# Panconnectedness of K-trees with Sufficiently Large Toughness

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**Abstract**

In this paper, we prove that if the toughness of a k-tree G is at least (k+1)/3, then G is panconnected for k $\geq$ 3, or G is vertex pancyclic for k = 2. This result improves a result of Broersma, Xiong and Yoshimoto.

Keywords: Panconnectedness, toughness, k-tree.

## 1. Introduction and terminology

The graphs in this paper are finite, undirected, connected and simple. For all terminology and notation not defined in this paper, the reader is referred to [4].

Let G = (V, E) be a graph. We denote $|V|$ by $\nu(G)$. If $P = u_0u_1u_2\cdots u_k$ is a path with $x = u_0$ and $y = u_k$, we say that $P$ is a path from $x$ to $y$ or $P$ is a $(x, y)$ path, denoted by $P(x, y)$ or $P$, and we say that $P$ is a path of length $k$. If $C = u_1u_2\cdots u_ku_1$ is a cycle, we say that $C$ is a cycle of length k. Let $P(x, y)$ and $Q(y, z)$ be two paths which are disjoint except at y, then $P(x, y) + Q(y, z)$ denotes the path from $x$ to $y$ along $P$ and then from $y$ to $z$ along $Q$. If $z = x$, $P(x, y) + Q(y, z)$ denotes the cycle from $x$ to $y$ along $P$ and then from $y$ to $z$ $(= x)$ along $Q$. Let $u$ and $v$ be two vertices of $G$. We denote by $d(x, y)$ the distance between $u$ and $v$ that is the length of a shortest $(u, v)$ path. A connected graph $G$ is said to be panconnected if , for any two vertices $u$ and $v$ in G, there is a path $P$ from $u$ to $v$ of length $L$ for each integer $L$ from $d(u, v)$ to $\nu(G) - 1$ in $G$. A graph $G$ is called pancyclic if there is a cycle $C$ of length $L$ for each integer $L$ from 3 to $\nu(G)$ in G. In particular, if $G$ has a cycle of length $\nu(G)$, then $G$ is called Hamiltonian. A graph $G$ is called vertex pancyclic if, for each vertex v in G, there is a cycle $C$ containing $v$ of length $L$ for each integer $L$ from 3 to $\nu(G)$ in $G$. A graph $G$ is called edge pancyclic if, for each edge $e$ in $G$, there is a cycle $C$ containing $e$ of length $L$ for each integer $L$ from 3 to $\nu(G)$ in $G$. Notice that a graph $G$ to be panconnected implies that $G$ is edge pancyclic, $G$ to be edge pancyclic implies that $G$ is vertex pancyclic, and $G$ to be vertex pancyclic implies that $G$ is pancyclic.

Let $G$ be a graph. Let $S \subseteq V(G)$ with $S \neq \emptyset$. The subgraph of $G$ with vertex set $S$ and edge set consisting of all edges in $G$ with both ends in $S$ is called the induced subgraph of $G$ on $S$, denoted by $G[S]$. Let $S \subseteq V(G)$ and $S \neq V(G)$, the $G - S = G[V(G)\backslash S]$. If $S = \{x\}$, we use $G - x$ to represent $G - \{x\}$. If $e \in E(G)$, we use $G - e$ to represent the subgraph of $G$ by deleting $e$ from $G$ with two ends remained. Let $\omega(G)$ denote the number of components of $G$. A graph $G$ is called t-tough if $|S| \geq t\omega(G-S)$ for each subset $S$ of $V(G)$ with $\omega(G - S) > 1$. The toughness of $G$ is denoted by $\tau(G)$ and is defined as follows: If $G$ is not complete, then $\tau(G)$ $= \min\{|S|/\omega(G - S)\}$, where the minimum is taken over all cutsets $S$ of vertices in $G$, otherwise $G = K_n$ and $\tau(G) = \infty$. We denote by $N(v)$ the set of all neighbours of vertex $v$ in $G$. Let $G$ and $H$ be two disjoint graphs. We denote by $G \oplus H$ the graph $F$ with $V(F) = V(G) \cup V(H)$ and $E(F) = E(G) \cup E(H) \cup \{ uv \mid u \in V(G) \text{ and } v \in V(H) \}$.

We define a graph $G$ to be chordal if $G$ contains no chordless cycle of length at least 4. Then we define a k-tree as follows: $K_k$ is the smallest k-tree, and a graph $G$ on at least $k + 1$ vertices is a k-tree if and only if it contains a vertex $v$ of degree $k$ such that the neighbours of $v$ are mutually adjacent and $G-v$ is a k-tree. We call $v$ a k-simplicial vertex, or a simplicial vertex for short. Obviously, a 1-tree is just a tree.

The concept of toughness was introduced by Chvátal [7] in 1973, it is clear that being 1-tough is a necessary condition for a graph to be Hamiltonian. Chvátal [7] conjectured that there exists a finite constant $t_0$ such that every $t_0$-tough graph is Hamiltonian. It had been a long standing conjecture for $t_0 = 2$ until Bauer et al. [1] showed that for every $\varepsilon > 0$, there exists a $(\frac{9}{4} - \varepsilon)$-tough nontraceable graph (a graph without Hamiltonian path), which disproved the conjecture. Chvátal [7] obtained $(\frac{3}{2} - \varepsilon)$-tough graphs without a 2-factor for arbitrary $\varepsilon > 0$. These examples are all chordal. Recently Bauer et al. [2] showed that every $\frac{3}{2}$-tough chordal graph has a 2-factor. Motivated by this result, Kratsch raised the question whether every $\frac{3}{2}$ -tough chordal graph is Hamiltonian. But, in [1], it is shown that there is an infinite class of chordal graphs with toughness close to $\frac{7}{4}$ having no Hamiltonian path, and hence no Hamiltonian cycle. However, Böhme et al. [3] showed that let $G$ be a chordal planar graph with $\tau(G) > 1$, then $G$ is Hamiltonian. Although $\frac{3}{2}$-tough chordal graphs are not necessarily Hamiltonian, Chen et al. [6] proved that every 18-tough chordal graph is Hamiltonian. Recently, Broersma, Xiong and Yoshimoto [5] showed that k-trees (a subclass of chordal graphs) are Hamiltonian if the toughness is at least $\frac{k+1}{3}$ for $k \geq 2$. The authors of this paper try to extend the result of Broersma et al. [5] to panconnectedness.

K-trees are not only a subclass of chordal graphs, but also an important class of graphs in computer science. Robertson and Seymour [9] introduced the concept of treewidth of graphs. According to [8], a graph $G$ has treewidth at most k if and only if $G$ is a partial k-tree, which is a subgraph of a k-tree. According to [8] ( Chapter 10 Graphs in Computer Science), graphs with treewidth at most k are a class of recursively constructed graphs. Many NP-complete problems can be solved in linear time on these graphs. Motivated also by this fact, the authors work on the property of k-trees.

## 2. Preliminary results

In this section, we introduce some basic properties and notation of k-trees obtained by Broersma et al. [5] first.

Let $S_1(K_k) = \emptyset$, and for a k-tree $G \neq K_k$, let $S_1(G)$ denote the set of k-simplicial vertices of $G$ if $G \neq K_{k+1}$ and a set of one arbitrary vertex of $G$ if $G = K_{k+1}$. Now we give a lemma.

**Lemma 1:** Let $G \neq K_k$ be a k-tree (k≥2). Then
(1) $S_1(G) \neq \emptyset$;
(2) $S_1(G)$ is an independent set;
(3) Every k-simplicial vertex (if any) of $G - S_1(G)$ is adjacent in $G$ to at least one vertex of $S_1(G)$;
(4) $\tau(G - v) \geq \tau(G)$ for any k-simplicial vertex $v \in S_1(G)$;
(5) $\tau(G - S_1(G)) \geq \tau(G)$.

**Proof.** See [5], Lemma 6.

Then we define $S_i(G)$. For a k-tree $G \neq K_k$, let $S_i(G)$ and $G_i$ be defined as follows: $G_1 = G$, $S_1(G)$ is defined before Lemma 1, $G_i = G_{i-1} - S_1(G_{i-1})$ and $S_i(G) = S_1(G_i)$ for $i = 2, 3, \cdots$ as long as $S_i(G) \neq \emptyset$ (i.e. $G_i \neq K_k$). We denote by $N_i(v)$ the set of neighbours of $v$ in $G_i$.

**Lemma 2:** For any vertex $u \in S_2(G)$ (if any), there exists a vertex $v \in S_1(G)$ such that $uv \in E(G)$, and $N_1(u) \backslash N_2(u) \subseteq S_1(G)$.

**Proof.** See [5], Lemma 8.

**Lemma 3:** If $u \in S_2(G)$, then $N_1(w) \subseteq N_2(u) \cup \{u\}$ for any $w \in N_1(u) \backslash N_2(u)$.

**Proof.** See [5], Lemma 9.

Next, we introduce some basic properties of k-trees obtained by the authors of this paper.

**Lemma 4:** If $G$ is a k-tree and $G \neq K_k$ or $K_{k+1}$, then $|S_1(G)| \geq 2$.

**Proof.** Since $G \neq K_k$ or $K_{k+1}$, $\nu(G) \geq k+2$. We proceed by induction on $\nu(G)$. Suppose $\nu(G) = k+2$. Then by the definition of a k-tree, $G$ is constructed by adding a k-simplicial vertex $v$ to $K_{k+1}$ and connecting $v$ to k vertices of $K_{k+1}$ by edges. Then $K_{k+1}$ has a vertex $u$ not adjacent to $v$, and $u$ and $v$ are two k-simplicial vertices in $S_1(G)$.

Assume that, when $\nu(G) = m$ ($m \geq k+2$), we have $|S_1(G)| \geq 2$.

Now suppose $\nu(G) = m+1$. We shall prove that $|S_1(G)| \geq 2$. By Lemma 1, there is a vertex $v \in S_1(G)$. Let $H = G - v$. Since $H$ is a k-tree and $\nu(H) \geq k+2$, by induction hypothesis, $|S_1(H)| \geq 2$. If $v$ is not adjacent to any vertex of $S_1(H)$, then $S_1(G) = S_1(H) \cup \{v\}$. So $|S_1(G)| = |S_1(H)| + 1 \geq 2 + 1 > 2$. If $v$ is adjacent to a vertex $u$ in $S_1(H)$, then $u \in S_2(G)$, by Lemma 3, $N_1(v) \subseteq N_2(u) \cup \{u\}$. Since $S_1(H)$ is independent by Lemma 1, $v$ is not adjacent to any vertex in $S_1(H)\backslash\{u\}$ as $v$ is a simplicial vertex of $G$. So $S_1(G) = (S_1(H)\backslash\{u\}) \cup \{v\}$, and $|S_1(G)| = |S_1(H)| - 1 + 1 = |S_1(H)| \geq 2$. Then this lemma is proved. □

**Lemma 5:** If $G$ is a k-tree, $G \neq K_k$ and $G - S_1(G) \neq K_k$ or $K_{k+1}$, then $|S_2(G)| \geq 2$ and $S_2(G)$ is an independent set.

**Proof.** Since $G \neq K_k$ and $G - S_1(G) \neq K_k$ or $K_{k+1}$, by Lemma 4, $H = G - S_1(G)$ satisfies that $|S_1(H)| \geq 2$. But $S_1(H) = S_2(G)$, by (2) of Lemma 1, $|S_2(G)| \geq 2$ and $S_2(G)$ is an independent set. □

**Lemma 6:** Let $H$ be a graph of $r$ independent vertices $v_1, v_2, \cdots, v_r$. If a k-tree $G = K_k \oplus H$ with $r < k$, then $G$ is panconnected, where $V(H) \cap V(K_k) = \emptyset$.

**Proof.** Let $V(K_k) = \{u_1, u_2, \cdots, u_k\}$. We just verify that, for any two vertices $x$ and $y$, there is a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x,y)$ to $\nu(G) - 1$ in $G$.

Case 1: $x = u_i$ and $y = v_j$ $(1 \le i \le k$ and $1 \le j \le r)$.

By symmetry, we can assume that $x = u_1$ and $y = v_r$ without loss of generality. We notice that $d(x,y) = 1$.

Then $P = (x =)u_1v_1u_2v_2u_3 \cdots u_mv_r(= y)$ is a path from $x$ to $y$ of length $2m - 1$ $(1 \le m \le r)$; $P = (x =)u_1v_1u_2v_2u_3 \cdots u_mu_{m+1}v_r(= y)$ is a path from $x$ to $y$ of length $2m$ $(1 \le m \le r)$; and $P = (x =)u_1v_1u_2v_2u_3 \cdots u_ru_{r+1} \cdots u_{r+s}v_r(= y)$ is a path from $x$ to $y$ of length $2r - 1 + s$ $(1 \le s \le k - r)$.

Case 2: $x = v_i$ and $y = v_j$ $(1 \le i < j \le r)$

By symmetry, we can assume that $x = v_1$ and $y = v_r$ without loss of generality. We notice that $d(x,y) = 2$.

Then $P = (x =)v_1u_1v_2u_2 \cdots v_mu_mv_r(= y)$ is a path from $x$ to $y$ of length $2m$ $(1 \le m \le r - 1)$; $P = (x =)v_1u_1v_2u_2 \cdots v_mu_mu_{m+1}v_r(= y)$ is a path from $x$ to $y$ of length $2m + 1$ $(1 \le m \le r - 1)$; and $P = (x =)v_1u_1v_2u_2 \cdots v_{r-1}u_{r-1}u_r \cdots u_{r-1+s}v_r(= y)$ is a path from $x$ to $y$ of length $2(r - 1) + s$ $(1 \le s \le k - r + 1)$.

Case 3: $x = u_i$ and $y = u_j$ $(1 \le i < j \le k)$.

By symmetry, we can assume that $x = u_1$ and $y = u_k$ without loss of generality. We notice that $d(x,y) = 1$.

Then $P = (x =)u_1v_1u_2v_2u_3 \cdots u_mv_mu_k(= y)$ is a path from $x$ to $y$ of length $2m$ $(1 \le m \le r)$; $P = (x =)u_1v_1u_2v_2 \cdots u_mu_k(= y)$ is a path from $x$ to $y$ of length $2m-1$ $(1 \le m \le r)$; and $P = (x =)u_1v_1u_2v_2u_3 \cdots v_ru_{r+1}u_{r+2} \cdots u_{r+s}u_k(= y)$ is a path from $x$ to $y$ of length $2r + s$ $(1 \le s \le k - r - 1)$.

Hence G is panconnected. The proof of this lemma is complete. □

**Lemma 7:** Let $G = (V,E)$ be a k-tree $(k \ge 3)$ such that $G - S_1(G) = K_{k+1}$, $V(K_{k+1}) = \{u_0, u_1, \cdots, u_k\}$ and $S_1(G) = \{v_1, v_2, v_3\}$. Suppose that $v_i$ is adjacent to all vertices of $u_0, u_1, \cdots, u_k$ but $u_i$ $(i = 1, 2, 3)$. Then $G$ is panconnected.

**Proof.** We just verify that, for any two vertices $x$ and $y$ in G, there is a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x,y)$ to $\nu(G) - 1$, and hence $G$ is panconnected. Let $P_r$ denote a path from $x$ to $y$ of length $r$.

Case 1: $x = u_i$ and $y = v_j$ $(0 \le i \le k$ and $1 \le j \le 3)$.

By symmetry, we can assume that $y = v_1$.

Case (1.1): $x = u_i$ $(i = 0, 4, 5, \cdots, k)$.

Without loss of generality, assume that $x = u_0$. Notice that $d(x,y) = 1$. Then $P_1 = (x =)u_0v_1(= y)$; $P_2 = (x =)u_0u_2v_1(= y)$; $P_3 = (x =)u_0v_2u_3v_1(= y)$; $P_4 = (x =)u_0v_2u_1u_2v_1(= y)$; $P_5 = (x =)u_0v_2u_1v_3u_2v_1(=$

$y$); $P_6 = (x =)u_0v_2u_1v_3u_2u_3v_1(= y)$; $P_{6+s} = (x =)u_0v_2u_1v_3u_2u_3 \cdots u_{3+s}v_1(= y)$ $(1 \le s \le k-3)$.

Case (1.2): $x = u_1$.

Notice that $d(x,y) = 2$. Then $P_2 = (x =)u_1u_0v_1(= y)$; $P_3 = (x =)u_1v_2u_0v_1(= y)$; $P_4 = (x =)u_1v_2u_3u_0v_1(= y)$; $P_5 = (x =)u_1v_2u_3u_2u_0v_1(= y)$; $P_6 = (x =)u_1v_2u_3u_2v_3u_0v_1(= y)$; $P_{6+s} = (x =)u_1v_2u_3u_2v_3u_0u_4u_5 \cdots u_{3+s}v_1(= y)$ $(1 \le s \le k-3)$.

Case (1.3): $x = u_2$ or $u_3$.

By symmetry, we assume that $x = u_2$ without loss of generality. Notice that $d(x,y) = 1$.

Then $P_1 = (x =)u_2v_1(= y)$; $P_2 = (x =)u_2u_0v_1(= y)$; $P_3 = (x =)u_2v_3u_0v_1(= y)$; $P_4 = (x =)u_2v_3u_0u_3v_1(= y)$; $P_5 = (x =)u_2v_3u_0u_1u_3v_1(= y)$; $P_6 = (x =)u_2v_3u_1v_2u_3u_0v_1(= y)$; $P_{6+s} = (x =)u_2v_3u_1v_2u_3u_0u_4u_5 \cdots u_{3+s}v_1(= y)$ $(1 \le s \le k-3)$.

Case 2: $x = v_i$ and $y = v_j$ $(1 \le i < j \le 3)$.

By symmetry, we assume that $x = v_1$ and $y = v_3$ without loss of generality. Notice that $d(x,y) = 2$.

Then $P_2 = (x =)v_1u_0v_3(= y)$; $P_3 = (x =)v_1u_0u_1v_3(= y)$; $P_4 = (x =)v_1u_0v_2u_1v_3(= y)$; $P_5 = (x =)v_1u_0v_2u_1u_2v_3(= y)$; $P_6 = (x =)v_1u_0v_2u_1u_3u_2v_3(= y)$; $P_{6+s} = (x =)v_1u_0v_2u_1u_3u_2u_4u_5 \cdots u_{3+s}v_3(= y)$ $(1 \le s \le k-3)$;

Case 3: $x = u_i$ and $y = u_j$ $(0 \le i < j \le k)$.

By symmetry, we have the following subcases. Notice that $d(x,y) = 1$.

Case (3.1): $x = u_i$ and $y = u_j$, $i, j \in \{0, 4, 5, \cdots, k\}$ and $i \ne j$.

By symmetry, we assume that $x = u_0$ and $y = u_k$ without loss of generality.

Then $P_1 = (x =)u_0u_k(= y)$; $P_2 = (x =)u_0v_1u_k(= y)$; $P_3 = (x =)u_0v_1u_2u_k(= y)$; $P_4 = (x =)u_0v_3u_1v_2u_k(= y)$; $P_5 = (x =)u_0v_1u_2v_3u_1u_k(= y)$; $P_6 = (x =)u_0v_1u_2v_3u_1v_2u_k(= y)$; $P_{6+s} = (x =)u_0v_1u_2v_3u_1v_2u_3u_4 \cdots u_{3+s-1}u_k(= y)$ $(1 \le s \le k-3)$.

Case (3.2): $x = u_i$ and $y = u_j$, $i \in \{0, 4, 5, \cdots, k\}$ and $j \in \{1, 2, 3\}$.

By symmetry, assume that $x = u_0$ and $y = u_1$.

Then $P_1 = (x =)u_0u_1(= y)$; $P_2 = (x =)u_0v_2u_1(= y)$; $P_3 = (x =)u_0v_1u_2u_1(= y)$; $P_4 = (x =)u_0v_1u_2v_3u_1(= y)$; $P_5 = (x =)u_0u_3v_1u_2v_3u_1(= y)$; $P_6 = (x =)u_0v_2u_3v_1u_2v_3u_1(= y)$; $P_{6+s} = (x =)u_0v_2u_3v_1u_2v_3u_4u_5 \cdots u_{3+s}u_1(= y)$ $(1 \le s \le k-3)$.

Case (3.3): $x = u_i$ and $y = u_j$ $(1 \le i < j \le 3)$.

By symmetry, assume that $x = u_1$ and $y = u_3$.

Then $P_1 = (x =)u_1u_3(= y)$; $P_2 = (x =)u_1v_2u_3(= y)$; $P_3 = (x =)u_1v_2u_0u_3(= y)$; $P_4 = (x =)u_1v_2u_0v_1u_3(= y)$; $P_5 = (x =)u_1v_2u_0v_1u_2u_3(= y)$; $P_6 = (x =)u_1v_3u_2v_1u_0v_2u_3(= y)$; $P_{6+s} = (x =)u_1v_3u_2v_1u_0v_2u_4u_5 \cdots u_{3+s}u_3(= y)$ $(1 \le s \le k-3)$.

Hence $G$ is panconnected. The proof of this lemma is complete. □

**Lemma 8:** Let $G = (V, E)$ be a k-tree ($k \geq 3$) such that $G - S_1(G) = K_{k+1}$, $V(K_{k+1}) = \{u_0, u_1, \cdots, u_k\}$ and $S_1(G) = \{v_1, v_2\}$. Suppose that $v_i$ is adjacent to all vertices of $u_0, u_1, \cdots, u_k$ but $u_i$ ($i = 1, 2$). Then $G$ is panconnected.

**Proof.** The proof is similar to that of Lemma 7. We verify that, for any two vertices $x$ and $y$ in $G$, there is a path $P_L$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G) - 1$ in $G$.

Case 1: $x = u_i$ and $y = v_j$ ($0 \leq i \leq k$ and $1 \leq j \leq 2$).

By symmetry, we can assume that $y = v_1$.

Case (1.1): $x = u_i$, $i \in \{0, 3, 4, \cdots, k\}$.

Without loss of generality, assume that $x = u_0$. Notice that $d(x, y) = 1$. Then $P_1 = (x =)u_0v_1(= y)$; $P_2 = (x =)u_0u_2v_1(= y)$; $P_3 = (x =)u_0v_2u_3v_1(= y)$; $P_4 = (x =)u_0u_1v_2u_3v_1(= y)$; $P_{4+s} = (x =)u_0u_2u_1v_2u_3u_4 \cdots u_{2+s}v_1(= y)$ $(1 \leq s \leq k-2)$;

Case (1.2): $x = u_1$.

Notice that $d(x, y) = 2$. Then $P_2 = (x =)u_1u_0v_1(= y)$; $P_3 = (x =)u_1v_2u_0v_1(= y)$; $P_4 = (x =)u_1v_2u_0u_2v_1(= y)$; $P_{4+s} = (x =)u_1v_2u_0u_2u_3 \cdots u_{2+s}v_1(= y)$ $(1 \leq s \leq k-2)$.

Case (1.3): $x = u_2$.

Notice that $d(x, y) = 1$. Then $P_1 = (x =)u_2v_1(= y)$; $P_2 = (x =)u_2u_0v_1(= y)$; $P_3 = (x =)u_2u_1u_0v_1(= y)$; $P_4 = (x =)u_2u_1v_2u_0v_1(= y)$; $P_{4+s} = (x =)u_2u_1u_0v_2u_3u_4 \cdots u_{2+s}v_1(= y)$ $(1 \leq s \leq k-2)$.

Case 2: $x = v_1$ and $y = v_2$.

Notice that $d(x, y) = 2$. Then $P_2 = (x =)v_1u_0v_2(= y)$; $P_3 = (x =)v_1u_0u_1v_2(= y)$; $P_4 = (x =)v_1u_0u_2u_1v_2(= y)$; $P_{4+s} = (x =)v_1u_0u_1u_2u_3 \cdots u_{2+s}v_2(= y)$ $(1 \leq s \leq k-2)$.

Case 3: $x = u_i$ and $y = u_j$ ($0 \leq i < j \leq k$).

By symmetry, we have the following subcases. Notice that $d(x, y) = 1$.

Case (3.1): $x = u_i$ and $y = u_j$, $i, j \in \{0, 3, 4, \cdots, k\}$ and $i \neq j$.

By symmetry, we assume that $x = u_0$ and $y = u_k$ without loss of generality.

Then $P_1 = (x =)u_0u_k(= y)$; $P_2 = (x =)u_0v_1u_k(= y)$; $P_3 = (x =)u_0u_2v_1u_k(= y)$; $P_4 = (x =)u_0v_1u_2u_1u_k(= y)$; $P_{4+s} = (x =)u_0v_1u_2u_1v_2u_3u_4 \cdots u_{2+s}u_k(= y)$ $(1 \leq s \leq k-3)$.

Case (3.2): $x = u_i$ and $y = u_j$, ($i \in \{0, 3, 4, \cdots, k\}$ and $j \in \{1, 2\}$).

By symmetry, we assume that $x = u_0$ and $y = u_1$.

Then $P_1 = (x =)u_0u_1(= y)$; $P_2 = (x =)u_0v_2u_1(= y)$; $P_3 = (x =)u_0v_2u_3u_1(= y)$; $P_4 = (x =)u_0v_1u_2u_3u_1(= y)$; $P_{4+s} = (x =)u_0v_1u_2u_3 \cdots u_{2+s}v_2u_1(= y)$ $(1 \leq s \leq k-2)$.

Case (3.3): $x = u_1$ and $y = u_2$.

Then $P_1 = (x =)u_1u_2(= y)$; $P_2 = (x =)u_1u_0u_2(= y)$; $P_3 = (x = )u_1v_2u_0u_2(= y)$; $P_4 = (x =)u_1v_2u_0v_1u_2(= y)$; $P_{4+s} = (x =)u_1v_2u_0v_1u_3u_4 \cdots u_{2+s}u_2(= y)$ $(1 \le s \le k-2)$.

Hence $G$ is panconnected. The proof of Lemma 8 is complete. □

## 3. The main theorems

In this section, we prove the main theorems of this paper that if the toughness of a k-tree $G$ is at least (k+1)/3, then G is panconnected for $k \ge 3$, or $G$ is vertex pancyclic for $k = 2$. We show the results for the cases $k \ge 3$ and $k = 2$ separately.

**Theorem 9:** If a k-tree $G$ $(k \ge 3)$ has toughness $\tau(G) \ge \frac{k+1}{3}$, then $G$ is panconnected.

**Proof.** Let $G$ be a k-tree $(k \ge 3)$ with toughness $\tau(G) \ge (k+1)/3$. We proceed by induction on $\nu(G)$ to prove that $G$ is panconnected.

When $\nu(G) = k$ or $k+1$, $G$ is $K_k$ or $K_{k+1}$, obviously $G$ is panconnected.

Assume that, when $\nu(G) < n$, $G$ is panconnected.

Now suppose that $\nu(G) = n \ge k+2$.

First, suppose that $S_2(G) = \emptyset$. Let $K = K_k$ and $H$ be a graph of $r$ independent vertices $v_1, v_2, \cdots, v_r$ such that $V(H) = S_1(G)$. Then $G = K \oplus H$. If $r \ge k$, let $S = V(K)$, then $\omega(G - S) = \omega(G - V(K)) = |\{v_1, v_2, \cdots, v_r\}| = r \ge k = |S|$, contradicting $\tau(G) \ge (k+1)/3$ for $k \ge 3$. So $r \le k-1$. Since $G = K \oplus H$ and $r \le k-1$, by Lemma 6, $G$ is panconnected.

Now suppose $S_2(G) \ne \emptyset$.

For any $u \in S_2(G)$, by Lemma 2, there is a $v \in S_1(G)$ such that $uv \in E(G)$ and $N_1(u) \backslash N_2(u) \subseteq S_1(G)$.

Since $u \in S_2(G)$, the clique $G[N_1(v)]$ contains $u$, $|N_2(u) \cap N_1(v)| = k-1$. Hence $|N_2(u) \backslash N_1(v)| = 1$ (1)

Case 1: $u$ has at least four neighbours in $S_1(G)$, i.e. $v_1, v_2, \cdots, v_r$ $(r \ge 4)$.

Then we delete all $k+1$ vertices of $S = N_2(u) \cup \{u\}$, we shall obtain that $\omega(G - S) \ge r = |\{v_1, v_2, \cdots, v_r\}|$ $(r \ge 4)$, and we have $(k+1)/4 \ge (k+1)/r \ge |S|/\omega(G-S) \ge \tau(G) \ge (k+1)/3$, which is a contradiction.

Case 2: $u$ has exactly three neighbours in $S_1(G)$, i.e. $v_1, v_2, v_3$.

Then for any two vertices $v_i$ and $v_j \in S_1(G) \cap N_1(u)$ $(1 \le i < j \le 3)$, $N_2(u) \backslash N_1(v_i) \ne N_2(u) \backslash N_1(v_j)$ (2).

Otherwise, suppose $N_1(v_i) = N_1(v_i) \cap (N_2(u) \cup \{u\}) = N_1(v_j) \cap (N_2(u) \cup \{u\}) = N_1(v_j)$. Let $S = N_1(v_i) = N_1(v_j)$ and let $\{u'\} = N_2(u) \setminus N_1(v_i) = N_2(u) \setminus N_1(v_j)$. Then $|S| = k$ and $\omega(G - S) \geq 3$ since $G - S$ has three components $v_i, v_j$ and the component containing $u'$, so $k/3 \geq |S|/\omega(G - S) \geq \tau(G) \geq (k+1)/3$, which is a contradiction.

By (1) and (2), let $\{u_i\} = N_2(u) \setminus N_1(v_i)$ $(i = 1, 2, 3)$ and let $u = u_0$ and the vertices of $(N_2(u) \cup \{u\}) \cap N_1(v_i) = N(v_i)$ be $u_0, u_1, \cdots, u_k$ except $u_i$ $(i = 1, 2, 3)$.

If $G - \{u_0, u_1, \cdots, u_k\}$ has a component besides $v_1, v_2, v_3$, let $S = \{u_0, u_1, \cdots, u_k\}$, then $\omega(G - S) \geq 4$ and $(k+1)/4 \geq |S|/\omega(G - S) \geq \tau(G) \geq (k+1)/3$, which is a contradiction. See Figure 1.

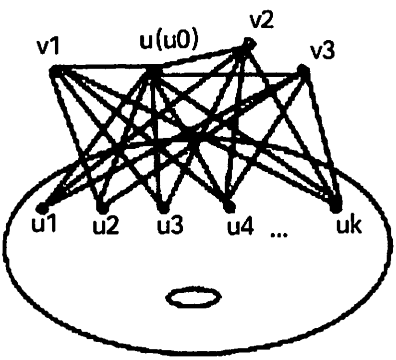


Figure 1

So suppose $G - \{u_0, u_1, \cdots, u_k\}$ has only components $v_1, v_2$, and $v_3$. Then $G$ satisfies the hypothesis of Lemma 7, by Lemma 7, we know that $G$ is panconnected.

Case 3: $u$ has exactly one neighbour in $S_1(G)$, i. e. $v_1$.

By (1), assume that $\{u_1\} = N_2(u) \setminus N_1(v_1)$ and $u_0 = u$, and the vertices of $(N_2(u) \cup \{u\}) \cap N_1(v_1) = N_1(v_1)$ are $u_0, u_1, \cdots, u_k$ except $u_1$. Now we prove that $G$ is panconnected. Let $x$ and $y$ be two arbitrary vertices of $G$. See Figure 2.

Case (3.1): $x, y \in V(G) \setminus \{v_1, u\}$.

Let $G_1 = G - v_1$. Since $v_1 \in S_1(G)$, by Lemma 1, $G_1$ is a k-tree and $\tau(G_1) \geq \tau(G)$. By induction hypothesis, there is a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G_1) - 1$ in $G_1$ ( and hence in $G$). (Notice that a shortest $(x, y)$ path in $G$ will not go through $v_1$, so $d_G(x, y) = d_{G_1}(x, y)$). In particular, the path $P_1$ from $x$ to $y$ of length $\nu(G_1) - 1$ in $G_1$ must go through two edges $uu_i$ and $uu_j$ such that one of $u_i$ and $u_j$ is not $u_1$. Assume that $u_i \neq u_1$ without loss of generality. By replacing $uu_i$ by $uv_1u_i$ on $P_1$, we obtain a path $P$ from $x$ to $y$ of length $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Case (3.2): $x = u$ and $y \in V(G)\backslash\{v_1, u\}$.

Let $G_1 = G - v_1$ and $G_2 = G_1 - u$. Since $v_1 \in S_1(G)$ and $u$ is a k-simplicial vertex of $G_1$, by Lemma 1, $G_1$ and $G_2$ are k-trees and $\tau(G_2) \geq \tau(G_1) \geq \tau(G)$.

By induction hypothesis, there is a path from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G_1) - 1$ in $G_1$ (and hence in $G$), and since $k \geq 3$, we have a vertex $x' = u_i$ such that $x' \neq y$ and $x' \neq u_1$, so by induction hypothesis, $G_2$ is panconnected, hence we have a path $P_1$ from $y$ to $x'$ of length $\nu(G_2) - 1$ in $G_2$, then we have a path $P = P_1 + x'v_1u$ from $y$ to $x(= u)$ of length $\nu(G_2) - 1 + 2 = \nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

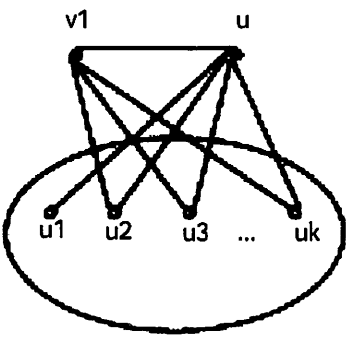


Figure 2

Case (3.3): $x = v_1$ and $y \in V(G)\backslash\{v_1, u\}$.

Let $G_1 = G - v_1$. By Lemma 1, $G_1$ is a k-tree and $\tau(G_1) \geq \tau(G)$.

If the shortest path $P_1$ from $x(= v_1)$ to $y$ goes through $xu_i$ and a shortest path from $u_i$ to $y$ ($i \neq 0$), then $P_1$ exists in $G$ and $P_1$ has length $d(x, y)$. By induction hypothesis, there is a path $P_2$ from $u$ to $y$ of length $L$ for each integer $L$ from $d(u, y) = d(x, y)$ to $\nu(G_1) - 1$ in $G_1$. Then $P = xu + P_2$ is a path from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y) + 1$ to $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

If the shortest path from $x(= v_1)$ to $y$ goes through $xu$ and a shortest path from $u$ to $y$, then $d(x, y) = d(u, y) + 1$. By induction hypothesis, there is a path $P_1$ from $u$ to $y$ of length $L$ for each integer $L$ from $d(u, y)$ to $\nu(G_1) - 1$ in $G_1$, then $P = xu + P_1$ is a path from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y) = d(u, y) + 1$ to $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Case (3.4): $x = v_1$ and $y = u$.

Let $G_1 = G - v_1$. By Lemma 1, $G_1$ is a k-tree and $\tau(G_1) \geq \tau(G)$.

First, $xy = v_1u$ is a path from $x$ to $y$ of length $d(x, y) = 1$ in $G$.

By induction hypothesis, there is a path $P_1$ from $u_2$ to $u(= y)$ of length $L$ for each integer $L$ from $d(u_2, u) = 1$ to $\nu(G_1) - 1$ in $G_1$. Then $P = xu_2 + P_1$ is a path from $x$ to $y(= u)$ of length $L$ for each integer $L$ from $d(x, y) + 1 = d(u_2, u) + 1 = 2$ to $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Hence in all subcases of Case 3, there is a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G) - 1$ in $G$.

Case 4: $u$ has exactly two neighbours in $S_1(G)$, i. e. $v_1$ and $v_2$.

By the same argument as (2) in Case (2), we have

$N_2(u)\backslash N_1(v_1) \neq N_2(u)\backslash N_1(v_2)$ (3)

By (1) and (3), let $\{u_i\} = N_2(u)\backslash N_1(v_i)$ $(i = 1, 2)$ and let $u = u_0$ and the vertices of $(N_2(u) \cup \{u\}) \cap N_1(v_i)$ be $u_0, u_1, \cdots, u_k$ except $u_i$ $(i = 1, 2)$. See Figure 3.

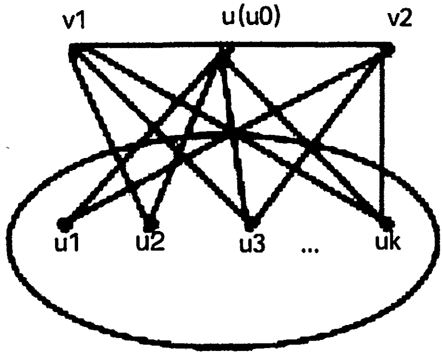


Figure 3

Since $u \in S_2(G)$, $G - S_1(G)$ cannot be $K_k$. If $G - S_1(G)$ is $K_{k+1}$, then the $K_{k+1} = G[\{u_0, u_1, \cdots, u_k\}]$, and all vertices of $S_1(G)\backslash\{v_1, v_2\}$ are adjacent to all of $u_1, u_2, \cdots, u_k$.

If $|S_1(G)\backslash\{v_1, v_2\}| \geq 2$, assume that $v_3, v_4 \in S_1(G)\backslash\{v_1, v_2\}$. Let $S = \{u_1, u_2, \cdots, u_k\}$. Then $G - S$ has at least 3 components $v_3, v_4$ and the component containing $\{u_0, v_1, v_2\}$, so $k/3 \geq |S|/\omega(G - S) \geq \tau(G) \geq (k + 1)/3$, which is a contradiction.

If $|S_1(G)\backslash\{v_1, v_2\}| = 1$, assume that $\{v_3\} = S_1(G)\backslash\{v_1, v_2\}$. Since $k \geq 3$, by the hypothesis of this case, $u_3$ is adjacent to $v_1, v_2$ and $v_3$. We relabel $u_0, u_1, u_2, \cdots, u_k$. Let $u'_0 = u_3, u'_1 = u_1, u'_2 = u_2, u'_3 = u_0$ and $u'_i = u_i$ $(i = 4, 5, \cdots, k)$. Then the graph satisfies the hypothesis of Lemma 7 with $u'_i$ substituting $u_i$ $(i = 0, 1, \cdots, k)$, by Lemma 7, we know that $G$ is panconnected. See Figure 4.

If $|S_1(G)\backslash\{v_1, v_2\}| = 0$, by Lemma 8, $G$ is panconnected.

Now suppose that $G - S_1(G)$ is not $K_{k+1}$. By Lemma 5, $|S_2(G)| \geq 2$ and there exists a $w \in S_2(G)$ such that $w \neq u$ and $wu \notin E(G)$, i. e. $w \notin \{u_1, u_2, \cdots, u_k\}$.

Applying the same argument of Cases 1 to 4 on $w$, the only remaining case to discuss is as following:

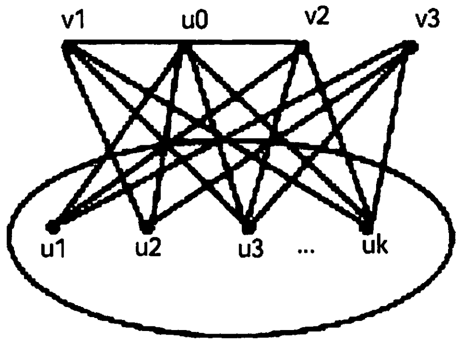


Figure 4

w has exactly two neighbours in $S_1(G)$, i.e. $z_1$ and $z_2$,
$N_2(w)\backslash N_1(z_1) \neq N_2(w)\backslash N_1(z_2)$ (4).

By (1) and (4), let $\{w_i\} = N_2(w)\backslash N_1(z_i)$ $(i = 1, 2)$ and let $w = w_0$ and the vertices of $(N_2(w) \cup \{w\}) \cap N_1(z_i)$ be $w_0, w_1, \cdots, w_k$ except $w_i$ $(i = 1, 2)$ and $G[\{w_0, w_1, \cdots, w_k\}]$ is a clique $K_{k+1}$.

Let $x$ and $y$ be two arbitrary vertices of $G$. By symmetry, we have the following subcases to discuss. Now the case is as Figure 3.

Case (4.1): $x, y \in V(G)\backslash\{u, v_1, v_2\}$.

Let $G_1 = G - v_1$. Since $v_1 \in S_1(G)$, by Lemma 1, $G_1$ is a k-tree and $\tau(G_1) \geq \tau(G)$.

By induction hypothesis, there is a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G_1) - 1$ in $G_1$ (and hence in $G$). (Notice that a shortest $(x, y)$ path in $G$ will not go through $v_1$). In particular, $P_1$ is a path from $x$ to $y$ of length $\nu(G_1) - 1$ in $G_1$ (and hence in $G$).

Suppose that $P_1$ goes through an edge uu$_i$ such that $u_i \neq u_1$. By replacing $uu_i$ by $uv_1u_i$ on $P_1$, we obtain a path $P$ from $x$ to $y$ of length $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

The only exceptional case for $P_1$ is that $P_1$ goes through $u_iv_2uu_1$ $(u_i \neq u_1)$. But, replacing $u_iv_2uu_1$ by $u_iuv_2u_1$ on $P_1$, we obtain a path $P_2$ from $x$ to $y$ of length $\nu(G_1) - 1$ in $G_1$, which goes through $uu_i$ $(u_i \neq u_1)$. Applying the above argument by substituting $P_1$ by $P_2$, we can obtain a path $P$ from $x$ to $y$ of length $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Case (4.2): $x \in V(G)\backslash\{u, v_1, v_2\}$ and $y = v_1$.

Let $G_1 = G - v_2$, $G_2 = G_1 - v_1$ and $G_3 = G_2 - u$. Since $v_1, v_2 \in S_1(G)$ and $u \in S_1(G_2)$, by Lemma 1, $G_1$, $G_2$ and $G_3$ are k-trees and $\tau(G_3) \geq \tau(G_2) \geq \tau(G_1) \geq \tau(G)$.

By induction hypothesis, there is a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G_1) - 1$ in $G_1$ (and hence in $G$).

Suppose $x = u_3$. Let $y' = u_1$. By induction hypothesis, $G_3$ is panconnected, so there is a path $P_1$ from $x$ to $y'$ of length $\nu(G_3) - 1$ in $G_3$. Then $P = P_1 + y'v_2uy$ is a path from $x$ to $y$ of length $\nu(G_3) - 1 + 3 = \nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Otherwise, suppose that $x \neq u_3$. Let $y' = u_3$. By induction hypothesis, there is a path $P_2$ from $x$ to $y'$ of length $\nu(G_3) - 1$ in $G_3$. Then $P = P_2 + y'v_2uy$ is a path from $x$ to $y$ of length $\nu(G_3) - 1 + 3 = \nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Case (4.3): $x = u$ and $y = v_1$.

Let $G_1 = G - v_1$. Since $v_1 \in S_1(G)$, by Lemma 1, $G_1$ is a k-tree and $\tau(G_1) \geq \tau(G)$.

Since $d(x, y) = 1$, $xy = uv_1$ is a path from $x$ to $y$ of length $d(x, y) = 1$. Let $y' = u_3$. By induction hypothesis, there is a path $P_1$ from $x$ to $y'$ of length $L$ for each integer $L$ from $d(x, y') = 1$ to $\nu(G_1) - 1$ in $G_1$ (and hence in $G$). Then $P = P_1 + y'y(u_3v_1)$ is a path from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y') + 1 = 2$ to $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Case (4.4): $x = v_1$ and $y = v_2$.

Let $G_1 = G - v_2$. Since $v_2 \in S_1(G)$, by Lemma 1, $G_1$ is a k-tree and $\tau(G_1) \geq \tau(G)$.

Notice that $d(x, y) = d(v_1, v_2) = 2$. Let $y' = u$. By induction hypothesis, there is a path $P_1$ from $x$ to $y'$ of length $L$ for each integer $L$ from $d(x, y') = d(v_1, u) = 1$ to $\nu(G_1) - 1$ in $G_1$ (and hence in $G$). Then $P = P_1 + y'y(uv_2)$ is a path from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y) = 2$ to $\nu(G_1) - 1 + 1 = \nu(G) - 1$ in $G$.

Case (4.5): $x \in V(G)\backslash\{v_1, v_2, u, u_1, u_2, \cdots, u_k\}$ and $y = u$.

Let $G_1 = G - v_1$. Since $v_1 \in S_1(G)$, by Lemma 1, $G_1$ is a k-tree and $\tau(G_1) \geq \tau(G)$.

By induction hypothesis, there is a path $P$ from $x$ to $y$ $(= u)$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G_1) - 1$ in $G_1$ (and hence in $G$). We only need to prove that there is a path $P$ from $x$ to $y$ of length $\nu(G) - 1 = \nu(G_1) - 1 + 1$ in $G$.

Let $G_2 = G - \{v_1, v_2\}$ and $G_3 = (G - \{v_1, u\}) + v_2u_2$. Notice that $G_2$ and $G_3$ are isomorphic. Since $v_1, v_2 \in S_1(G)$, by Lemma 1, $G_2$ is a k-tree and $\tau(G_2) \geq \tau(G)$, so $G_3$ is a k-tree and $\tau(G_3) \geq \tau(G)$. Let $y' = u_3$. By

induction hypothesis, $G_3$ is panconnected, so there is a path $P_1$ from $x$ to $y'$ of length $\nu(G_3)-1$ in $G_3$. Then we have the following subcases.
Case (4.5.1): $P_1 = Q_1(x, u_i) + u_i v_2 y'(= u_3)$, where $u_i \neq u_2, u_3(y')$, and $Q_1$ is a path from $x$ to $u_i$ in $G_3$.

Then there is a path $P = Q_1(x, u_i) + u_i v_2 u_3 v_1 y(= u)$ from $x$ to $y$ of length $\nu(G_3) - 1 + 2 = \nu(G) - 1$ in $G$.
Case (4.5.2): $P_1 = Q_1(x, u_2) + u_2 v_2 y'(= u_3)$, where $Q_1$ is a path from $x$ to $u_2$ in $G_3$.

Then there is a path $P = Q_1(x, u_2) + u_2 v_1 u_3 v_2 y(= u)$ from $x$ to $y$ of length $\nu(G_3) - 1 + 2 = \nu(G) - 1$ in $G$.
Case (4.5.3): $P_1 = Q_1(x, u_i) + u_i v_2 u_j + Q_2(u_j, y'(= u_3))$, where $u_i, u_j \neq u_2$ nor $u_3(= y')$, and $Q_1$ is a path from $x$ to $u_i$ and $Q_2$ is a path from $u_j$ to $y'$ in $G_3$.

Then there is a path $P = Q_1(x, u_i) + u_i v_2 u_j + Q_2(u_j, u_3) + u_3 v_1 y(= u)$ from $x$ to $y$ of length $\nu(G_3) - 1 + 2 = \nu(G) - 1$ in $G$.
Case (4.5.4): $P_1 = Q_1(x, u_i) + u_i v_2 u_2 + Q_2(u_2, y'(= u_3))$, where $u_i \neq u_2$ nor $u_3(= y')$, $Q_1$ is a path from $x$ to $u_i$ and $Q_2$ is a path from $u_2$ to $y'$ in $G_3$.

Then there is a path $P = Q_1(x, u_i) + u_i v_2 u_3 + Q_2(u_3, u_2) + u_2 v_1 y(= u)$ from $x$ to $y$ of length $\nu(G_3) - 1 + 2 = \nu(G) - 1$ in $G$.
Case (4.5.5): $P_1 = Q_1(x, u_2) + u_2 v_2 u_i + Q_2(u_i, y'(= u_3))$, where $u_i \neq u_2$ nor $u_3(= y')$, $Q_1$ is a path from $x$ to $u_2$ and $Q_2$ is a path from $u_i$ to $y'$ in $G_3$.

Then there is a path $P = Q_1(x, u_2) + u_2 v_1 u_3 + Q_2(u_3, u_i) + u_i v_2 y(= u)$ from $x$ to $y$ of length $\nu(G_3) - 1 + 2 = \nu(G) - 1$ in $G$.
Case (4.6): $x \in \{u_1, u_2, \cdots, u_k\}$ and $y = u$.

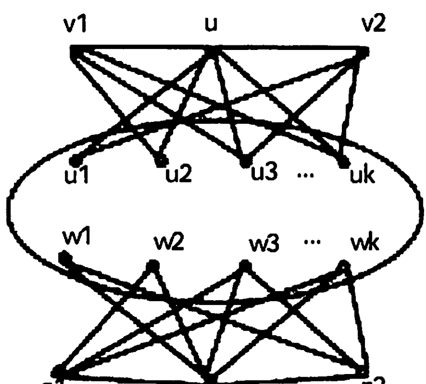


Figure 5

Now we consider $w, z_1, z_2, w_1, w_2, \cdots, w_k$. See Figure 5. By the argument of Case 4 above, $w, z_1, z_2, w_1, w_2, \cdots, w_k$ are in the same situation as $u, v_1, v_2, u_1, u_2, \cdots, u_k$. However, $x \in \{u_1, u_2, \cdots, u_k\}$, $y = u$, and $w \neq u$ and $wu \notin E(G)$ (i.e. $w \notin \{u_0, u_1, \cdots, u_k\}$) by the argument before. Substituting $u, v_1, v_2, u_1, u_2, \cdots, u_k$ by $w, z_1, z_2, w_1, w_2, \cdots, w_k$, we have Case (4.1) such that $x, y \in V(G)\backslash\{w, z_1, z_2\}$. By the argument of Case (4.1), we know that there is a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G) - 1$ in $G$.

In all cases, by the argument before, there is always a path $P$ from $x$ to $y$ of length $L$ for each integer $L$ from $d(x, y)$ to $\nu(G) - 1$ in $G$. Hence $G$ is panconnected. The proof of this theorem is complete. □

Now we prove that, if a k-tree $G$ has toughness $\tau(G) \geq (k+1)/3$ for $k = 2$, then $G$ is vertex pancyclic. An edge $e = uv$ of a 2-connected graph $G$ is called a cutting edge if $G - \{u, v\}$ is not connected. An edge not to be a cutting edge is called a noncutting edge.

**Theorem 10:** A 1-tough 2-tree $G$ with $\nu(G) \geq 3$ is vertex pancyclic.

**Proof.** Let $G$ be a 1-tough 2-tree with $\nu(G) \geq 3$. We proceed by induction on $\nu(G)$. When $\nu(G) = 3$, $G$ is a triangle, and obviously $G$ is vertex pancyclic and the Hamiltonian cycle $C$ of $G$ contains all noncutting edges of $G$.

Now suppose that $\nu(G) \geq 4$ and assume that every 1-tough 2-tree $H$ of order $\nu(G) - 1$ is vertex pancyclic and any Hamiltonian cycle $C$ of $H$ contains all noncutting edges of $H$.

By Lemma 4, $S_1(G)$ has at least two 2-simplicial vertex $w_1$ and $w_2$. Then $G[N(w_1)]$ and $G[N(w_2)]$ are single edges $e_1$ and $e_2$ respectively. Let $G_1 = G - w_1$ and $G_2 = G - w_2$. By (4) of Lemma 1, $\tau(G_1) \geq 1$ and $\tau(G_2) \geq 1$. By induction hypothesis, every vertex $v$ in $G_1$ is contained in a cycle of length $L$ for each integer $L$ from 3 to $\nu(G_1) = \nu(G) - 1$ in $G_1$ (and hence in $G$), particularly, $v$ is contained in a cycle $C$ of length $\nu(G) - 1$ ( an Hamiltonian cycle in $G_1$). By induction hypothesis, $C$ contains the edge $e_1 = xy$. (Notice that $e_1$ is a noncutting edge of $G_1$, otherwise $\tau(G) < 1$). Now $v$ is contained in a cycle $C' = (C - e_1) + xw_1y$ of length $\nu(G)$ in $G$, and $C'$ goes through every noncutting edge of $G$. (Notice that $e_1 = xy$ is not a noncutting edge of $G$). So every vertex $v$ in $G_1$ is contained in a cycle of length $L$ for each integer $L$ from 3 to $v(G)$ in $G$.

By similar argument on $G_2$, we can prove that every vertex $v$ (particularly $w_1$) in $G_2$ is contained in a cycle of length $L$ for each integer $L$ from 3 to $\nu(G)$ in $G$.

Then by above conclusions, every vertex $v$ of $G$ is contained in a cycle $C$ of length $L$ for each integer $L$ from 3 to $\nu(G)$. So $G$ is vertex pancyclic. □

**Remark:** A 1-tough 2-tree is not necessarily edge pancyclic, and hence not panconnected. We construct counterexamples as follows: Let H be a 1-tough 2-tree with $\nu(H) \geq 3$ constructed by starting from an edge $e = xy$ and keeping $e$ a noncutting edge of $H$. Let $G_1$ and $G_2$ be two copies of $H$ and we label the vertices of $G_1$ differently from those of $G_2$ except $x$ and $y$. We construct $G = G_1 \cup G_2$ by identifying the $x$ , $y$ in $G_1$ and the $x, y$ in $G_2$. Then any cycle containing $e = xy$ in $G$ is contained in either $G_1$ or $G_2$. So there is not any cycle of length greater than $\nu(G_1)$ containing $e = xy$ in $G$ and $G$ is not edge pancyclic.

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