Multiplicity of Stars in Complete Graphs and Complete r-partite Graphs

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Abstract

Let K_r be the complete graph on r vertices in which there exist an edge between every pair of vertices, $K_{m,n}$ be the complete bipartite graph with m vertices in one partition and n vertices in the other partition and each vertex in one partition is adjacent to each vertex in the other partition and $K(n,r)$ be the complete $r-$ partite graph $K_{n,n,n,\dots,n}$ where each partition has n vertices. In this paper, we determine the minimum number of monochromatic stars $K_{1,p}$ $\forall p \geq 2$ in any t ($t \geq 2$) coloring of edges of K_r, $K_{m,n}$ and $K(n,r)$. Also, we prove that these lower bounds are sharp for all values of m, n, p, r and t by giving explicit constructions.

1 Introduction and Backround results

If F and G are graphs, define $M(G,F,t)$ to be the minimum number of monochromatic copies of G that occur in any t coloring of the edges of F. $M(G,F,t)$ is called the multiplicity of G in F with t colors. A graph G is said to be monochromatic if all its edges are of same color.

A $p-$star $K_{1,p}$ at a vertex v in a graph G is a subgraph with v as the centre vertex and p edges incident at v say $vv_1, vv_2, vv_3, \dots, vv_p$. Two $p-$stars S_1 and S_2 at a vertex v in a graph G are said to be distinct if atleast one edge of S_1 is distinct from the edges of S_2.

Suppose that the edges of a graph F are colored with t colors. In this paper, to obtain the minimum number of monochormatic copies of stars $K_{1,p}$ $\forall p \geq 2$ in any t coloring of edges of F, we minimize such monochromatic copies of $K_{1,p}$ at each vertex of F. We use combinatorial arguements for minimizing the monochromatic copies of $K_{1,p}$ at each vertex.

We use the decompositions of complete graphs K_r and complete $r-$ partite graphs $K(n,r)$ into edge disjoint Hamilton cycles [4] or edge disjoint perfect matchings in the theorems 3.1 and 4.1 in sections 3 and 4 respectively. For all odd $r \geq 3$, edges of K_r are decomposed into $\frac{r-1}{2}$ edge disjoint Hamilton cycles and for all even $r \geq 2$, edges of K_r are decomposed into $r-1$ edge disjoint perfect matchings.

We use the same decomposition for complete $r-$ partite graph with equal partitions as in the case of K_r by giving $1-1$ correspondence between the vertices of K_r and the partite sets of $K(n,r)$. Let $v_1, v_2, v_3, ..., v_r$ be the vertices of K_r. Corresponding to each edge v_iv_j in K_r, there is a set edges between the i^{th} and the j^{th} partite sets in $K(n,r)$.

For the case $r = odd$ ($r \geq 3$), each Hamilton cycle in K_r corresponds to n edge disjoint Hamilton cycles in $K(n,r)$. Hence, the edges of $K(n,r)$ are decomposed into $\frac{n(r-1)}{2}$ edge disjoint Hamilton cycles. Similarly, for the case $r = even$ ($r \geq 2$), each edge disjoint perfect matchings in K_r corresponds to n edge disjoint perfect matchings in $K(n,r)$. Hence, there are $n(r-1)$ edge disjoint perfect matchings in $K(n,r)$.

2 The minimum number of monochromatic stars

Let F be a graph. Our aim is to determine the minimum number of monochromatic stars $K_{1,p}$ $\forall p \geq 2$ in any t coloring of edges of F. To obtain this minimum number, we use the following lemma.

Lemma 2.1

$$\sum_{i=1}^{t} \binom{n_i}{p} \geq r\binom{\lceil \frac{n}{t} \rceil}{p} + (t-r)\binom{\lfloor \frac{n}{t} \rfloor}{p}$$

where $n_1, n_2, n_3, \ldots, n_t, n, t, p, r$ *are nonnegative integers,* $\sum_{i=1}^{t} n_i = n$, $n \equiv r \pmod t$, $\lceil \frac{n}{t} \rceil$ *denotes the ceiling of* $\frac{n}{t}$ *and* $\lfloor \frac{n}{t} \rfloor$ *denotes the floor of* $\frac{n}{t}$.

Proof

Without loss of generality, assume that $0 \le n_1 \le n_2 \le n_3 \le \ldots \le n_t$. If $n_1, n_2, n_3, \ldots, n_t$ are almost equal, then equality arises. If not, let $n_j - n_i \ge 2$ for some i and j.
We show that $\binom{n_i}{p} + \binom{n_j}{p} \ge \binom{n_i+1}{p} + \binom{n_j-1}{p}$.

Consider $\binom{n_i}{p} - \binom{n_i+1}{p} + \binom{n_j}{p} - \binom{n_j-1}{p}$

$= \binom{n_i}{p} - [\binom{n_i}{p-1}\binom{1}{1} + \binom{n_i}{p}\binom{1}{0}] + [\binom{n_j-1}{p-1}\binom{1}{1} + \binom{n_j-1}{p}\binom{1}{0}] - \binom{n_j-1}{p}$

$= \binom{n_j-1}{p-1} - \binom{n_i}{p-1}$

$\ge \binom{n_i+1}{p-1} - \binom{n_i}{p-1}$ since $n_j - n_i \ge 2$

≥ 0

Thus by stepwise increasing the smallest and decreasing the largest of any two n_i and n_j,$(1 \le i \le j \le t)$, we get the required inequality.

3 Complete Graphs

Theorem 3.1

$$M(K_{1,p}, K_r, t) = \begin{cases} t(r-1)\binom{\frac{r-1}{t}}{p} + \frac{t}{2}\left[\binom{\frac{r-1}{t}-1}{p} + \binom{\frac{r-1}{t}+1}{p}\right] & \textit{if } r \textit{ and } \frac{r-1}{t} \textit{ are odd integers} \\ r\left[r_1\binom{\lceil\frac{r-1}{t}\rceil}{p} + (t-r_1)\binom{\lfloor\frac{r-1}{t}\rfloor}{p}\right] & \textit{otherwise} \end{cases}$$

where $t \ge 2$, $p \ge 2$, $\lceil\frac{r-1}{t}\rceil$ *is the ceiling of* $\frac{r-1}{t}$, $\lfloor\frac{r-1}{t}\rfloor$ *is the floor of* $\frac{r-1}{t}$ *and* $r - 1 \equiv r_1 \pmod{t}$, $0 \le r_1 \le t-1$.

Proof

Using Lemma 2.1, we observe that

$$M(K_{1,p}, K_r, t) \ge r\left[r_1\binom{\lceil\frac{r-1}{t}\rceil}{p} + (t-r_1)\binom{\lfloor\frac{r-1}{t}\rfloor}{p}\right]$$

This bound is not attainable when r is odd and $\frac{r-1}{t}$ is an odd integer since we can't construct a graph with the color degree sequence $(\frac{r-1}{t}, \frac{r-1}{t}, \ldots, \frac{r-1}{t})$

at each vertex. So, the next possible minimum is attained when exactly one vertex is of color degree sequence $(\frac{r-1}{t}+1, \frac{r-1}{t}-1, ..., \frac{r-1}{t}+1, \frac{r-1}{t}-1)$ and the remaining vertices are of color degree sequence $(\frac{r-1}{t}, \frac{r-1}{t}, ..., \frac{r-1}{t})$.
Hence, $M(K_{1,p}, K_r, t) \geq (r-1)t\binom{\frac{r-1}{t}}{p} + \frac{t}{2}\left[\binom{\frac{r-1}{t}-1}{p} + \binom{\frac{r-1}{t}+1}{p}\right]$.
Now to prove that the bounds are sharp in all the above cases, we give the following explicit constructions.

Case 1: r is even.

Partition the edges of K_r into $r-1$ edge disjoint perfect matchings say $P_1, P_2, P_3, \ldots, P_{r-1}$. Partiton the sets $P_1, P_2, P_3, \ldots, P_{r-1}$ into t disjoint sets of almost equal cardinality. Let the sets be $C_1, C_2, C_3, ..., C_t$. For example, if $r-1 = 10$ and $t = 3$ then, $P_1, P_2, ..., P_{10}$ is partitioned into $C_1 = \{P_1, P_2, P_3, P_4\}$, $C_2 = \{P_5, P_6, P_7\}$ and $C_3 = \{P_8, P_9, P_{10}\}$. Assign color i to the edges in C_i, $1 \leq i \leq t$. Now we have the number of edges of each color at each vertex almost equal.

Case 2: r is odd.

In this case the degree at each vertex is $r-1 =$ even. Using Lemma 2.1, if $r_1 = 0$, color degree sequence at each vertex is either $(2l, 2l, ..., 2l)$ where $l \geq 1$ or $(2l+1, 2l+1, ..., 2l+1)$ where $l \geq 0$ and if $r_1 \neq 0$, color degree sequence at each vertex in K_r is either $(2l+1, 2l+1, ..., 2l+1, 2l, 2l, ..., 2l)$ where $l \geq 0$ or $(2l-1, 2l-1, ..., 2l-1, 2l, 2l, ..., 2l)$ where $l \geq 1$.

Case 2(a): The color degree sequence at each vertex be $(2l, 2l, ..., 2l)$, $l \geq 1$.

The number of edge disjoint Hamilton cycles in K_r is lt. Partition these lt Hamilton cycles into t sets, each containing distinct l Hamilton cycles. Let the t sets be $H_1, H_2, H_3, ..., H_t$. Color the edges of Hamilton cycles in H_i with color i, $1 \leq i \leq t$.

Case 2(b): The color degree sequence at each vertex be $(2l+1, 2l+1, ..., 2l+1)$, $l \geq 0$.

We cannot construct a graph with this color degree sequence at each vertex, since the number of vertices in K_r is odd. So, to attain the next possible minimum, one vertex (say v_1) of K_r must have the color degree sequence $(2l+2, 2l. 2l+2, 2l, ..., 2l+2, 2l)$. The number of edge dijoint Hamilton cycles in K_r is $lt + \frac{t}{2}$ where t is even since $r-1$ is even. Color the edges of lt Hamilton cycles as given in the case 2(a), so that color degree sequence at each vertex will be $(2l, 2l, ..., 2l)$. Let the remaining

Hamilton cycles be $h_1, h_2, h_3, ..., h_{\frac{t}{2}}$. Color the edges of each Hamilton cycle h_i alternatively with the colors $2i-1, 2i$ where $i = 1$ to $\frac{t}{2}$ starting from the vertex v_1 in each cycle. Now at each vertex other than v_1 we have the number of edges of each color almost equal. Hence,

$$M(K_{1,p}, K_r, t) \geq t(r-1)\binom{2l+1}{p} + \frac{t}{2}\left[\binom{2l+2}{p} + \binom{2l}{p}\right]$$

Case 2(c): The color degree sequence at each vertex be $(2l+1, 2l+1, ..., 2l+1, 2l, 2l, ..., 2l)$, $l \geq 0$.

Here $r-1 = r_1(2l+1) + (t-r_1)2l$ where r_1 is even since $r-1$ is even. The number of edge disjoint Hamilton cycles is $lt + \frac{r_1}{2}$. Color the edges of lt Hamilton cycles as given in the case 2(a), so that the color degree sequence at each vertex is $(2l, 2l, ..., 2l)$. Let the remaining $\frac{r_1}{2}$ Hamilton cycles be $h_1, h_2, h_3, ..., h_{\frac{r_1}{2}}$. It is possible to find the set of independent edges in K_r say $e_1, e_2, e_3, ..., e_{\frac{r_1}{2}}$ such that $e_i \in h_i$, $1 \leq i \leq \frac{r_1}{2}$. Color the edges $e_1, e_2, e_3, ..., e_{\frac{r_1}{2}}$ with color t. Let $e_i = (u_i, v_i)$, $1 \leq i \leq \frac{r_1}{2}$. Color the edges of h_i alternatively with the colors $2i-1, 2i$, $1 \leq i \leq \frac{r_1}{2}$ starting and ending with the vertices u_i and v_i respectively. Now for $1 \leq i \leq \frac{r_1}{2}$, degree of each vertex is increased by 1 in the $(2i-1)^{th}$ position and in the $(2i)^{th}$ position of its color degree sequence $(2l, 2l, ..., 2l)$ other than the vertices u_i and v_i whereas degree of u_i is increased by 1 in the $(2i-1)^{th}$ position and in the t^{th} position and degree of v_i is increased by 1 in the $(2i)^{th}$ position and in the t^{th} position of its color degree sequence $(2l, 2l, ..., 2l)$. Hence, the number of edges of each color is almost equal at each vertex.

Case 2(d): The color degree sequence at each vertex be $(2l-1, 2l-1, ..., 2l-1, 2l, 2l, ..., 2l)$, $l \geq 1$.

Here $r-1 = r_1 2l + (t-r_1)(2l-1)$ where $t - r_1$ is even since $r-1$ is even. The number of edge disjoint Hamilton cycles is $(l-1)t + r_1 + (\frac{t-r_1}{2})$. Partiton the $(l-1)t$ Hamilton cycles into t sets of equal cardinality. Let the sets be $H_1, H_2, H_3, ..., H_t$. Color the edges of all cycles in H_i with color i, $(1 \leq i \leq t)$, so that the color degree sequence at each vertex is $(2l-2, 2l-2, ..., 2l-2)$. Let the remaining $(r_1 + \frac{t-r_1}{2})$ Hamilton cycles be $h_1, h_2, ..., h_{r_1}, h_{r_1+1}, ..., h_{\frac{t-r_1}{2}+r_1}$. Choose a set of $\frac{t-r_1}{2}$ independent edges in h_1, say (x_i, y_i), $1 \leq i \leq \frac{t-r_1}{2}$. Color the edge (x_i, y_i) in h_1 with the color $r_1 + 2i$ for each $i = 1$ to $\frac{t-r_1}{2}$ and the remaining edges are colored with color 1. Color the edges of h_i with color i, $2 \leq i \leq r_1$. Color the edges of h_{r_1+i} alternatively with colors $r_1 + 2i - 1, r_1 + 2i$ starting and ending with a vertex x_i, $\forall i = 1$ to $\frac{t-r_1}{2}$. Hence, the number of edges of each color is almost equal at each vertex.

4 Complete r-partite graphs

Theorem 4.1

$$M(K_{1,p},K(n,r),t)=\begin{cases} t(nr-1)\binom{\frac{n(r-1)}{t}}{p}+\frac{t}{2}\left[\binom{\frac{n(r-1)}{t}-1}{p}+\binom{\frac{n(r-1)}{t}+1}{p}\right] & \\ \qquad \text{if } nr \text{ and } \frac{n(r-1)}{t} \text{ are odd integers} & \\ nr\left[r_1\binom{\lceil\frac{n(r-1)}{t}\rceil}{p}+(t-r_1)\binom{\lfloor\frac{n(r-1)}{t}\rfloor}{p}\right] & \text{otherwise} \end{cases}$$

where $r \geq 2$, $p \geq 2$, $t \geq 2$, $\lceil\frac{n(r-1)}{t}\rceil$ is the ceiling of $\frac{n(r-1)}{t}$, $\lfloor\frac{n(r-1)}{t}\rfloor$ is the floor of $\frac{n(r-1)}{t}$ and $n(r-1) \equiv r_1 \pmod{t}$, $0 \leq r_1 \leq t-1$.

Proof

Using Lemma 2.1, we observe that

$$M(K_{1,p},K(n,r),t) \geq nr\left[(t-r_1)\binom{\lfloor\frac{n(r-1)}{t}\rfloor}{p}+r_1\binom{\lceil\frac{n(r-1)}{t}\rceil}{p}\right]$$

Using the argument given in theorem 3.1, this bound is not attainable when nr is odd and $\frac{n(r-1)}{t}$ is an odd integer. The next possible minimum would be

$$M(K_{1,p},K(n,r),t) \geq (nr-1)t\binom{\frac{n(r-1)}{t}}{p}+\frac{t}{2}\left[\binom{\frac{n(r-1)}{t}-1}{p}+\binom{\frac{n(r-1)}{t}+1}{p}\right].$$

Now to prove that the bounds in all cases are sharp, we give the following explicit constructions.

Case 1: r is even and n is any positive integer.

Partition the edges of $K(n,r)$ into $n(r-1)$ edge disjoint perfect matchings say $P_1, P_2, ..., P_{n(r-1)}$. Partition the sets $P_1, P_2, ..., P_{n(r-1)}$ into t disjoint sets of almost equal cardinality. Let the sets be $C_1, C_2, ..., C_t$. Color the set of edges in C_i with color i where $1 \leq i \leq t$. Now we have the number of edges of each color at each vertex almost equal.

Case 2: r is odd and n is even

In this case the number of Hamilton cycles in $K(n,r)$ is $\frac{n(r-1)}{2}$. Let $n(r-1) = k_1r_1 + k_2(t-r_1)$, $0 \le r_1 \le t-1$. If $k_1 = 2l+1$ and $k_2 = 2l$ ($l \ge 0$), the number of Hamilton cycles is $lt + \frac{r_1}{2}$. Since $n(r-1)$ is even, r_1 should be even. Divide lt Hamilton cycles into t sets of equal cardinality, say $H_1, H_2, ..., H_t$. Color the edges of H_i with color i, $1 \le i \le t$. Let the remaining cycles be $h_1, h_2, ..., h_{\frac{r_1}{2}}$. Color the edges of h_i alternatively with colors $2i-1, 2i$, $1 \le i \le \frac{r_1}{2}$. Now, if $k_1 = 2l$ and $k_2 = 2l-1$, the number of Hamilton cycles is $(l-1)t + \frac{t-r_1}{2} + r_1$ where $t-r_1$ is even since $n(r-1)$ is even. Divide $(l-1)t$ Hamilton cycles into t sets of equal cardinality, say $H_1, H_2, ..., H_t$. Color the edges of H_i with color i, $1 \le i \le t$. Let the remaining cycles be $h_1, h_2, ..., h_{\frac{t-r_1}{2}+r_1}$. Color the edges of h_i alternatively with colors $2i-1, 2i$, $1 \le i \le \frac{t-r_1}{2}$. Color the edges of $h_{\frac{t-r_1}{2}+i}$ with color $t-r_1+i$, $1 \le i \le r_1$. Now in both the cases we have the number of edges of each color at each vertex almost equal.

Case 3: r is odd and n is odd

The possible color degree sequence at each vertex is same as given in the case 2 of Theorem 3.1 and the construction remains the same.

5 Complete bipartite graphs

In this section, we determine the minimum number of monochromatic stars in complete bipartite graph with partitions of any size using t colors.

Theorem 5.1
$M(K_{1,p}, K_{m,n}, t) = m\left[(t-r_1)\binom{\lfloor \frac{n}{t} \rfloor}{p} + r_1\binom{\lceil \frac{n}{t} \rceil}{p}\right] + n\left[(t-r_2)\binom{\lfloor \frac{m}{t} \rfloor}{p} + r_2\binom{\lceil \frac{m}{t} \rceil}{p}\right]$
where $n \equiv r_1 (mod\ t), m \equiv r_2 (mod\ t)$ *and* $0 \le r_1, r_2 \le t-1$.

Proof

Using Lemma 2.1, we observe that
$M(K_{1,p}, K_{m,n}, t) \ge m\left[(t-r_1)\binom{\lfloor \frac{n}{t} \rfloor}{p} + r_1\binom{\lceil \frac{n}{t} \rceil}{p}\right] + n\left[(t-r_2)\binom{\lfloor \frac{m}{t} \rfloor}{p} + r_2\binom{\lceil \frac{m}{t} \rceil}{p}\right]$

To prove that the bound is sharp, we give the following explicit construction in which the edges colored with each of the t colors are almost equal at each vertex. Let $x_1, x_2, \ldots, x_m$ be the vertices in one partition and

$y_1, y_2, \ldots, y_n$ be the vertices in other partition of $K_{m,n}$. Let $c_1, c_2, \ldots, c_t$ be the t different colors used to color the edges of $K_{m,n}$. The edge $x_i y_j$ is colored with color c_{q+1} if $i \equiv p(mod\, \mathrm{t})$ and $j \equiv p + q(mod\, \mathrm{t})$ where $0 \leq p, q \leq t-1$.

References

1. Stefan A.Burr, Vera Rosta, On the Ramsey Multiplicities of Graphs-Problems and recent results. Journal of Graph theory, Vol 4 (1980), 347-361.
2. A.W.Goodman, On sets of acquaintances and strangers at any party, Amer. Math. Monthly 66(1959), 778-783.
3. A.W.Goodman, Triangles in a complete chromatic graph, J.Austral. Math. Soc. (Ser.A)39 (1985), 86-93.
4. Renu Laskar, Bruce Auerbach, On decomposition of r-partite graphs into edge-disjoint Hamilton circuits, Discrete Mathematics, Volume 14, Issue 3 (1976), 265-268.