mathbb{Z}_2^2){:}\mathbb{Z}_2.$$

By Magma [1], $S_5 \leq \mathbb{Z}_2\times S_5$ and $A_5\times\mathbb{Z}_2 \leq \mathbb{Z}_2^2\times A_5$. Furthermore, we have $\mathbb{Z}_5.S_5 \cong \mathbb{Z}_5\times S_5$ or $(\mathbb{Z}_5\times A_5){:}\mathbb{Z}_2$ and $\mathbb{Z}_5.(A_5\times\mathbb{Z}_2) \cong D_{10}\times A_5$ or $\mathbb{Z}_{10}\times A_5$, where $(\mathbb{Z}_5\times A_5){:}\mathbb{Z}_2$ is isomorphic to SmallGroup(600, 145). It implies that A contains an arc-transitive subgroup isomorphic to $\mathbb{Z}_5\times S_5$, $(\mathbb{Z}_5\times A_5){:}\mathbb{Z}_2$, $D_{10}\times A_5$ or $\mathbb{Z}_{10}\times A_5$. By Magma [1], $\Gamma \cong \mathcal{C}_{120}^1$ or $\mathcal{C}_{120}^2$ in Table 1.

Now we suppose that A has no soluble minimal normal subgroup. Then, by Lemma 3.2, $N = T \trianglelefteq A$, where T is a $\{2,3,5\}$-nonabelian simple group. By Proposition 2.3, T is isomorphic to one of the following groups: A_5, A_6 or PSU(4, 2). If $N \cong A_6$ or PSU(4, 2), then Lemma 3.1 implies that $2^2\cdot 3\cdot 5^2 \mid |N|$, which is impossible as $|A_6| = 2^3\cdot 3^2\cdot 5$ and $|\mathrm{PSU}(4,2)| = 2^6\cdot 3^4\cdot 5$. If $N \cong A_5$, then $A/C_A(N) \lesssim \mathrm{Aut}(N) \cong S_5$. If $C_A(N) = 1$, then $N \leq A \leq S_5$. It follows that $A \cong A_5$ or S_5 and $|A_v| = \frac{|A|}{|V\Gamma|} = \frac{1}{2}$ or 1, which is impossible. Thus, we have $C_A(N) \neq 1$. Since A has no soluble minimal normal subgroup, we have $C_A(N)$ is insoluble. On the other hand, $C_A(N) \cap N = Z(N) = 1$, we have $C_A(N)N = C_A(N)\times N$. Furthermore, $C_A(N)$ contains an insoluble normal subgroup isomorphic to A_5 as A is $\{2,3,5\}$-group and the insoluble minimal normal subgroup of A is not isomorphic to A_6 or PSU(4, 2). Hence A contains a normal subgroup isomorphic to A_5^2. Since $C_A(A_5^2) = 1$, by 'N/C' theorem, we have $A \leq \mathrm{Aut}(A_5^2) \cong \mathrm{Aut}(A_5) \wr S_2$. By Magma [1], there is no pentavalent symmetric graph of order 80 for this case. ∎

Lemma 3.5. *If $p = 5$, then there is no pentavalent symmetric graph of order* 200.

Proof. Let N be a minimal normal subgroup of A. Suppose first that N is soluble. Then $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_5$ or $\mathbb{Z}_5^2$. If $N \cong \mathbb{Z}_2^3$, then Lemma 2.1 implies that Γ_N is a pentavalent symmetric graph of odd order, a contradiction. If $N \cong \mathbb{Z}_5$ or $\mathbb{Z}_5^2$, then Lemma 2.1 implies that Γ_N is a pentavalent symmetric graph of order 40 or 8. However, by Lemma 2.5, there is no pentavalent symmetric graph of order 40. Further, by [12, p.1112], $|A_v| = 2$ or 16 of a pentavalent vertex-transitive graph of order 8. It implies that there is no pentavalent symmetric graph of order 8.

For the case $N \cong \mathbb{Z}_2$, we first prove the following claim:

Claim: There is no pentavalent symmetric graph of order 100.

Let Σ be a pentavalent symmetric graph of order 100 and let $L = \mathrm{Aut}\Sigma$. Suppose first that L has a soluble minimal normal subgroup M. With similar discussion as above, we have $M \cong \mathbb{Z}_2$ and Σ_M is pentavalent symmetric graph with order 50. By Lemma 2.5, Σ_M is isomorphic to $\mathcal{C}_{50}$ and $\mathsf{Aut}(\mathcal{C}_{50}) \cong G{:}(\mathbb{Z}_4^2{:}\mathbb{Z}_2)$ is soluble. Then L is soluble because $A/M \lesssim \mathsf{Aut}(\mathcal{C}_{50})$. Let F be the Fitting subgroup of L, the subgroup generated by all the normal nilpotent subgroups of L. Since L is soluble, we have $F \neq 1$ and $\mathrm{C}_L(F) \leq F$ (see [16, 5.4.4] for example). Since L has no nontrivial normal 5-subgroup and F is not isomorphic to $\mathbb{Z}_2^2$, we have $F = \mathrm{O}_2(L) \cong \mathbb{Z}_2$. Thus, F is abelian and $\mathrm{C}_L(F) = F$. It follows that $L/F = L/\mathrm{C}_L(F) \lesssim \mathsf{Aut}(F) = 1$, which is impossible. Now we suppose that L has no soluble minimal normal subgroup. By Lemma 3.2, $M = T \trianglelefteq L$ is isomorphic to A_5, A_6 or $\mathrm{PSU}(4,2)$. By Lemma 3.1, M has at most two orbits on $V\Gamma$, which implies that $2 \cdot 5^2 \mid |M|$, a contradiction with $|M| = 2^2 \cdot 3 \cdot 5$, $2^3 \cdot 3^2 \cdot 5$ or $2^6 \cdot 3^4 \cdot 5$, as we claim.

If $N \cong \mathbb{Z}_2$, then Γ_N is a pentavalent symmetric graph of order 100. By the above claim, this is impossible.

If $N \cong \mathbb{Z}_2^2$, then Γ_N is a pentavalent symmetric graph of order 50. Arguing as the above, A is soluble and the Fitting subgroup F of A is isomorphic to $\mathbb{Z}_2^2$. It follows that $A/F = A/\mathrm{C}_A(F) \lesssim \mathsf{Aut}(F) \cong \mathrm{GL}(2,2)$, which is impossible.

Now we suppose that A has no soluble minimal normal subgroup. By Lemma 3.2, $N = T \trianglelefteq A$ is isomorphic to A_5, A_6 or $\mathrm{PSU}(4,2)$. This is impossible since N has at most two orbits on $V\Gamma$ which implies that $2^2 \cdot 5^2 \mid |N|$. ∎

Now we consider the case when $p > 5$. First we suppose that A contains a soluble minimal normal subgroup N, then we have the following lemma.

Lemma 3.6. *If A has a soluble minimal normal subgroup N, then no graph appears.*

Proof. Let N be a soluble minimal normal subgroup, then $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_5$ or $\mathbb{Z}_p$. If $N \cong \mathbb{Z}_2$, then Γ_N is a pentavalent symmetric graph of order $20p$. However, by [15], there is no pentavalent symmetric graph of order $20p$, a contradiction. If $N \cong \mathbb{Z}_2^3$, then Γ_N is a pentavalent symmetric graph of odd order, which is impossible. If $N \cong \mathbb{Z}_p$, then Γ_N is a pentavalent symmetric graph of order 40, by Lemma 2.5, which is also impossible. Hence suppose first $N \cong \mathbb{Z}_2^2$. Then Γ_N is a pentavalent symmetric graph of order $10p$. By Lemma 2.5, we have $\Gamma_N \cong \mathcal{C}_{170}$ or $\mathcal{CD}_{10p}^l$.

If $\Gamma \cong \mathcal{CD}_{10p}^l$, then $A/N \leq \mathsf{Aut}\Gamma \cong \mathrm{D}_{10p}{:}\mathbb{Z}_5$. Since A/N is arc-transitive on Γ_N, we have $A/N \cong \mathrm{D}_{10p}{:}\mathbb{Z}_5$, which follows that $A \cong \mathbb{Z}_2^2{:}(\mathrm{D}_{10p}{:}\mathbb{Z}_5)$. Since $\mathbb{Z}_p$ is a normal subgroup of $\mathrm{D}_{10p}{:}\mathbb{Z}_5$ and $\mathbb{Z}_p$ centralizes $\mathbb{Z}_2^2$, we have $\mathbb{Z}_p$ is a normal subgroup of A. It implies that the corresponding normal quotient graph is a pentavalent symmetric graph of order 40, which is impossible by Lemma 2.5.

If $\Gamma_N \cong C_{170}$, then $A/N \leq \mathrm{Aut}\Gamma_N \cong \mathrm{Aut}(\mathrm{PSp}(4,4))$ and $p = 17$. Since A/N is arc-transitive on Γ_N, we have $5 \cdot 170 \mid |A/N|$. By Atlas [2], A/N contains a normal subgroup M/N isomorphic to $\mathrm{PSp}(4,4)$. By Atlas [2], the Schur multiplier of $\mathrm{PSp}(4,4)$ is 1, Lemma 2.7 implies that $M = \mathbb{Z}_2^2 \times \mathrm{PSp}(4,4)$. Then $\mathrm{PSp}(4,4) \trianglelefteq A$ because $M' = \mathrm{PSp}(4,4)$ is a characteristic subgroup of M and $M \trianglelefteq A$. By Lemma 3.1, $\mathrm{PSp}(4,4)$ has at most two orbits on $V\Gamma$. Hence $|M'_v| = \frac{|M'|}{40 \cdot 17} = 1440$ or $|M'_v| = \frac{|M'|}{20 \cdot 17} = 2880$, which is a contradiction as $\mathrm{PSp}(4,4)$ has no subgroup of order 1440 or 2880 by Magma [1].

Suppose now $N \cong \mathbb{Z}_5$. Then, by Lemma 2.6, Γ_N is isomorphic to C_{248} and $p = 31$. Furthermore, $A/N \leq \mathrm{Aut}\Gamma_N \cong \mathrm{PSL}(2,31)$. Note that A/N acts arc-transitively on Γ_N and so $5 \cdot 248 \mid |A/N|$. By checking the maximal subgroup of $\mathrm{PSL}(2,31)$, we have $A/N \cong \mathrm{PSL}(2,31)$. On the other hand, by Atlas [2], the Schur multiplier of $\mathrm{PSL}(2,31)$ is isomorphic to $\mathbb{Z}_2$, Lemma 2.7 implies that $A = \mathbb{Z}_5 \times \mathrm{PSL}(2,31)$. Since $A' = \mathrm{PSL}(2,31) \trianglelefteq A$, Lemma 3.1 implies that $2^2 \cdot 5^2 \cdot 31 \mid |\mathrm{PSL}(2,31)| = 2^5 \cdot 3 \cdot 5 \cdot 31$, a contradiction. ∎

Now we may treat the case that A has no soluble minimal normal subgroup and the next lemma completes the proof of Theorem 1.1.

Lemma 3.7. *If A has no soluble minimal normal subgroup, then no graph appears.*

Proof. Let $N = T^d$ be an insoluble minimal normal subgroup of A. By Lemma 3.2, $d = 1$, and so $N = T \trianglelefteq A$. By Lemma 3.1, N has at most two orbits on $V\Gamma$ and so $20p \mid |N|$. Since $p > 5$, we have $120 \nmid |N|$ and Lemma 3.1 implies that $2^2 \cdot 5^2 \cdot p \mid |T|$. Since $|A| \mid 2^{12} \cdot 3^2 \cdot 5^2 \cdot p$, we have $|T|$ is a divisor of $2^{12} \cdot 3^2 \cdot 5^2 \cdot p$. By Proposition 2.4, T is isomorphic to $\mathrm{PSL}(2,25)$, $\mathrm{PSU}(3,4)$ or $\mathrm{PSp}(4,4)$. Note that T has at most two orbits on $V\Gamma$, hence $|T_v| = \frac{|T|}{40p}$ or $|T_v| = \frac{|T|}{20p}$.

Suppose that $T \cong \mathrm{PSU}(3,4)$. Then $p = 13$ and $|T_v| = 120$ or 240. However, by Atlas [2], $\mathrm{PSU}(3,4)$ has no subgroup of order 120 or 240. Suppose that $T \cong \mathrm{PSp}(4,4)$. Then $p = 17$ and $|T_v| = 1440$ or 2880. However, $\mathrm{PSp}(4,4)$ has no subgroup of order 1440 or 2880. Suppose that $T \cong \mathrm{PSL}(2,25)$. Then $p = 13$ and $|T_v| = 15$ or 30. However, by Atlas [2], $\mathrm{PSL}(2,25)$ has no subgroup of order 15 or 30. ∎

Appendices

Magma codes

```
/*
Input : a positive integer n and two finite groups G, N
Output: all groups X of order n, which has the quotient group X/N iso-
morphic to G
*/
f:=function(n,G,N);
P:=SmallGroupProcess(n);
X:=[];
repeat GG:=Current(P);
NN:=NormalSubgroups(GG);
for i in [1..#NN] do
if IsIsomorphic(NN[i]`subgroup,N) eq true then
F:=quo<GG|NN[i]`subgroup>;
if IsIsomorphic(F,G) eq true then
_,a:=CurrentLabel(P);
Append(~X,SmallGroup(n,a));
end if;
end if;
end for;
Advance(~P);
until IsEmpty(P);
return X;
end function;
```

```
/*
Input : a finite group G and a positive integer n
Output: all graphs of order |G|/n, which admit G as an arc-transitive au-
tomorphism group
*/
Graph:=function(G,n);
graph:=[];
i:=0;
H:=Subgroups(G:OrderEqual:=n);
for j in [1..#H] do
HH:=H[j]`subgroup;
CA:=CosetAction(G,HH);
O:=Orbits(CA(HH));
for k in [1..#O] do
```

```
OO:=SetToSequence(O[k]);
GR:=OrbitalGraph(CA(G),1,OO[1]);
if (IsConnected(GR) eq true) and (Valence(GR) eq 5) and (not exists{t:t
in graph|IsIsomorphic(GR,t) eq true}) then
Append(~graph,GR);
i:=i+1;
end if;
end for;
end for;
return i,graph;
end function;
```

REFERENCES

[1] W. Bosma, C. Cannon, C. Playoust, The MAGMA algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235-265.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Cambridge Press, Oxford, 1985.

[3] Y. Cheng, J. Oxley, On the weakly symmetric graphs of order twice a prime, *J. Combin. Theory, Ser. B* **42** (1987), 196-211.

[4] S. T. Guo, Y. Q. Feng, A note on pentavalent s-transitive graphs, *Discrete Math.* **312** (2012), 2214-2216.

[5] S. T. Guo, Y. Q. Feng, C. H. Li, The finite edge-primitive pentavalent graphs, *J. Algebraic Combin.* **38** (2013), 491-497.

[6] S. T. Guo, J. X. Zhou, Y. Q. Feng, Pentavalent symmetric graphs of order $12p$, *Electronic J. Combin.* **18** (2011), #P233.

[7] D. Gorenstein, *Finite Simple Group*, Plenum Press, New York, 1982.

[8] X. H. Hua, Y. Q. Feng, Pentavalent symmetric graphs of order $8p$, *J. Beijing Jiaotong University* **35** (2011), 132-135, 141.

[9] X. H. Hua, Y. Q. Feng, Pentavalent symmetric graphs of order $2pq$, *Discrete Math.* **311** (2011), 2259-2267.

[10] C. H. Li, J. M. Pan, Finite 2-arc-transitive abelian Cayley graphs, *European J. Combin.* **29** (2008), 148-158.

[11] Y. T. Li, Y. Q. Feng, Pentavalent one-regular graphs of square-free order, *Algebraic Colloquium* **17** (2010), 515-524.

[12] B. D. Mckay, Transitive graphs with fewer than 20 vertices, *Math. Comp.* **33** (1979), 1101-1121.

[13] J.M. Pan, Y. Liu, Z.H. Huang and C.L. Liu, Tetravalent edge-transitive graphs of order p^2q, *Sci. China Math.* **57** (2014), 293-302.

[14] J.M. Pan, Z.Liu and X.F. Yu, Pentavalent graphs of order twice a prime square, *Algebraic Colloquium*, **22**(2015), 383-394.

[15] J. M. Pan, B. G. Lou, C. F. Liu, Arc-transitive pentavalent graphs of order $4pq$, *Electronic J. Combin.* **20(1)** (2013), #P36.

[16] D. J. S. Robinson. *A course in the theory of groups*, Springer-Verlag New York Berlin Heidelberg, 1995.

[17] J. X. Zhou, Y. Q. Feng, On symmetric graphs of valency five, *Discrete Math.* **310** (2010), 1725-1732.

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