Exact Wirelength of Embedding Circulant Networks into Necklace and Windmill Graphs

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Abstract

Graph embedding has been known as a powerful tool for implementation of parallel algorithms and simulation of different interconnection networks. In this paper, we obtain minimum wirelength of embedding circulant networks into necklace and windmill graphs. The algorithms for obtaining the same are of $O(2n)$-linear time.

Keywords: Embedding, congestion, wirelength, edge isomperimetric problem, circulant networks

1 Introduction

Interconnection networks provide an effective mechanism for exchanging data between processors in a parallel computing system. An interconnection network is often represented as a graph, where nodes and edges correspond to processors and communication links between processors, respectively. In the design and analysis of an interconnection network, its graph embedding ability is a major concern. An ideal interconnection network (host graph) is expected to possess excellent graph embedding ability which helps efficiently execute parallel algorithms with regular task graphs (guest graphs) on this network [1].

An embedding of a guest graph G into a host graph H is a one-to-one mapping of the vertex set of G into that of H. The dilation of an embedding is defined as the maximum distance between a pair of vertices of H that are images of adjacent vertices of G. The study of graph embeddings is an important topic in the theory of parallel computation: the existence of such an embedding demonstrates the ability of a parallel computer, whose interconnection network is represented by the host graph, to simulate a parallel algorithm, whose communication structure is described by the guest graph. The dilation can then serve as one of natural measures of the communication delay [2].

The circulant graph (network) is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [3]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [4]. It is also used in VLSI design and distributed computation [5, 6, 7]. The term circulant comes from the nature of its adjacency matrix. A matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have been employed for designing binary codes [8]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [5]. Every circulant graph is a vertex transitive graph and a Cayley graph [9]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [4, 5].

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [9, 10].

Graph embeddings have been well studied for meshes into crossed cubes [11], binary trees into paths [10], binary trees into hypercubes [2, 12], complete binary trees into hypercubes [13], incomplete hypercube in books [14], tori and grids into twisted cubes [15], meshes into locally twisted cubes [16], meshes into faulty crossed cubes [1], generalized ladders into hypercubes [17], grids into grids [18], binary trees into grids [19], hypercubes into cycles [20, 21], star graph into path [22], snarks into torus [23], generalized wheels into arbitrary trees [24], hypercubes into grids [25], m-sequencial k-ary trees into hypercubes [26], meshes into möbius cubes [27], ternary tree into hypercube [28], enhanced and augmented hypercube into complete binary tree [29], circulant into arbitrary trees, cycles, certain multicyclic graphs and ladders [30], hypercubes into cylinders, snakes and caterpillars [31], hypercubes into necklace, windmill and snake graphs [32], embedding of special classes of circulant networks, hypercubes and generalized Petersen graphs [33].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [20, 34]. The embeddings discussed in this paper produce exact wirelength.

2 Preliminaries

In this section we give the basic definitions and preliminaries related to embedding problems.

Definition 2.1. [34] *Let G and H be finite graphs with n vertices. An embedding f of G into H is defined as follows:*

1. *f is a bijective map from $V(G) \to V(H)$*
2. *f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.*

Definition 2.2. [34] *The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in H. In other words,*

$$EC_f(G, H(e)) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}|$$

where $P_f(u,v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G,H)$ is the minimum, over all embeddings $f : V(G) \to V(H)$, of the maximum number of wires that cross any edge of H [35].

Definition 2.3. [25] *The wirelength of an embedding f of G into H is given by*

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} d_H(f(u), f(v)) = \sum_{e\in E(H)} EC_f(G, H(e))$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u,v)$ in H. Then, the wirelength of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

The *wirelength problem* [19, 20, 24, 25, 34, 35] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$.

The following version of the edge isoperimetric problem of a graph $G(V,E)$ has been considered in the literature [36], and is NP-complete [37].

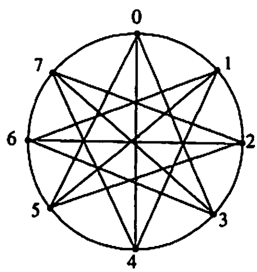

Figure 1: Circulant graph $G(8; \pm\{1,3,4\})$

Edge Isoperimetric Problem : Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A \subseteq V,\, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u,v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

Lemma 2.4. (Congestion Lemma) [25] Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ *and* $G_2 = f^{-1}(H_2)$. *Also* S satisfies the following conditions:

(i) *For every edge* $(a,b) \in G_i$, $i = 1, 2, P_f(a,b)$ *has no edges in* S.

(ii) *For every edge* (a,b) *in* G *with* $a \in G_1$ *and* $b \in G_2$, $P_f(a,b)$ *has exactly one edge in* S.

(iii) G_1 *is an optimal set.*

Then $EC_f(S)$ *is minimum and* $EC_f(S) = r\,|V(G_1)| - 2\,|E(G_1)|$. □

Lemma 2.5. (Partition Lemma) [25] Let $f : G \rightarrow H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i). \quad \square$$

Lemma 2.6. (2-Partition Lemma) [38] Let $f : G \rightarrow H$ be an embedding. Let $[2E(H)]$ denote a collection of edges of H repeated exactly 2 times. Let $\{S_1, S_2, \ldots, S_m\}$ be a partition of $[2E(H)]$ such that each S_i is an edge cut of H. Then

$$WL_f(G,H) = \frac{1}{2}\sum_{i=1}^{m} EC_f(S_i). \quad \square$$

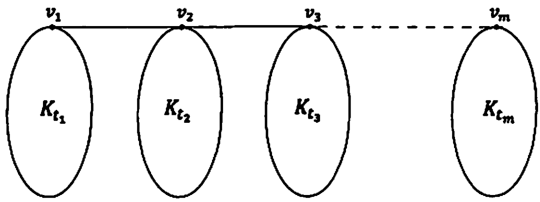

Figure 2: Necklace $N(P_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$

3 Embedding of Circulant Networks into Necklace Graphs

In this section, we compute the exact wirelength of embedding circulant networks into necklace graphs.

Definition 3.1. [5, 33] *The undirected circulant graph $G(n; S)$, $S \subseteq \pm\{1, 2, \ldots, j\}$, $1 \leq j \leq \lfloor n/2 \rfloor$ is a graph with vertex set $V = \{0, 1, \ldots, n-1\}$ and the edge set $E = \{(i, k) : |k - i| \equiv s(mod\ n),\ s \in S\}$.*

The circulant graph shown in Figure 1 is $G(8; \pm\{1, 3, 4\})$. It is clear that $G(n; \pm 1)$ is the undirected cycle C_n and $G(n; \pm\{1, 2, \ldots, \lfloor n/2 \rfloor\})$ is the complete graph K_n. The cycle $G(n; \pm 1) \simeq C_n$ contained in $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$ is sometimes referred to as the outer cycle C of G.

Theorem 3.2. [30] *A set of k consecutive vertices of $G(n; \pm 1)$, $1 \leq k \leq n$ induces a maximum subgraph of $G(n; \pm S)$, where $S = \{1, 2, \ldots, j\}$, $1 \leq j < \lfloor n/2 \rfloor$, $n \geq 3$.* □

Theorem 3.3. [30] *The number of edges in a maximum subgraph on k vertices of $G(n; \pm S)$, $S = \{1, 2, \ldots, j\}$, $1 \leq j < \lfloor n/2 \rfloor$, $1 \leq k \leq n$, $n \geq 3$ is given by*

$$\xi = \begin{cases} k(k-1)/2 & ;\ k \leq j+1 \\ kj - j(j+1)/2 & ;\ j+1 < k \leq n-j \\ \frac{1}{2}\{(n-k)^2 + (4j+1)k - (2j+1)n\} & ;\ n-j < k \leq n. \end{cases}$$ □

Definition 3.4. *Let $P = v_1 v_2 \ldots v_m$ be a path. Let K_{t_i} be a complete graph on t_i vertices such that $P \uplus K_{t_i}$ has just v_i as a cut-vertex, $i = 1, 2, \ldots, m$. The resultant graph $P \uplus (\bigcup_{i=1}^{m} K_{t_i})$ is a necklace denoted by $N(P_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$.*

Remark 3.5. *$N(P_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$ has $n = \sum_{i=1}^{m} t_i$ vertices. We denote $\sum_{i=0}^{k} t_i$ by s_k, $0 \le k \le m$, where $t_0 = 0$. See Figure 2. For brevity, the necklace $N(P_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$ will be represented by $N(P_m, K)$.*

Embedding Algorithm A

Input : A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j \le \lfloor n/2 \rfloor$ and a necklace $N(P_m, K)$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$ as $0, 1, 2, \ldots, n-1$ in the clockwise sense. Label the vertices of K_{t_i} in $N(P_m, K)$ as $s_{i-1} + j$, $j = 0, 1, 2, \ldots, t_i - 1$ such that s_{i-1} is the label of v_i, $1 \le i \le m$. See Figure 3.

Output : The exact wirelength of embedding circulant network $G(n; \pm\{1, 2, \ldots, j\})$ into necklace $N(P_m, K)$ given by the mapping $f(x) = x$.

Proof of correctness : We assume that the labels represent the vertices to which they are assigned. For $1 \le i \le m-1$, let $S_i = S_{i'} = \{(s_{i-1}, s_i)\}$. For $1 \le i \le m$, let $S'_i = \{(s_{i-1}, s_{i-1}+j) : 1 \le j \le t_i - 1\}$. For $1 \le i \le m$ and $1 \le j \le t_i - 1$, let $S_i^j = \{(s_{i-1}+j, s_{i-1}+k) : 0 \le k \le t_i - 1 \text{ and } j \ne k\}$. See Figure 3. Then $\{S_i, S_{i'} : 1 \le i \le m-1\} \cup \{S'_i : 1 \le i \le m\} \cup \{S_i^j : 1 \le i \le m$ and $1 \le j \le t_i - 1\}$ is a partition of $[2E(N(P_m, K))]$.

For each i, $1 \le i \le m-1$, $E(N(P_m, K)) \setminus S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{0, 1, 2, \ldots, s_i - 1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. Similarly, $EC_f(S_{i'})$ is minimum.

For each i, $1 \le i \le m$, $E(N(P_m, K)) \setminus S'_i$ has two components H'_{i1} and H'_{i2}, where $V(H'_{i1}) = \{s_{i-1}+1, s_{i-1}+2, \ldots, s_i - 1\}$. Let $G'_{i1} = f^{-1}(H'_{i1})$ and $G'_{i2} = f^{-1}(H'_{i2})$. By Theorem 3.2, G'_{i1} is an optimal set, each S'_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S'_i)$ is minimum.

For each i, j, $1 \le i \le m$, and $1 \le j \le t_i - 1$, $E(N(P_m, K)) \setminus S_i^j$ has two components H_{i1}^j and H_{i2}^j, where $V(H_{i1}^j) = \{s_{i-1} + j\}$. Let $G_{i1}^j = f^{-1}(H_{i1}^j)$ and $G_{i2}^j = f^{-1}(H_{i2}^j)$. By Theorem 3.2, G_{i1}^j is an optimal set, each S_i^j satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i^j)$ is minimum. The 2-Partition Lemma implies that the wirelength is minimum. □

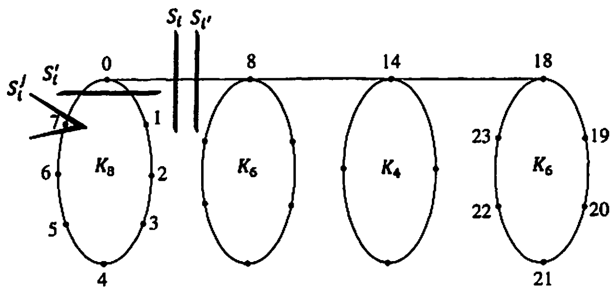

Figure 3: The edge cuts of necklace $N(P_4; K_8, K_6, K_4, K_6)$

Theorem 3.6. *The exact wirelength of circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j < \lfloor n/2 \rfloor$ into $N(P_m, K)$, is given by*

$$WL(G, N(P_m, K)) = \sum_{i=1}^{m-1} \theta_G(s_i) + \frac{1}{2}\sum_{i=1}^{m} \theta_G(t_i - 1) + j(s_m - m).$$

Proof. By Congestion Lemma,

(i) $EC_f(S_i) = EC_f(S_{i'}) = \theta_G(s_i)$, $1 \le i \le m-1$

(ii) $EC_f(S_i') = \theta_G(t_i - 1)$, $1 \le i \le m$ and

(iii) $EC_f(S_i^j) = 2j$, $1 \le i \le m$ and $1 \le j \le t_i - 1$.

Then by 2-Partition Lemma,

$$\begin{aligned} WL(G, N(P_m, K)) &= \frac{1}{2}[2\sum_{i=1}^{m-1} \theta_G(s_i) + \sum_{i=1}^{m} \theta_G(t_i - 1) + 2j(s_m - m)] \\ &= \sum_{i=1}^{m-1} \theta_G(s_i) + \frac{1}{2}\sum_{i=1}^{m} \theta_G(t_i - 1) + j(s_m - m). \quad \square \end{aligned}$$

Definition 3.7. *Let $C = v_1 v_2 \ldots v_m v_1$ be a cycle. Let K_{t_i} be a complete graph on t_i vertices such that $C \uplus K_{t_i}$ has just v_i as a cut-vertex, $i = 1, 2, \ldots, m$. The resultant graph $C \uplus (\bigcup_{i=1}^{m} K_{t_i})$ is a necklace denoted by $N(C_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$.*

Remark 3.8. *$N(C_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$ has $n = \sum_{i=1}^{m} t_i$ vertices. We denote $\sum_{i=0}^{k} t_i$ by s_k, $0 \le k \le m$, where $t_0 = 0$. For brevity, the necklace $N(C_m; K_{t_1}, K_{t_2}, \ldots, K_{t_m})$ will be represented by $N(C_m, K)$.*

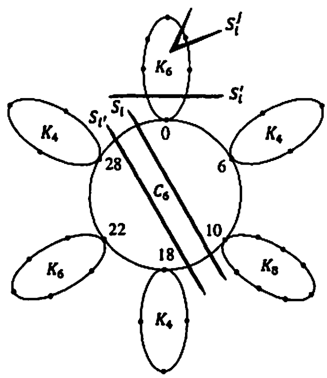

Figure 4: The edge cuts of necklace $N(C_6; K_6, K_4, \ldots, K_4)$

Embedding Algorithm B

Input : A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$ and a necklace $N(C_m, K)$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$ as $0, 1, 2, \ldots, n-1$ in the clockwise sense. Label the vertices of K_{t_i} in $N(C_m, K)$ as $s_{i-1} + j$, $j = 0, 1, 2, \ldots, t_i - 1$ such that s_{i-1} is the label of v_i, $1 \leq i \leq m$. See Figure 4.

Output : The exact wirelength of embedding circulant network $G(n; \pm\{1, 2, \ldots, j\})$ into necklace $N(C_m, K)$ given by the mapping $f(x) = x$.

Proof of correctness:

We assume that the labels represent the vertices to which they are assigned. **Case 1 (m is even):** For $1 \leq i \leq \frac{m}{2}$, let $S_i = S_{i'} = \{(s_{i-1}, s_i), (s_{\frac{m}{2}+i-1}, s_{\frac{m}{2}+i})\}$. For $1 \leq i \leq m$, let $S_i' = \{(s_{i-1}, s_{i-1} + j) : 1 \leq j \leq t_i - 1\}$. For $1 \leq i \leq m$ and $1 \leq j \leq t_i - 1$, let $S_i^j = \{(s_{i-1} + j, s_{i-1} + k) : 0 \leq k \leq t_i - 1$ and $j \neq k$ $\}$. See Figure 4. Then $\{S_i, S_{i'} : 1 \leq i \leq \frac{m}{2}\} \cup \{S_i' : 1 \leq i \leq m\} \cup \{S_i^j : 1 \leq i \leq m$ and $1 \leq j \leq t_i - 1\}$ is a partition of $[2E(N(C_m, K))]$.

For each i, $1 \leq i \leq \frac{m}{2}$, $E(N(C_m, K)) \backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{s_i, s_i + 1, \ldots, s_{\frac{m}{2}+i} - 1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. Similarly, $EC_f(S_{i'})$ is minimum.

For each i, $1 \leq i \leq m$, $E(N(C_m, K))\backslash S'_i$ has two components H'_{i1} and H'_{i2}, where $V(H'_{i1}) = \{s_{i-1}+1, s_{i-1}+2, \ldots, s_i - 1\}$. Let $G'_{i1} = f^{-1}(H'_{i1})$ and $G'_{i2} = f^{-1}(H'_{i2})$. By Theorem 3.2, G'_{i1} is an optimal set, each S'_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S'_i)$ is minimum.

For each i, j, $1 \leq i \leq m$ and $1 \leq j \leq t_i - 1$, $E(N(C_m, K))\backslash S^j_i$ has two components H^j_{i1} and H^j_{i2}, where $V(H^j_{i1}) = \{s_{i-1}+j\}$. Let $G^j_{i1} = f^{-1}(H^j_{i1})$ and $G^j_{i2} = f^{-1}(H^j_{i2})$. By Theorem 3.2, G^j_{i1} is an optimal set, each S^j_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S^j_i)$ is minimum. The 2-Partition Lemma implies that the wirelength is minimum.

Case 2 (m is odd): For $1 \leq i \leq m$, let $S_i = \{(s_{i-1}, s_i), (s_{\lfloor \frac{m}{2} \rfloor + i - 1}, s_{\lfloor \frac{m}{2} \rfloor + i})\}$.

For each i, $1 \leq i \leq m$, $E(N(C_m, K))\backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{s_i, s_i + 1, \ldots, s_{\frac{m}{2}+i} - 1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum.

The other cuts $\{S'_i, S^j_i\}$ are similar from Case 1. The 2-Partition Lemma implies that the wirelength is minimum. □

Theorem 3.9. *The exact wirelength of circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$ into $N(C_m, K)$, is given by*

$$
\begin{aligned}
WL(G, N(C_m, K)) &= \sum_{i=1}^{m/2} \theta_G(t_i + t_{i+1} + \cdots + t_{i+\frac{m}{2}-1}) + \frac{1}{2}\sum_{i=1}^{m} \theta_G(t_i - 1) \\
&\quad + j(s_m - m).
\end{aligned}
$$

Proof. By Congestion Lemma,

(i) $EC_f(S_i) = EC_f(S_{i'}) = \theta_G(t_i + t_{i+1} + \cdots + t_{i+\frac{m}{2}-1})$, $1 \leq i \leq \frac{m}{2}$

(ii) $EC_f(S'_i) = \theta_G(t_i - 1)$, $1 \leq i \leq m$ and

(iii) $EC_f(S^j_i) = 2j$, $1 \leq i \leq m$ and $1 \leq j \leq t_i - 1$.

Then by 2-Partition Lemma,

$$
\begin{aligned}
WL(G, N(C_m, K)) &= \frac{1}{2}[2\sum_{i=1}^{m/2} \theta_G(t_i + t_{i+1} + \cdots + t_{i+\frac{m}{2}-1}) + \sum_{i=1}^{m} \theta_G(t_i - 1) \\
&\quad + 2j(s_m - m)] \\
&= \sum_{i=1}^{m/2} \theta_G(t_i + t_{i+1} + \cdots + t_{i+\frac{m}{2}-1}) + \frac{1}{2}\sum_{i=1}^{m} \theta_G(t_i - 1) \\
&\quad + j(s_m - m). \quad \square
\end{aligned}
$$

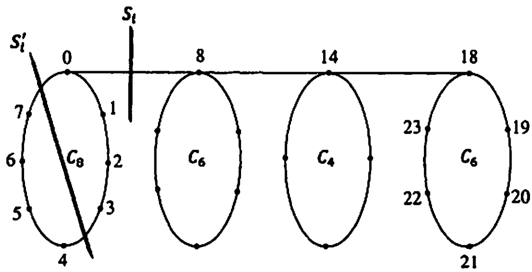

Figure 5: The edge cuts of necklace $N(P_4; C_8, C_6, C_4, C_6)$

Definition 3.10. *Let $P = v_1 v_2 \dots v_m$ be a path. Let C_{t_i} be a cycle on t_i vertices such that $P \uplus C_{t_i}$ has just v_i as a cut-vertex, $i = 1, 2, \dots, m$. The resultant graph $P \uplus (\bigcup_{i=1}^{m} C_{t_i})$ is a necklace denoted by $N(P_m; C_{t_1}, C_{t_2}, \dots, C_{t_m})$.*

Remark 3.11. *$N(P_m; C_{t_1}, C_{t_2}, \dots, C_{t_m})$ has $n = \sum_{i=1}^{m} t_i$ vertices. We denote $\sum_{i=0}^{k} t_i$ by s_k, $0 \leq k \leq m$, where $t_0 = 0$. For brevity, the necklace $N(P_m; C_{t_1}, C_{t_2}, \dots, C_{t_m})$ will be represented by $N(P_m, C)$.*

Embedding Algorithm C

Input : A circulant network $G(n; \pm\{1, 2, \dots, j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$ and a necklace $N(P_m, C)$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \dots, j\})$ as $0, 1, 2, \dots, n-1$ in the clockwise sense. Label the consecutive vertices of C_{t_i} in $N(P_m, C)$ as $s_{i-1} + j$, $j = 0, 1, 2, \dots, t_i - 1$ such that s_{i-1} is the label of v_i, $1 \leq i \leq m$. See Figure 5.

Output : The exact wirelength of embedding circulant network $G(n; \pm\{1, 2, \dots, j\})$ into necklace $N(P_m, C)$ given by the mapping $f(x) = x$.

Proof of correctness:

We assume that the labels represent the vertices to which they are assigned for $1 \leq i \leq m$, depending on the number of vertices in C_{t_i}, the following two cases arise.

Case 1 (t_i is even): For $1 \leq i \leq m-1$, let $S_i = \{(s_{i-1}, s_i)\}$. For $1 \leq i \leq m$, let $S_i' = \{(s_{i-1} + j, s_{i-1} + j + 1), (s_{i-1} + \frac{t_i}{2} + j, s_{i-1} + \frac{t_i}{2} + j + 1) :$

$0 \leq j \leq \frac{t_i}{2}-1\}$. See Figure 5. Then $\{S_i : 1 \leq i \leq m-1\} \cup \{S'_i : 1 \leq i \leq m\}$ is a partition of $[E(N(P_m, C))]$.

For each i, $1 \leq i \leq m-1$, $E(N(P_m, C)) \backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{0, 1, 2, \ldots, s_i - 1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum.

For each i, $1 \leq i \leq m$, $E(N(P_m, C)) \backslash S'_i$ has two components H'_{i1} and H'_{i2}, where $V(H'_{i1}) = \{s_{i-1} + j + 1, s_{i-1} + j + 2, \ldots, s_{i-1} + j + \frac{t_i}{2}\}$. Let $G'_{i1} = f^{-1}(H'_{i1})$ and $G'_{i2} = f^{-1}(H'_{i2})$. By Theorem 3.2, G'_{i1} is an optimal set, each S'_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S'_i)$ is minimum.

Case 2 (t_i is odd): For $1 \leq i \leq m-1$, let $S_i = S_{i'} = \{(s_{i-1}, s_i)\}$. For $1 \leq i \leq m$, let $S'_i = \{(s_{i-1}+j, s_{i-1}+j+1), (s_{i-1}+\lfloor \frac{t_i}{2} \rfloor+j, s_{i-1}+\lfloor \frac{t_i}{2} \rfloor+j+1) : 0 \leq j \leq t_i - 1\}$. Then $\{S_i, S_{i'} : 1 \leq i \leq m-1\} \cup \{S'_i : 1 \leq i \leq m\}$ is a partition of $[2E(N(P_m, C))]$.

For each i, $1 \leq i \leq m-1$, $E(N(P_m, C)) \backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{0, 1, 2, \ldots, s_i - 1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. Similarly, $EC_f(S_{i'})$ is minimum.

For each i, $1 \leq i \leq m$, $E(N(P_m, C)) \backslash S'_i$ has two components H'_{i1} and H'_{i2}, where $V(H'_{i1}) = \{s_{i-1} + j + 1, s_{i-1} + j + 2, \ldots, s_{i-1} + j + \lfloor \frac{t_i}{2} \rfloor\}$. Let $G'_{i1} = f^{-1}(H'_{i1})$ and $G'_{i2} = f^{-1}(H'_{i2})$. By Theorem 3.2, G'_{i1} is an optimal set, each S'_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S'_i)$ is minimum. The Partition Lemma and 2-Partition Lemma imply that the wirelength is minimum. □

Theorem 3.12. *The exact wirelength of circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$ into $N(P_m, C)$, is given by*

$$WL(G, N(P_m, C)) = \sum_{i=1}^{m-1} \theta_G(s_i - 1) + \sum_{i=1}^{m} \frac{t_i}{2} \theta_G(\lfloor \frac{t_i}{2} \rfloor).$$

Proof. By Congestion Lemma,

(i) $EC_f(S_i) = EC_f(S_{i'}) = \theta_G(s_i - 1)$, $1 \leq i \leq m-1$ and

(ii) $EC_f(S'_i) = t_i\ \theta_G(\lfloor \frac{t_i}{2} \rfloor)$, $1 \leq i \leq m$.

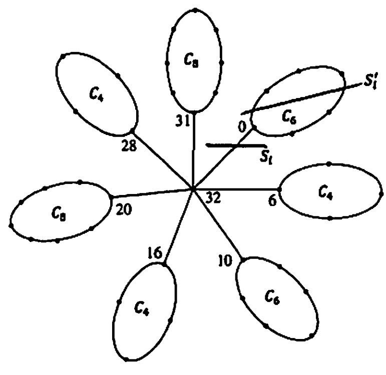

Figure 6: The edge cuts of necklace $N(K_{1,7}; C_6, C_4, \ldots, C_8)$

Then by Partition Lemma and 2-Partition Lemma,

$$\begin{aligned} WL(G, N(P_m, C)) &= \frac{1}{2}[2\sum_{i=1}^{m-1}\theta_G(s_i - 1) + \sum_{i=1}^{m} t_i\, \theta_G(\lfloor\frac{t_i}{2}\rfloor)] \\ &= \sum_{i=1}^{m-1}\theta_G(s_i - 1) + \sum_{i=1}^{m}\frac{t_i}{2}\, \theta_G(\lfloor\frac{t_i}{2}\rfloor). \quad \square \end{aligned}$$

Definition 3.13. *Let $K_{1,m}$ be a star graph on $m+1$ vertices $v_0, v_1, \ldots, v_m$. Let C_{t_i} be a cycle on t_i vertices such that $K_{1,m} \cup C_{t_i}$ has just v_i as a cut-vertex, $i = 1, 2, \ldots, m$. The resultant graph $K_{1,m} \cup (\biguplus_{i=1}^{m} C_{t_i})$ is a necklace denoted by $N(K_{1,m}; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$.*

Remark 3.14. *$N(K_{1,m}; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$ has $n = \sum_{i=1}^{m} t_i + 1$ vertices. We denote $\sum_{i=o}^{k} t_i$ by s_k, $0 \leq k \leq m$, where $t_0 = 0$. For brevity, the necklace $N(K_{1,m}; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$ will be represented by $N(K_{1,m}, C)$.*

Embedding Algorithm D

Input : A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$ and a necklace $N(K_{1,m}, C)$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$ as $0, 1, 2, \ldots, n-1$ in the clockwise sense. Label the consecutive vertices of C_{t_i} in $N(K_{1,m}, C)$ as $s_{i-1} + j$, $j = 0, 1, 2, \ldots, t_i - 1$ such that s_{i-1} is the label of v_i, $1 \leq i \leq m$. See Figure 6.

Output : The exact wirelength of embedding circulant network $G(n; \pm\{1, 2, \ldots, j\})$ into necklace $N(K_{1,m}, C)$ given by the mapping $f(x) = x$.

Proof of correctness:

We assume that the labels represent the vertices to which they are assigned for $1 \le i \le m$, depending on the number of vertices in C_{t_i}, the following two cases arise.

Case 1 (t_i is even): For $1 \le i \le m$, let $S_i = \{(s_{i-1}, n)\}$. For $1 \le i \le m$, let $S_i' = \{(s_{i-1}+j, s_{i-1}+j+1), (s_{i-1}+\frac{t_i}{2}+j, s_{i-1}+\frac{t_i}{2}+j+1) : 0 \le j \le \frac{t_i}{2}-1\}$. See Figure 6. Then $\{S_i, S_i' : 1 \le i \le m\}$ is a partition of $[E(NN(K_{1,m}, C))]$.

For each i, $1 \le i \le m$, $E(N(K_{1,m}, C))\backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{t_{i-1}, t_{i-1}+1, \ldots, t_{i-1}+t_i-1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum.

For each i, $1 \le i \le m$, $E(N(K_{1,m}, C))\backslash S_i'$ has two components H_{i1}' and H_{i2}', where $V(H_{i1}') = \{s_{i-1}+j+1, s_{i-1}+j+2, \ldots, s_{i-1}+j+\frac{t_i}{2}\}$. Let $G_{i1}' = f^{-1}(H_{i1}')$ and $G_{i2}' = f^{-1}(H_{i2}')$. By Theorem 3.2, G_{i1}' is an optimal set, each S_i' satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i')$ is minimum.

Case 2 (t_i is odd): For $1 \le i \le m$, let $S_i = S_{i'} = \{(s_{i-1}, n)\}$. For $1 \le i \le m$, let $S_i' = \{(s_{i-1}+j, s_{i-1}+j+1), (s_{i-1}+\lfloor\frac{t_i}{2}\rfloor+j, s_{i-1}+\lfloor\frac{t_i}{2}\rfloor+j+1) : 0 \le j \le t_i-1\}$. Then $\{S_i, S_{i'}, S_i' : 1 \le i \le m\}$ is a partition of $[2E(N(K_{1,m}, C))]$.

For each i, $1 \le i \le m$, $E(N(K_{1,m}, C))\backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{t_{i-1}, t_{i-1}+1, \ldots, t_{i-1}+t_i-1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. Similarly $EC_f(S_{i'})$ is minimum.

For each i, $1 \le i \le m$, $E(N(K_{1,m}, C))\backslash S_i'$ has two components H_{i1}' and H_{i2}', where $V(H_{i1}') = \{s_{i-1}+j+1, s_{i-1}+j+2, \ldots, s_{i-1}+j+\lfloor\frac{t_i}{2}\rfloor\}$. Let $G_{i1}' = f^{-1}(H_{i1}')$ and $G_{i2}' = f^{-1}(H_{i2}')$. By Theorem 3.2, G_{i1}' is an optimal set, each S_i' satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i')$ is minimum. The Partition Lemma and 2-Partition Lemma imply that the wirelength is minimum. □

Theorem 3.15. *The exact wirelength of circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j < \lfloor n/2 \rfloor$ into $N(K_{1,m}, C)$, is given by*

$$WL(G, N(K_{1,m}, C)) = \sum_{i=1}^{m} \theta_G(t_i) + \sum_{i=1}^{m} \frac{t_i}{2} \theta_G(\lfloor \frac{t_i}{2} \rfloor).$$

Proof. By Congestion Lemma,

(i) $EC_f(S_i) = EC_f(S_{i'}) = \theta_G(t_i)$, $1 \le i \le m$ and

(ii) $EC_f(S_i') = t_i\ \theta_G(\lfloor \frac{t_i}{2} \rfloor)$, $1 \le i \le m$.

Then by Partition Lemma and 2-Partition Lemma,

$$\begin{aligned} WL(G, N(K_{1,m}, C)) &= \frac{1}{2}[2\sum_{i=1}^{m}\theta_G(t_i) + \sum_{i=1}^{m} t_i\ \theta_G(\lfloor \frac{t_i}{2} \rfloor)] \\ &= \sum_{i=1}^{m}\theta_G(t_i) + \sum_{i=1}^{m}\frac{t_i}{2}\theta_G(\lfloor \frac{t_i}{2} \rfloor). \quad \square \end{aligned}$$

Definition 3.16. *Let K_m be a complete graph on m vertices. Let C_{t_i} be a cycle on t_i vertices such that $K_m \uplus C_{t_i}$ has just v_i as a cut-vertex, $i = 1, 2, \ldots, m$. The resultant graph $K_m \uplus (\bigcup_{i=1}^{m} C_{t_i})$ is a necklace denoted by $N(K_m; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$.*

Remark 3.17. *$N(K_m; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$ has $n = \sum_{i=1}^{m} t_i$ vertices. We denote $\sum_{i=0}^{k} t_i$ by s_k, $0 \le k \le m$, where $t_0 = 0$. For brevity, the graph $N(K_m; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$ will be represented by $N(K_m, C)$.*

Embedding Algorithm E

Input : A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j \le \lfloor n/2 \rfloor$ and a necklace $N(K_m, C)$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$ as $0, 1, 2, \ldots, n-1$ in the clockwise sense. Label the vertices of C_{t_i} in $N(K_m, C)$ as $s_{i-1} + j$, $j = 0, 1, 2, \ldots, t_i - 1$ such that s_{i-1} is the label of v_i, $1 \le i \le m$. See Figure 7.

Output : The exact wirelength of embedding circulant network $G(n; \pm\{1, 2, \ldots, j\})$ into necklace $N(K_m, C)$ given by the mapping $f(x) = x$.

Proof of correctness:

We assume that the labels represent the vertices to which they are assigned for $1 \le i \le m$, depending on the number of vertices in C_{t_i}, the following two cases arise.

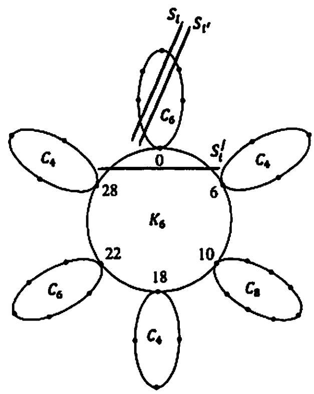

Figure 7: The edge cuts of necklace $N(K_6; C_6, C_4, \ldots, C_4)$

Case 1 (t_i is even): For $1 \leq i \leq m$, let $S_i = S_{i'} = \{(s_{i-1} + j, s_{i-1} + j + 1), (s_{i-1} + \frac{t_i}{2} + j, s_{i-1} + \frac{t_i}{2} + j + 1) : 1 \leq j \leq \frac{t_i}{2} - 1\}$. For $1 \leq i \leq m$ and $0 \leq j \leq m-1$, let $S_i^j = \{(s_{i-1} + s_j, s_{i-1} + s_k) : 0 \leq k \leq m-1$ and $j \neq k\}$. See Figure 7. Then $\{S_i, S_{i'} : 1 \leq i \leq m\} \cup \{S_i^j : 1 \leq i \leq m$ and $0 \leq j \leq m-1\}$ is a partition of $[2E(N(K_m, C))]$.

For each i, $1 \leq i \leq m$, $E(N(K_m, C)) \backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{s_{i-1} + j + 1, s_{i-1} + j + 2, \ldots, s_{i-1} + j + \frac{t_i}{2}\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. Similarly, $EC_f(S_i')$ is minimum.

For each i, j, $1 \leq i \leq m$ and $0 \leq j \leq m-1$, $E(N(K_m, C)) \backslash S_i^j$ has two components H_{i1}^j and H_{i2}^j, where $V(H_{i1}^j) = \{s_{i-1}, s_{i-1} + 1, \ldots, s_i - 1\}$. Let $G_{i1}^j = f^{-1}(H_{i1}^j)$ and $G_{i2}^j = f^{-1}(H_{i2}^j)$. By Theorem 3.2, G_{i1}^j is an optimal set, each S_i^j satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i^j)$ is minimum. The 2-Partition Lemma implies that the wirelength is minimum.

Case 2 (t_i is odd): For $1 \leq i \leq m$, let $S_i = \{(s_{i-1} + j, s_{i-1} + j + 1), (s_{i-1} + \lfloor \frac{t_i}{2} \rfloor + j, s_{i-1} + \lfloor \frac{t_i}{2} \rfloor + j + 1) : 0 \leq j \leq t_i - 1\}$. For $1 \leq i \leq m$ and $0 \leq j \leq m-1$, let $S_i^j = \{(s_{i-1} + s_j, s_{i-1} + s_k) : 0 \leq k \leq m-1$ and $j \neq k\}$. Then $\{S_i : 1 \leq i \leq m\} \cup \{S_i^j : 1 \leq i \leq m$ and $0 \leq j \leq m-1\}$ is a partition of $[2E(N(K_m, C))]$.

For each i, $1 \leq i \leq m$, $E(N(K_m, C)) \backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{s_{i-1} + j + 1, s_{i-1} + j + 2, \ldots, s_{i-1} + j + \lfloor \frac{t_i}{2} \rfloor\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum.

For each i, j, $1 \leq i \leq m, 0 \leq j \leq m-1$, $E(N(K_m, C))\backslash S_i^j$ has two components H_{i1}^j and H_{i2}^j, where $V(H_{i1}^j) = \{s_{i-1}, s_{i-1}+1, \ldots, s_i - 1\}$. Let $G_{i1}^j = f^{-1}(H_{i1}^j)$ and $G_{i2}^j = f^{-1}(H_{i2}^j)$. By Theorem 3.2, G_{i1}^j is an optimal set, each S_i^j satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i^j)$ is minimum. The 2-Partition Lemma implies that the wirelength is minimum. □

Theorem 3.18. *The exact wirelength of circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$ into $N(K_m, C)$, is given by*

$$WL(G, N(K_m, C)) = \sum_{i=1}^{m} \frac{t_i}{2} \theta_G(\lfloor \frac{t_i}{2} \rfloor) + \frac{1}{2} \sum_{i=1}^{m} \theta_G(t_i).$$

Proof. By Congestion Lemma,

(i) $EC_f(S_i) = t_i\ \theta_G(\lfloor \frac{t_i}{2} \rfloor)$, $1 \leq i \leq m$ and

(ii) $EC_f(S_i^j) = \theta_G(t_i)$, $1 \leq i \leq m$ and $0 \leq j \leq m-1$.

Then by 2-Partition Lemma,

$$\begin{aligned} WL(G, N(K_m, C)) &= \frac{1}{2}[\sum_{i=1}^{m} t_i\ \theta_G(\lfloor \frac{t_i}{2} \rfloor) + \sum_{i=1}^{m} \theta_G(t_i)] \\ &= \sum_{i=1}^{m} \frac{t_i}{2} \theta_G(\lfloor \frac{t_i}{2} \rfloor) + \frac{1}{2} \sum_{i=1}^{m} \theta_G(t_i). \quad \square \end{aligned}$$

4 Embedding of Circulant Networks into Windmill graphs

In this section, we compute the exact wirelength of embedding circulant networks into windmill graphs.

Definition 4.1. *Let C_{t_i} be a cycle on t_i vertices such that $\uplus C_{t_i}$ has just v_1 is a cut-vertex. The resultant graph $P_1 \uplus (\bigcup_{i=1}^{m} C_{t_i})$ is a windmill graph incident with a common vertex v_1 denoted by $WM(P_1; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$.*

Remark 4.2. *$WM(P_1; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$ has $n = \sum_{i=1}^{m} t_i - m + 1$ vertices.*

We denote $\sum_{i=0}^{k} t_i$ by s_k, $0 \leq k \leq m$, where $t_0 = 0$. See Figure 8. For brevity, the graph $WM(P_1; C_{t_1}, C_{t_2}, \ldots, C_{t_m})$ will be represented by $WM(P, C)$.

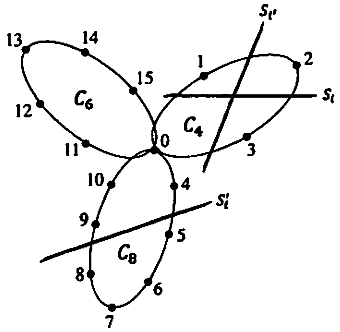

Figure 8: The edge cuts of $WM(P_1; C_4, C_8, C_6)$

Embedding Algorithm F

Input : A circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$ and $WM(P, C)$.

Algorithm : Label the consecutive vertices of $G(n; \pm 1)$ in $G(n; \pm\{1, 2, \ldots, j\})$ as $0, 1, 2, \ldots, n-1$ in the clockwise sense. Label the consecutive vertices of C_{t_i} in $WM(P, C)$ as $s_{i-1} + j$, $j = 0, 1, 2, \ldots, t_i - i$ such that $s_1 - 1$ is the label of v_1, $1 \leq i \leq m$. See Figure 8.

Output : The exact wirelength of embedding circulant network $G(n; \pm\{1, 2, \ldots, j\})$ into windmill $WM(P, C)$ given by the mapping $f(x) = x$.

Proof of correctness:

We assume that the labels represent the vertices to which they are assigned.
Case 1 (t_i is even for all i): For $1 \leq i \leq m$, let $S_i = \{(0, s_{i-1} + 2 - i), (s_{i-1} + \frac{t_i}{2} + 1 - i, s_{i-1} + \frac{t_i}{2} + 2 - i)\}$. For $1 \leq i \leq m$, let $S_{i'} = \{(0, s_i - 1), (s_{i-1} + \frac{t_i}{2} - i, s_{i-1} + \frac{t_i}{2} + 1 - i)\}$. For $1 \leq i \leq m$, let $S_i' = \{(s_{i-1} + 2 - i + j, s_{i-1} + 3 - i + j), (s_{i-1} + \frac{t_i}{2} + 2 - i + j, s_{i-1} + \frac{t_i}{2} + 3 - i + j) : 0 \leq j \leq \frac{t_i}{2} - 3\}$. See Figure 8. Then $\{S_i, S_{i'}, S_i' : 1 \leq i \leq m\}$, is a partition of $[E(WM(P, C))]$.

For each i, $1 \leq i \leq m$, $E(WM(P, C)) \backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{s_{i-1} + 2 - i, s_{i-1} + 3 - i, \ldots, s_{i-1} + 1 - i + \frac{t_i}{2}\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Theorem 3.2, G_{i1} is an optimal set, each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum.

For each i, $1 \le i \le m$, $E(WM(P,C))\backslash S_{i'}$ has two components $H_{i'1}$ and $H_{i'2}$, where $V(H_{i'1}) = \{s_{i-1}+1-i+\frac{t_i}{2}, s_{i-1}-i+\frac{t_i}{2}, \ldots, s_i - 1\}$. Let $G_{i'1} = f^{-1}(H_{i'1})$ and $G_{i'2} = f^{-1}(H_{i'2})$. By Theorem 3.2, $G_{i'1}$ is an optimal set, each S'_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S'_i)$ is minimum.

For each i, $1 \le i \le m$, $E(WM(P,C))\backslash S'_i$ has two components H'_{i1} and H'_{i2}, where $V(H'_{i1}) = \{s_{i-1}+3-i+j, \ldots, s_{i-1}+2-i+j+\frac{t_i}{2}\}$. Let $G'_{i1} = f^{-1}(H'_{i1})$ and $G'_{i2} = f^{-1}(H'_{i2})$. By Theorem 3.2, G'_{i1} is an optimal set, each S'_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S'_i)$ is minimum. The Partition Lemma implies that the wirelength is minimum.

Case 2 (t_i is odd for all i): Using the proof techniques employed in Case 2 of Embedding Algorithm E, the wirelength is minimum. □

Theorem 4.3. *The exact wirelength of circulant network $G(n; \pm\{1,2,\ldots,j\})$, $1 \le j < \lfloor n/2 \rfloor$ into $WM(P,C)$, is given by*

$$WL(G, WM(P,C)) = \sum_{i=1}^{m} \frac{t_i}{2}\theta_G(\lfloor \frac{t_i}{2} \rfloor).$$

Proof. By Congestion Lemma,

(i) $EC_f(S_i) = EC_f(S_{i'}) = t_i\ \theta_G(\lfloor \frac{t_i}{2} \rfloor)$, $1 \le i \le m$ and

(ii) $EC_f(S'_i) = t_i\ \theta_G(\lfloor \frac{t_i}{2} \rfloor)$, $1 \le i \le m$.

Then by Partition Lemma and 2-Partition Lemma,

$$\begin{aligned} WL(G, WM(P,C)) &= \frac{1}{2}[\sum_{i=1}^{m} \frac{t_i}{2}\ \theta_G(\lfloor \frac{t_i}{2} \rfloor) + \frac{t_i}{2}\ \theta_G(\lfloor \frac{t_i}{2} \rfloor)] \\ &= \sum_{i=1}^{m} \frac{t_i}{2}\theta_G(\lfloor \frac{t_i}{2} \rfloor). \quad \square \end{aligned}$$

5 Time Complexity

In computer science, the time complexity of an algorithm quantifies the amount of time taken by an algorithm to run as a function of the size of the input to the problem. An algorithm is said to take linear time, or $O(n)$ time, if its time complexity is $O(n)$. Informally, this means that for large enough input sizes the running time increases linearly with the size of the input.

Linear time is often viewed as a desirable attribute for an algorithm. Much research has been invested into creating algorithms exhibiting (nearly) linear time or better. This research includes both software and hardware methods. In the case of hardware, some algorithms which, mathematically speaking, can never achieve linear time with standard computation models are able to run in linear time. There are several hardware technologies which exploit parallelism to provide this. An example is content-addressable memory. This concept of linear time is used in string matching algorithms such as the Boyer-Moore Algorithm and Ukkonen's Algorithm [39, 40].

In this Section, we compute the time complexity of finding the exact wirelength of embedding circulant networks into necklace graphs using Embedding Algorithm A. The algorithm is formally presented as follows.

Time Complexity Algorithm

Input : The circulant network $G(n; \pm\{1, 2, \ldots, j\})$, $1 \le j \le \lfloor n/2 \rfloor$ and a necklace $N(P_m, K)$.

Algorithm : Embedding Algorithm A.

Output : The time taken to compute the minimum wirelength of embedding $G(n; \pm\{1, 2, \ldots, j\})$ into $N(P_m, K)$ is $O(2n)$, which is linear.

Method : We know that, $N(P_m, K)$ contains $n = \sum_{i=1}^{m} t_i$ vertices. For assigning the labels of n vertices, we spend n time units. By Embedding Algorithm A, we have $m(t_i + 2)$ edge cuts. Since for each cut, we need one unit of time. Thus, we need $m(t_i + 2)$ time units. Then, we need $m(t_i + 2)$ time for finding the wirelength by using 2-Partition Lemma.

$$\begin{aligned} \text{Hence the total time taken is } &= n + 2m(t_i + 2) \\ &\le 2n \end{aligned}$$

Hence, the time taken to compute the exact wirelength of embedding $G(n; \pm \{1, 2, \ldots, j\})$ into $N(P_m, K)$ is $O(2n)$, which is linear. □

Proceeding along the same lines, we can compute the exact wirelength of embedding ciruclant networks into all other graphs with linear time.

6 Concluding Remarks

In this paper, we compute the exact wirelength of embedding circulant networks into certain necklace graphs. Also, we obtain the exact wirelength of embedding circulant networks into windmill graphs. Moreover, we provide an $O(2n)$-linear time algorithm to compute the exact wirelength of embedding circulant networks into certain necklace and windmill graphs.

We also note that the necklace and windmill graphs are constructed from the complete graphs and cycles, which are important architectures for interconnection networks. Finding the dilation of embedding circulant networks into necklace and windmill graphs is under investigation.

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