RESIDUAL CLOSENESS OF SPLITTING NETWORKS

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ABSTRACT. Networks are important structures and appear in many different applications and settings. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. Centrality parameters play an important role in the field of network analysis. Numerous studies have proposed and analyzed several centrality measures. These concept measures the importance of a node's position in a network. In this paper, vertex residual closeness(VRC) and normalized vertex residual closeness($NVRC$) of some Splitting networks modeling by splitting graph are obtained.

1. INTRODUCTION

Complex networks describe a wide range of systems in nature and society including examples the Internet, metabolic networks, electric power grids, supply chains and the world trade Web among many others. The study of networks has become an important area of multidisciplinary research involving computer science, mathematics, chemistry, social sciences, informatics and other theoretical and applied sciences [8-10, 12]. The stability and reliability of a network are of prime importance to network designers. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links [12].

Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. As usual, a network is described by an undirected simple graph. There are several types of graph theoretical parameters depending upon the distance such as vertex and edge betweenness, average vertex and edge betweenness, normalized average vertex and edge betweenness [7, 11, 13], closeness, vertex residual closeness, normalized vertex residual closeness [1, 3, 4, 14, 16].

Let $G = (V(G), E(G))$ be a simple undirected graph of order n. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, the *open neighborhood* of v is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$.

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The *degree of* v in G denoted by $d_G(v)$, is the size of its open neighborhood. The *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$[6].

Our aim in this paper is to consider the computing the vertex residual closeness(VRC) and normalized vertex residual closeness($NVRC$) of Splitting networks that are modeled by Splitting graphs. In section 2, definitions and well-known basic results are given for closeness, VRC and $NVRC$, respectively. In section 3, definitions of Splitting graphs are given and VRC and $NVRC$ of some Splitting graphs are determined.

2. Residual Closeness and Basic Results

The concept of closeness, vertex residual closeness(VRC) and normalized vertex residual closeness($NVRC$) were introduced on 2006 by Chavdar Dangalchev [3, 4] and has been further studied by Aytac et al. [1, 16] and Turaci et al. [14]. The aim of residual closeness is to measure the vulnerability even when the actions (removal of the vertices) do not disconnect the graph.

The closeness of a graph G is defined as: $C(G) = \sum_{v_i} C(v_i)$, where $C(v_i)$ is the closeness of a vertex v_i, and it is defined as: $C(v_i) = \sum_{v_i \neq v_j} \frac{1}{2^{d(v_i,v_j)}}$ [3]. Let $d_{v_k}(v_i,v_j)$ be the distance between vertices v_i and v_j in the graph G, received from the original graph where all links of vertex v_k are deleted. Then the closeness after removing vertex v_k is defined as: $C_{v_k} = \sum_{v_i} \sum_{v_i \neq v_j} \frac{1}{2^{d_{v_k}(v_i,v_j)}}$ [3]. The vertex residual closeness (VRC) of the graph G is defined as: $R(G) = \min_{v_k}\{C_{v_k}\}$. The normalized vertex residual closeness ($NVRC$) of the graph G is defined as dividing the residual closeness by the closeness $C(G)$: $R'(G) = R(G)/C(G)$ [3].

Theorem 2.1. [3] *The VRC and NVRC of*
(a) *If* $G = K_n$, *then* $R(G) = ((n-1)(n-2))/2$ *and* $R'(G) = (n-2)/n$.
(b) *If* $G = S_n$, *then* $R(G) = 0$ *and* $R'(G) = 0$.

Theorem 2.2. [3] *For a graph* G, $0 \leq R'(G) < 1$.

Theorem 2.3. [3] *Let* G *be a graph of order* n. *If* H *is a proper subgraph of* G, *then* $R(H) < R(G)$.

Theorem 2.4. [1, 16] *The vertex residual closeness (VRC) of*
(a) *the cycle graph* C_n *with* n *vertices is* $R(C_n) = 2n - 6 + 1/2^{n-3}$;
(b) *the friendship graph* f_n *with* $2n+1$ *vertices is* $R(f_n) = n$;
(c) *the fan graph* F_n *with* n *vertices is* $R(F_n) = 2n - 6 + 1/2^{(n-3)}$;

(d) *the wheel graph* W_n *with* $n+1$ *vertices is*

$$R(W_n) = \begin{cases} n(\sum_{i=1}^{\lfloor n/2 \rfloor} 1/2^{(i-1)}) & , \text{if } n \text{ is odd;} \\ n((\sum_{i=1}^{(n/2)-1} 1/2^{(i-1)}) + 1/2^{(n/2)}) & , \text{if } n \text{ is even.} \end{cases}$$

(e) *the gear graph* G_n *with* $2n+1$ *vertices is*

$$R(G_n) = \begin{cases} 2n((\sum_{i=1}^{n-1} 1/2^{(i-1)}) + 1/2^n) & , \text{if } n \geq 5; \\ (9n^2 + 37n - 38)/16 & , \text{if } n < 5. \end{cases}$$

3. Calculation of Residual Closeness Centrality of Some Splitting Graphs

In this section, fistly the definitions of Mycielski graph and Splitting graph and well-known basic results have been given for the closeness, *VRC* and *NVRC*, respectively. Then, we have computed *VRC* and *NVRC* of the some splitting graphs.

Definition 3.1. [5] *For a graph* G *on vertices* $V(G) = V = \{v_1, v_2, ..., v_n\}$ *and edges* $E(G) = E$, *let splitting graph* $S'(G)$ *be the graph on vertices and edges* $V \cup V' = \{v_1, v_2, ..., v_n, v_1', v_2', ..., v_n'\}$ *and* $E \cup \{v_i v_j' | v_i v_j \in E\}$, *respectively.*

Theorem 3.1. [15] *The closeness centrality of Splitting graphs of*

(a) *the complete graph* K_n *with* n *vertices is* $C(S'(K_n)) = (7n^2 - 5n)/4$;

(b) *the star graph* S_n *with* $n+1$ *vertices is* $C(S'(S_n)) = (4n^2 + 11n + 2)/4$;

(c) *the wheel graph* W_n *with* $n+1$ *vertices is*

$$C(S'(W_n)) = (2n^2 + 9n + 1)/2;$$

(d) *the path graph* P_n *with* n *vertices is*

$$C(S'(P_n)) = 4(2n - 4 + 1/2^{n-2}) - (n-3)/4;$$

(e) *the cycle graph* C_n *with* n *vertices is*

$$C(S'(C_n)) = \begin{cases} n((31/4) - (12/2^{n/2})) & , \text{if } n \text{ is even;} \\ n((31/4) - (8/2^{(n-1)/2})) & , \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3.2. *For* $n \geq 3$; *If* $G = S_n$, *then the vertex residual closeness (VRC) of* $S'(G)$ *of order (2n+2) is defined as:* $R(S'(G)) = (n^2 + 3n)/4$.

Proof. Let the vertex set $S'(G)$ be $V(S'(G)) = \{v_c\} \cup V_1 \cup \{v_c'\} \cup V_2$, where let v_c be center vertex of G and the vertex v_c' be corresponding to the vertex v_c by the definition of splitting graph. Moreover, let $V_1 = \{v_i \in V(G) \setminus \{v_c\}, 1 \leq i \leq n\}$ and $V_2 = \{v_i' \in V(G') \setminus \{v_c'\}, 1 \leq i \leq n\}$. We have four cases depending on the vertices of $S'(G)$.

Case 1. Removing the central vertex v_c of the graph $S'(G)$. If the vertex v_c is removed from the graph $S'(G)$, there are n-vertices whose degree 1, and a vertex whose degree is n and n-vertices whose degree 0 in the remaining subgraph $S'(G) \setminus \{v_c\}$. Then, we have three subcases depending on the vertices of $S'(G) \setminus \{v_c\}$.

Subcase 1.1. For the vertices V_1 whose degree 1 of $S'(G) \setminus \{v_c\}$. Let v_i be any vertex in this case. Clearly, $|N_{S'(G)\setminus\{v_c\}}(v_i)| = 1$, where $N_{S'(G)\setminus\{v_c\}}(v_i) = \{v'_c\}$. Furthermore, the distance from the vertex v_i to any other vertices whose degree 1 is two. Finally, the distance from the vertex v_i to any vertices whose degree 0 is ∞. Thus,

$$(n)C_{v_c}(v_i) = n(2^{-1} + ((n-1)2^{-2}) + n(2^{-\infty})) = (n^2+n)/4.$$

Subcase 1.2. For the vertex v'_c of $S'(G) \setminus \{v_c\}$. The vertex v'_c is adjacent to vertices V_1 whose degree is 1 in the graph $S'(G) \setminus \{v_c\}$. Furthermore, there is not link to vertices between v'_c and any vertex of V_2 in the graph $S'(G) \setminus \{v_c\}$. Thus,

$$C_{v_c}(v'_c) = n(2^{-1}) + n(2^{-\infty}) = n/2.$$

Subcase 1.3. For the vertices V_2 of whose degree 0 in the graph $S'(G) \setminus \{v_c\}$. Let v'_i be any vertex in this case. Since the definition of closeness, we have

$$(n)C_{v_c}(v'_i) = n(2n(2^{-\infty})) = 0.$$

By summing Subcases 1.1, 1.2 and 1.3, we have $C_{v_c} = (n^2+3n)/4$.

Case 2. Removing a vertex $v_i \in V_1$ in the graph $S'(G)$. We have four subcases depending on the vertices of $S'(G) \setminus \{v_i\}$.

Subcase 2.1. For the central vertex v_c of the graph G in the graph $S'(G) \setminus \{v_i\}$. The vertex v_c is adjacent to $(2n-1)$-vertices which in sets of either V_1 or V_2. The distance from the vertex v_c to remaining 1-vertex is 2. Thus,

$$C_{v_i}(v_c) = (2n-1)(2^{-1}) + 2^{-2} = (4n-1)/4.$$

Subcase 2.2. For a vertex $v_j \in V_1 \setminus \{v_i\}$ in the graph $S'(G) \setminus \{v_i\}$. The vertex v_j is adjacent to vertices v_c and v'_c in the graph $S'(G) \setminus \{v_i\}$. So, $d_{v_i}(v_j, v_c) = d_{v_i}(v_j, v'_c) = 1$. Then, it is clear that the distance from the vertex v_j to remaining $(2n-2)$-vertices is 2. Thus,

$$(n-1)C_{v_i}(v_j) = (n-1)(2(2^{-1}) + (2n-2)(2^{-2})) = (2n^2-2)/4.$$

Subcase 2.3. For the vertex v'_c of the survival subgraph $S'(G) \setminus \{v_i\}$. The vertex v'_c is adjacent to $(n-1)$-vertices of V_1. Moreover, the distance from the vertex v'_c to the vertex v_c is 2. Finally, the distance from the vertex v'_c to any vertices of V_1 is 3 in the graph $S'(G) \setminus \{v_i\}$. Thus,

$$C_{v_i}(v'_c) = (n-1)(2^{-1}) + 2^{-2} + n(2^{-3}) = (5n-2)/8.$$

Subcase 2.4. For a vertex $v'_i \in V_2$ in the graph $S'(G) \setminus \{v_i\}$. The vertex v'_i is adjacent to the vertex v_c in the graph $S'(G) \setminus \{v_i\}$. So, $d_{v_i}(v'_i, v_c) = 1$. Then, it is clear that the distance from the vertex v'_i to the vertex v'_c is 3 in the graph $S'(G)\setminus\{v_i\}$. Finally, the distance from the vertex v_i' to remaining $(2n-2)$-vertices is 2. Thus,

$$(n)C_{v_i}(v_i') = (n)(2^{-1} + (2n-2)(2^{-2}) + 2^{-3}) = (4n^2+n)/8.$$

By summing Subcases 2.1, 2.2, 2.3 and 2.4, we obtain

$$C_{v_i} = (8n^2+14n-8)/8.$$

Case 3. If the vertex v_c' is removed from the graph $S'(G)$, then the survival subgraph $S'(G)\setminus\{v_c'\}$ is star graph S_{2n+1} with $(2n+1)$-vertices. By the Theorem 2.1.b, $C_{v_c'} = (4n^2+6n)/4$ is obtained.

Case 4. Removing a vertex $v_i' \in V_2$ in the graph $S'(G)$. We have four subcases depending on the vertices of $S'(G)\setminus\{v_i'\}$.

Subcase 4.1. For the central vertex v_c of the graph G in the graph $S'(G)\setminus\{v_i'\}$. It is clear that $|N_{S'(G)\setminus\{v_i'\}}(v_c)| = 2n-1$, and then $d_{v_i'}(v_c, v_c') = 2$ in the survival subgraph $S'(G)\setminus\{v_i'\}$. Thus,

$$C_{v_i'}(v_c) = (2n-1)(2^{-1}) + 2^{-2} = (4n-1)/4.$$

Subcase 4.2. For a vertex $v_i \in V_1$ in the survival subgraph $S'(G)\setminus\{v_i'\}$. The vertex v_i is adjacent to vertices v_c and v_c' in the graph $S'(G)\setminus\{v_i'\}$. Clearly, the distance from the vertex v_i to remaining $(2n-2)$-vertices is 2. Thus, $(n)C_{v_i'}(v_i) = (n)(2(2^{-1}) + (2n-2)(2^{-2})) = (2n^2+2n)/4$.

Subcase 4.3. For the vertex v_c' in the survival subgraph $S'(G)\setminus\{v_i'\}$. The vertex v_c' is adjacent to vertices of V_1 in the graph $S'(G)\setminus\{v_i'\}$. Since the definitions of star graph and splitting graph, we have $d_{v_i'}(v_c, v_c') = 2$. Finally, the distance from the vertex v_c' to $(n-1)$-vertices of V_2 is 3. Thus,

$$C_{v_i'}(v_c') = n(2^{-1}) + 2^{-2} + (n-1)(2^{-3}) = (5n+1)/8.$$

Subcase 4.4. For a vertex $v_j' \in V_2\setminus\{v_i'\}$ in the graph $S'(G)\setminus\{v_i'\}$. The proof is similar to the Subcase 4 of Case 2 for vertices of V_2. Thus,

$$(n-1)C_{v_i'}(v_j') = (n-1)(2^{-1} + (2n-2)(2^{-2}) + 2^{-3}) = (4n^2-3n-1)/8.$$

By summing Subcases 4.1,4.2,4.3 and 4.4, we obtain $C_{v_i'} = (8n^2+14n-2)/8$. By the definition of VRC of $S'(G)$, $R(S'(G)) = min\{C_{v_c}, C_{v_i}, C_{v_c'}, C_{v_i'}\}$ is obtained. Thus, VRC of $S'(G)$ is
$= min\{(n^2+3n)/4, (8n^2+14n-8)/8, (4n^2+6n)/4, (8n^2+14n-2)/8\}$
$= (n^2+3n)/4$. □

Theorem 3.3. *For $n \geq 7$; If $G = W_n$, then the vertex residual closeness (VRC) of $S'(G)$ of order (2n+2) is defined as:*

$$R(S'(G)) = \begin{cases} (2n^2+5n-8)/4 & , \text{ if } n \leq 35; \\ (3n^2+45n)/8 & , \text{ if } n > 35. \end{cases}$$

Proof. Let the vertex set $S'(G)$ be $V(S'(G)) = \{v_c\} \cup V_1 \cup \{v_c'\} \cup V_2$, where let v_c be center vertex of G and the vertex v_c' be corresponding to the vertex v_c by the definition of splitting graph. Moreover, let $V_1 = \{v_i \in V(G) \setminus \{v_c\}, 1 \le i \le n\}$ and $V_2 = \{v'_i \in V(G') \setminus \{v'_c\}, 1 \le i \le n\}$. We have four cases depending on the vertices of $S'(G)$.

Case 1. Removing the central vertex v_c of the graph $S'(G)$. If the vertex v_c is removed from the graph $S'(G)$, the diameter of survival graph $S'(G) \setminus \{v_c\}$ is 4. Then, we have three subcases depending on the vertices of $S'(G) \setminus \{v_c\}$.

Subcase 1.1. For the vertices of V_1 whose degree 5 of $S'(G) \setminus \{v_c\}$. Let v_i be any vertex in V_1. Clearly, $|N_{S'(G)\setminus\{v_c\}}(v_i)| = 5$. Thus, the number of distance with 1 is five. Furthermore, there are 5 paths which distance from the vertex v_i to other vertices. Since the structure of the graph $S'(G) \setminus \{v_c\}$ we have $(n-3)$-paths of length 3 and $(n-7)$-paths of length 4. Thus,

$$(n)C_{v_c}(v_i) = n(5(2^{-1})+5(2^{-2})+(n-3)(2^{-3})+(n-7)(2^{-4})) = (3n^2+47n)/16.$$

Subcase 1.2. For the vertex v'_c of $S'(G) \setminus \{v_c\}$. The vertex v'_c is adjacent to vertices of V_1 in the graph $S'(G) \setminus \{v_c\}$. Furthermore, there are n-paths which vertices between v'_c and any vertex of V_2 in the graph $S'(G) \setminus \{v_c\}$. Thus, $C_{v_c}(v'_c) = n(2^{-1}) + n(2^{-2}) = 3n/4$.

Subcase 1.3. The proof is similar to Subcase 1.1 of Case 1 of Theorem 3.3, but there are 2 paths, 6 paths, $(n-1)$-paths and $(n-7)$-paths with length of one, two, three and four, respectively. Thus,

$$(n)C_{v_c}(v'_i) = n(2(2^{-1})+6(2^{-2})+(n-1)(2^{-3})+(n-7)(2^{-4})) = (3n^2+31n)/16.$$

By summing Subcases 1.1, 1.2 and 1.3, we obtain $C_{v_c} = (3n^2 + 45n)/8$.

Case 2. Removing a vertex $v_i \in V_1$ in the graph $S'(G)$. We have four subcases depending on the vertices of $S'(G) \setminus \{v_i\}$.

Subcase 2.1. For the vertex v_c of the graph G in the graph $S'(G) \setminus \{v_i\}$. It is adjacent to $(n-1)$-vertices which belongs to $V_1 \setminus \{v_i\}$ and V_2. Clearly, $d_{v_i}(v_c, v'_c) = 2$. Thus, $C_{v_i}(v_c) = (2n-1)(2^{-1}) + 2^{-2} = (4n-1)/4$.

Subcase 2.2. For the vertex $v_j \in V_1 \setminus \{v_i\}$ in the graph $S'(G) \setminus \{v_i\}$. If the vertex v_j is adjacent to the vertex v_i in V_1, then the vertex v_j is adjacent to 5-vertices which include the vertex v_c. So, the distance from the vertex v_j to remaining $(2n-5)$-vertices is 2. If the vertex v_j is not adjacent to the vertex v_i in V_1, then the vertex v_j is adjacent 6-vertices which include the vertex v_c. So, the distance from the vertex v_j to remaining $(2n-6)$-vertices are 2. Since the vertex v_i has 2-adjacent vertices in V_1, then

$$\begin{aligned} C_{v_i}(v_j) &= (n-3)(6(2^{-1}) + (2n-6)(2^{-2})) + 2(5(2^{-1}) + (2n-5)(2^{-2})) \\ &= (n^2 + 2n - 4)/2. \end{aligned}$$

Subcase 2.3. For the vertex v'_c of the survival subgraph $S'(G) \setminus \{v_i\}$. The vertex v'_c is adjacent to $(n-1)$-vertices of V_1. Moreover, the distance from the vertex v'_c to the vertex v_c and the any vertex of V_2 is 2. Thus,

$$C_{v_i}(v'_c) = (n-1)(2^{-1}) + (n+1)(2^{-2}) = (3n-1)/4.$$

Subcase 2.4. For a vertex $v_i' \in V_2$ in the graph $S'(G) \setminus \{v_i\}$. The proof is similar to the Subcase 2.2 of Case 2 of Theorem 3.3. Thus,

$$\begin{aligned} C_{v_i}(v_i') &= (n-2)(3(2^{-1}) + (2n-3)(2^{-2})) + 2(2(2^{-1}) + (2n-2)(2^{-2})) \\ &= (2n^2 - n - 6)/4. \end{aligned}$$

By summing Subcases 2.1, 2.2, 2.3 and 2.4, we obtain $C_{v_i} = (2n^2+5n-8)/4$.

Case 3. Removing the vertex v_c' of the graph $S'(G)$. We have three subcases depending on the vertices of graph $S'(G) \setminus \{v_c'\}$.

Subcase 3.1. For the vertex v_c of the graph $S'(G) \setminus \{v_c'\}$. Due to $d_{S'(G)\setminus\{v_c'\}}(v_c) = 2n$, we have $C_{v_c'}(v_c) = 2n(2^{-1}) = n$.

Subcase 3.2. For a vertex $v_i \in V_1$ in the survival subgraph $S'(G) \setminus \{v_c'\}$. The vertex v_i is adjacent to 5-vertices which has the vertex v_c. So, the distance of remaining $(2n-5)$-vertices is 2. Thus,

$$(n)C_{v_c'}(v_i) = (n)(5(2^{-1}) + (2n-5)(2^{-2})) = (2n^2 + 5n)/4.$$

Subcase 3.3. For a vertex $v_i' \in V_2$ in the survival subgraph $S'(G) \setminus \{v_c'\}$. It is clear that the vertex v_i' is adjacent 3-vertices which has the vertex v_c. Then, the distance of remaining $(2n-3)$-vertices is 2. Thus,

$$(n)C_{v_c'}(v_i') = (n)(3(2^{-1}) + (2n-3)(2^{-2})) = (2n^2 + 3n)/4.$$

By summing Subcases 3.1, 3.2 and 3.3, we obtain $C_{v_c'} = n^2 + 3n$.

Case 4. Removing a vertex $v_i' \in V_2$ in the graph $S'(G)$. We have four subcases depending on the vertices of $S'(G) \setminus \{v_i'\}$.

Subcase 4.1. For the vertex v_c of the graph $S'(G) \setminus \{v_i'\}$. It is adjacent $(2n-1)$-vertices which belongs to $V_2 \setminus \{v_i'\}$ and V_1. Clearly, $d_{v_i'}(v_c, v_c') = 2$. Thus,

$$C_{v_i'}(v_c) = (2n-1)(2^{-1}) + 2^{-2} = (4n-1)/4.$$

Subcase 4.2. For a vertex $v_i \in V_1$ in the survival subgraph $S'(G) \setminus \{v_i'\}$. The proof is similar to the Subcase 2 of Case 2 of Theorem 3.3, then we have

$$\begin{aligned} C_{v_i'}(v_i) &= (n-2)(6(2^{-1}) + (2n-6)(2^{-2})) + 2(5(2^{-1}) + (2n-5)(2^{-2})) \\ &= (n^2 + 3n - 1)/2. \end{aligned}$$

Subcase 4.3. For the vertex v_c' of the survival subgraph $S'(G) \setminus \{v_i'\}$. The vertex v_c' is adjacent to n-vertices of V_1. Moreover, the distance from the vertex v_c' to the vertex v_c and any vertices of V_2 is 2. Thus,

$$C_{v_i'}(v_c') = n(2^{-1}) + n(2^{-2}) = 3n/4.$$

Subcase 4.4. For the vertex $v_j' \in V_2 \setminus \{v_i'\}$ in the graph $S'(G) \setminus \{v_i'\}$. The vertex v_j' is adjacent to 2-vertices of V_1 and the vertex v_c. It is clear that the distance of remaining $(2n-3)$-vertices is 2. Thus,

$$(n-1)C_{v_i'}(v_j') = (n-1)(3(2^{-1}) + (2n-3)(2^{-2})) = (2n^2 + n - 3)/4.$$

By summing Subcases 4.1, 4.2, 4.3 and 4.4, we obtain $C_{v'_i} = (2n^2+7n-3)/2$. Then we have $R(S'(G)) = min\{C_{v_c}, C_{v_i}, C_{v'_c}, C_{v'_i}\}$. Thus, VRC of $S'(G)$ is $= min\{(3n^2+45n)/8, (2n^2+5n-8)/4, n^2+3n, (2n^2+7n-3)/2\}$.
As a result, if $n \leq 35$, then we obtain $R(S'(G)) = (2n^2+5n-8)/4$. Otherwise, if $n > 35$ then we have $R(S'(G)) = (3n^2+45n)/8$. □

Theorem 3.4. *For $n \geq 3$; If $G = K_n$, then the vertex residual closeness (VRC) of $S'(G)$ of order $2n$ is defined as: $R(S'(G)) = (7n^2-13n+6)/4$.*
Proof. Let the vertex set $S'(G)$ be $V(S'(G)) = V_1 \cup V_2$, where let $V_1 = \{v_i \in V(G), 1 \leq i \leq n\}$ and $V_2 = \{v'_i \in V(G'), 1 \leq i \leq n\}$. We have two cases depending on the vertices of graph $S'(G)$.

Case 1. Removing a vertex $v_i \in V_1$ in the graph $S'(G)$. We have three subcases depending on the vertices of $S'(G) \setminus \{v_i\}$.

Subcase 1.1. For the vertex $v_j \in V_1 \setminus \{v_i\}$ in the graph $S'(G) \setminus \{v_i\}$. Since the definition of splitting graph and complete graph, the vertex v_j is adjacent $(2n-3)$-vertices. Moreover, $d_{v_i}(v_j, v'_j) = 2$. Thus,

$$(n-1)C_{v_i}(v_j) = (n-1)((2n-3)2^{-1} + 2^{-2}) = (4n^2-9n+5)/4.$$

Subcase 1.2. For the vertex v'_i in the graph $S'(G) \setminus \{v_i\}$. Since the definition of splitting graph and complete graph, the vertex v'_i is adjacent $(n-1)$-vertices. Then, the distance of remaining $(n-1)$-vertices is 2. Thus,

$$C_{v_i}(v'_i) = (n-1)2^{-1} + (n-1)2^{-2} = (3n-3)/4.$$

Subcase 1.3. For the vertex $v_j \in V_2 \setminus \{v'_i\}$ in the graph $S'(G) \setminus \{v_i\}$. Since the definition of splitting graph and complete graph, the vertex v'_j is adjacent $(n-2)$-vertices. Then, the distance of remaining n-vertices is 2. Thus,

$$C_{v_i}(v_j) = (n-1)((n-2)2^{-1} + (n)2^{-2}) = (3n^2-7n+4)/4.$$

By summing Subcases 1.1, 1.2 and 1.3, we have $C_{v_i} = (7n^2-13n+6)/4$.

Case 2. Removing a vertex $v'_i \in V_2$ in the graph $S'(G)$. We have three subcases depending on the vertices of $S'(G) \setminus \{v'_i\}$.

Subcase 2.1. For the vertex v_i in the graph $S'(G) \setminus \{v'_i\}$. Due to $d_{S'(G)\setminus\{v'_i\}}(v_i) = 2n-2$, then we have $C_{v'_i}(v_i) = (2n-2)2^{-1} = n-1$.

Subcase 2.2. For the vertex $v_j \in V_1 \setminus \{v_i\}$ in the graph $S'(G) \setminus \{v'_i\}$. Since the definition of splitting graph and complete graph, the vertex v_j is adjacent $(2n-3)$-vertices. Then, the distance of remaining a vertex is 2. Thus,

$$(n-1)C_{v'_i}(v_j) = (n-1)((2n-3)2^{-1} + 2^{-2}) = (4n^2-9n+5)/4.$$

Subcase 2.3. For the vertex $v'_j \in V_2 \setminus \{v'_i\}$ in the graph $S'(G) \setminus \{v'_i\}$. Since the definition of splitting graph and complete graph, the vertex v'_j is adjacent $(n-1)$-vertices. Then, the distance of remaining $(n-1)$-vertices is 2. Thus,

$$(n-1)C_{v_i'}(v_j') = (n-1)((n-1)2^{-1} + (n-1)2^{-2}) = (3n^2 - 6n + 3)/4.$$

By summing Subcases 2.1, 2.2 and 2.3, we obtain $C_{v_i'} = (7n^2 - 11n + 4)/4$. By the definition of VRC of the graph $S'(G)$, we obtain

$$\begin{aligned} R(S'(G)) &= min\{C_{v_i}, C_{v_i'}\} = \{(7n^2 - 13n + 6)/4, (7n^2 - 11n + 4)/4\} \\ &= (7n^2 - 13n + 6)/4. \square \end{aligned}$$

Theorem 3.5. *For $n \geq 3$; If $G = C_n$, then the vertex residual closeness (VRC) of $S'(G)$ of order $2n$ is defined as:*

$$C(S'(G)) = \begin{cases} C(S'(P_{n-1})) + 8 - 8(1/2)^{n/2} & , \text{ if } n \text{ is odd;} \\ C(S'(P_{n-1})) + (13/2) - 12(1/2)^{n/2} & , \text{ if } n \text{ is even.} \end{cases}$$

Proof. Let the vertex set $S'(G)$ be $V(S'(G)) = V_1 \cup V_2$, where let $V_1 = \{v_i \in V(G), 1 \leq i \leq n\}$ and $V_2 = \{v_i' \in V(G'), 1 \leq i \leq n\}$. We have two cases depending on the vertices of $S'(G)$.

Case 1. Removing a vertex of $v_i \in V_1$ in the graph $S'(G)$. Moreover, we remove the vertex v_i' in the graph $S'(G) \setminus \{v_i\}$. It is clear that the survival subgraph $S'(G) \setminus \{v_i, v_i'\}$ is $S'(P_{n-1})$ of order $2n-2$. So, we have

$$\begin{aligned} C_{v_i} &= C(S'(P_{n-1})) + 2C_{v_i}(v_i') \\ &= C(S'(P_{n-1})) + 2(\textstyle\sum_{v_i' \in V_1 \setminus \{v_i\}} 2^{-d(v_i', v_j)} + \sum_{v_j' \in V_2 \setminus \{v_i'\}} 2^{-d(v_i', v_j')}). \end{aligned}$$

We have two subcases depending on the number of vertices of G.

Subcase 1.1. Let n be odd. To calculate first sum, say *Sum1*, we get

$$Sum1 = 2/2^1 + 2/2^2 + ... + 2/2^{(n-1)/2}.$$

To calculate the above *Sum1*, we use the formula of finite geometric series. Thus, we get $Sum1 = 2 - 2(1/2)^{(n-1)/2}$.
To calculate second sum, say *Sum2*, we get

$$Sum2 = 2/2^1 + 2/2^2 + ... + 2/2^{(n-1)/2} + 2/2^3 = 2 - 2(1/2)^{(n-1)/2} + 1/4.$$

Thus, when n is odd we have

$$\begin{aligned} C_{v_i} &= C(S'(P_{n-1})) + 2((2 - 2(1/2)^{(n-1)/2}) + (2 - 2(1/2)^{(n-1)/2} + 1/4)) \\ &= C(S'(P_{n-1})) + (17/2) - 8(1/2)^{(n-1)/2}. \end{aligned}$$

Subcase 1.2. Let n be even. To calculate first sum, say *Sum1*, we get

$$Sum1 = 2/2^1 + 2/2^2 + ... + 2/2^{(n/2)-1} + 1/2^{n/2}$$

To calculate the above *Sum1*, we use the formula of the finite geometric series. Thus, we get $Sum1 = 2 - 3(1/2)^{n/2}$.
To calculate second sum, say *Sum2*, we get

$$Sum2 = 2/2^2 + 2/2^3 + ... + 2/2^{(n/2)-1} + 2/2^3 + 1/2^{n/2} = (5/4) - 3(1/2)^{n/2}.$$

Thus, when n is even we have

$$C_{v_i} = C(S'(P_{n-1})) + 2((2 - 3(1/2)^{n/2}) + ((5/4) - 3(1/2)^{n/2}))$$
$$= C(S'(P_{n-1})) + (13/2) - 12(1/2)^{n/2}.$$

Case 2. Removing a vertex of $v'_i \in V_2$ in the graph $S'(G)$. Moreover, we remove the vertex v_i in the graph $S'(G) \setminus \{v'_i\}$. It is clear that the survival subgraph $S'(G) \setminus \{v_i, v'_i\}$ is $S'(P_{n-1})$ of order $2n - 2$. So, we have

$$C_{v_i} = C(S'(P_{n-1})) + 2C_{v'_i}(v_i)$$
$$= C(S'(P_{n-1})) + 2(\sum_{v_j \in V_1 \setminus \{v_i\}} 2^{-d(v_i,v_j)} + \sum_{v'_j \in V_2 \setminus \{v'_i\}} 2^{-d(v_i,v'_j)}).$$

We have two subcases depending on the number of vertices of G.

Subcase 2.1. Let n be odd. First sum and second sum is similar to the *Sum1* of Subcase 1.1 of Case 1. So, $Sum1 = Sum2 = 2 - 2(1/2)^{(n-1)/2}$. Thus, when n is odd we have

$$C_{v'_i} = C(S'(P_{n-1})) + 2((2 - 2(1/2)^{(n-1)/2)}) + (2 - 2(1/2)^{(n-1)/2)}))$$
$$= C(S'(P_{n-1})) + 8 - 8(1/2)^{(n-1)/2}.$$

Subcase 2.2. Let n be even. First sum and second sum is similar to the *Sum1* of Subcase 1.2 of Case 1. So, $Sum1 = Sum2 = 2 - 3(1/2)^{n/2}$. Thus, when n is even we have

$$C_{v'_i} = C(S'(P_{n-1})) + 2((2 - 3(1/2)^{n/2}) + (2 - 3(1/2)^{n/2}))$$
$$= C(S'(P_{n-1})) + 8 - 12(1/2)^{n/2}.$$

By the definition of vertex residual closeness *(VRC)* of the graph $S'(G)$ as follows: $R(S'(G)) = min\{C_{v_i}, C_{v'_i}\}$.

When n is odd:

$$R(S'(G)) = min\{C(S'(P_{n-1})) + (17/2) - 8(1/2)^{(n-1)/2}, C(S'(P_{n-1})) + 8 - 8(1/2)^{(n-1)/2}\}$$
$$= C(S'(P_{n-1})) + 8 - 8(1/2)^{(n-1)/2} \text{ is obtained.}$$

When n is even:

$$R(S'(G)) = min\{C(S'(P_{n-1})) + (13/2) - 12(1/2)^{n/2}, C(S'(P_{n-1})) + 8 - 12(1/2)^{n/2}\}$$
$$= C(S'(P_{n-1})) + 8 - 12(1/2)^{n/2} \text{ is obtained. } \square$$

Theorem 3.6. *The NVRC of Splitting graphs of*

(a) *the cycle graph C_n of order n is*

$$R'(S'(C_n)) = \begin{cases} \frac{C(S'(P_{n-1}))+8-8/2^{n/2}}{n((31/4)-(12/2^{n/2}))} & , \text{ if } n \text{ is odd;} \\ \frac{C(S'(P_{n-1}))+(13/2)-12/2^{n/2}}{n((31/4)-(8/2^{(n-1)/2}))} & , \text{ if } n \text{ is even.} \end{cases}$$

(b) *the complete graph K_n of order n is* $R'(S'(K_n)) = 1 - \frac{8n-6}{7n^2-5n}$.

(c) *the star graph S_n of order $n+1$ is* $R'(S'(S_n)) = \frac{n^2+3n}{4n^2+11n+2}$.

(d) *the wheel graph W_n of order $n+1$ is*

$$R'(S'(W_n)) = \begin{cases} \frac{2n^2+5n-8}{4n^2+18n+2} & , \text{ if } n \le 35; \\ \frac{3n^2+45n}{8n^2+32n+4} & , \text{ if } n > 35. \end{cases}$$

4. Conclusion

We have investigated the residual closeness of networks as a measure of network analysis. Residual closeness is a new characteristic measure for graph vulnerability. Calculation of residual closeness for simple graph types is important because we get which vertices in the network are responsible for fast communication flow. Thus, vertices giving the residual closeness of a graph are important and fast in distributing information through the network.

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